Equations to know for exam 1

- M(Z,A) = A*amu
  - 1amu = atomic mass unit = u = 931.494MeV/$c^2$
- R(Z,A) ≈ (1.2fm)*$A^{1/3}$
  - fm = femtometer (a.k.a. *fermi*) = $10^{-15}$m
- ME(Z,A=Z+N) ≡ M(Z,N) - (Z+N)*u
- $BE(Z,A) = a_v f_v(A) - a_s f_s(A) - a_c f_c(Z,A) - a_a f_a(Z,A) + i a_p f_p(A)$
  - $f_v(A) = A$
  - $f_s(A) = A^{2/3}$
  - $f_c(Z,A) = \frac{Z(Z-1)}{A^{1/3}}$
  - $f_a(Z,A) = \frac{\left(Z-\frac{A}{2}\right)^2}{A}$
  - $f_p(A) = (\sqrt{A})^{-1}$ ;Even-Z, Even-N: $i = +1$; Odd-Z, Odd-N: $i = -1$; Even-Odd: $i = 0$
- $Q = \sum_{reactants} ME(Z,A) - \sum_{products} ME(Z,A)$
- $S_p(Z,N) = ME(Z-1,N) + ME(p) - ME(Z,N)$
- $S_n(Z,N) = ME(Z,N-1) + ME(n) - ME(Z,N)$
- $S_{2p}(Z,N) = ME(Z-2,N) + 2 * ME(p) - ME(Z,N)$
- $S_{2n}(Z,N) = ME(Z,N-2) + 2 * ME(n) - ME(Z,N)$
- $\rho_{ch,Fermi}(r) = \frac{\rho_0}{1+\exp\left(\frac{r-R}{a}\right)}$
- For particles: $\pi = (-1)^{\ell}$; $\pi = \prod_i \pi_i$
- $E_{rotor} = \frac{\hbar^2 j(j+1)}{2I}$
- $E_{n,oscillator} = (n + \frac{1}{2})\hbar\omega$

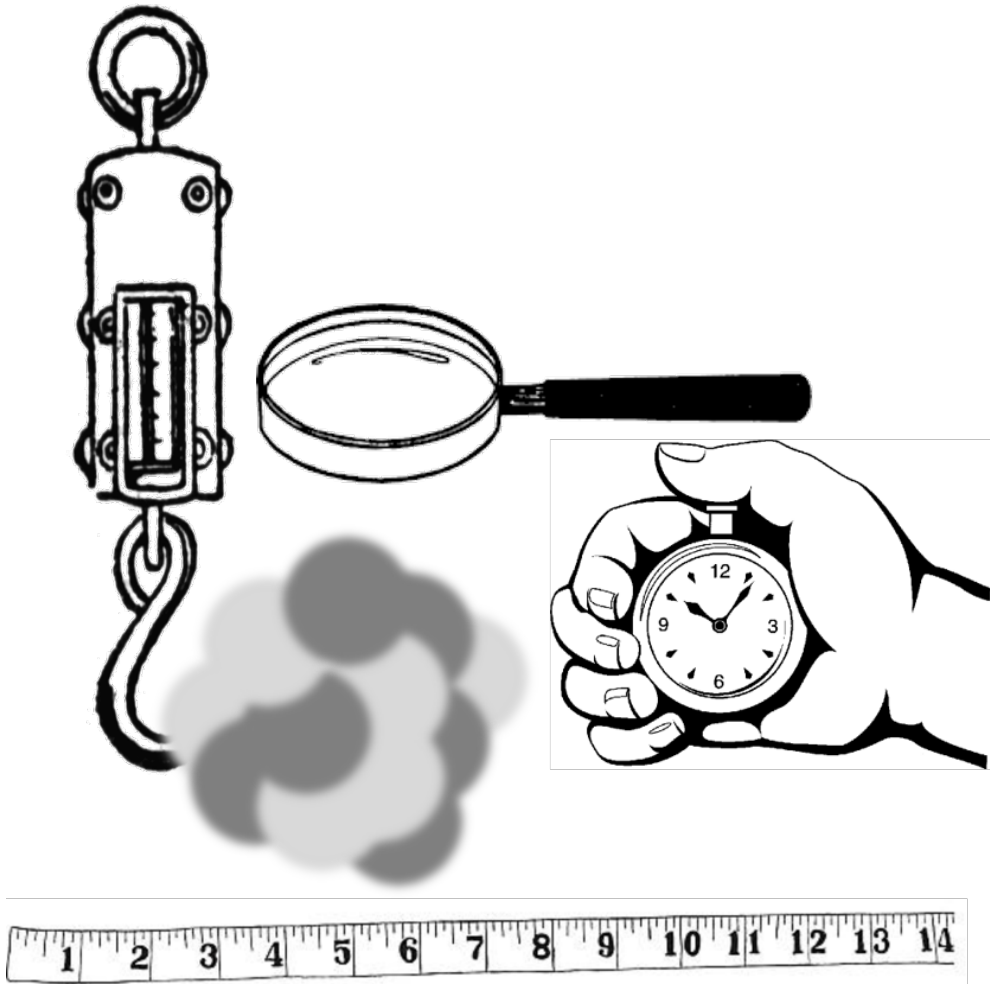
- $\lambda = \frac{h}{p}$
- $\hbar c \approx 197 MeV;\ \frac{e^2}{\hbar c} \sim 1/137$
- $\rho(E^*) \propto \exp\left(\sqrt{E^*}\right)$
- $N(t) = N(0)e^{-\lambda t}$
- $A = \lambda N$
- $t_{½} = \frac{\ln(2)}{\lambda} = \ln(2)\tau$
- $\Delta E \Delta t \geq \hbar$ ...e.g. $\Gamma = \frac{\hbar}{\tau}$
- $\lambda = \lambda_1 + \lambda_2 + \cdots = \sum_{i=1}^{N} \lambda_i$
- $V_c = \frac{Z_1 Z_2}{R_1 + R_2} \frac{e^2}{\hbar c} \hbar c$
- $t_{½,\alpha} \propto e^{Z/\sqrt{Q_\alpha}}$
- $Q_{\beta^-} = ME(Z, A) - ME(Z + 1, A)$
- $Q_{\beta^+} = ME(Z, A) - ME(Z - 1, A) - 2m_e$
- $Q_{EC} = ME(Z, A) - ME(Z - 1, A)$
- $\frac{\lambda_K}{\lambda_{\beta+}} \propto Z^3$
- $\lambda_\gamma \propto {E_\gamma}^{2l_\gamma + 1}$
- $\alpha_{IC} = \frac{\lambda_{IC}}{\lambda_\gamma} \propto Z^3$

- $x = \frac{E_C}{2E_S} = \frac{a_c}{2a_s}\frac{Z^2}{A}$
- $TKE = \frac{Z_1 Z_2 \alpha \hbar c}{1.8\left(A_1^{1/3} + A_2^{1/3}\right)}$

Equations to know for exam 1

# Lecture 1: *Nuclear Properties*

- General descriptors of nuclei
- Nuclear phenomenology
- The nuclear landscape



# The Atomic Nucleus

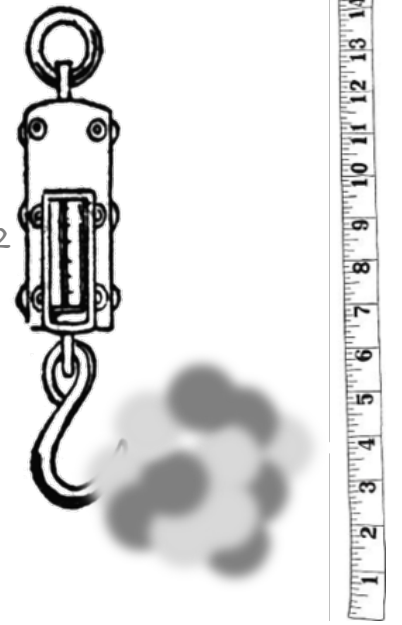


- A conglomeration of *nucleons*, a.k.a. neutrons and protons
- Convention for referring to a particular configuration, a.k.a *nuclide*
  - $^{A}_{Z}(chemical\ symbol)_{N}$
  - A = # of nucleons, Z = # of protons, N = # of neutrons.
  - The chemical symbol is from the periodic table that corresponds to Z
  - Since A=Z+N and Z is indicated by the chemical symbol, more common notation is: $^{A}chemical\ symbol$
  - For example, the most common type of carbon (Z=6) has 6 neutrons: $^{12}C$ Conversationally, you would call this "carbon twelve"
- Nuclides with the same Z but different N are *isotopes* *(though this term is often used in lieu of nuclides)*
- Nuclides with the same N but different Z are *isotones*
- Nuclides with the same A are *isobars*
- Nuclides with nicknames:
  - $^{1}H$ = proton, $^{2}H$ = deuteron (d), $^{3}H$ = triton (t), $^{4}He$ = α, $^{3}He$ = helion *(rarely used)*

An "m" after the mass-number of a nucleus indicates an isomer, a long-lived excited state. E.g. $^{26m}Al$

# How big is a nucleus?

*The definition of u being based on $^{12}C$ means it is a valuable tool for high-precision mass measurements, e.g. C.Scheidenberger et al., Nuc. Phys. A 2002*



Phenomenological estimates:

- A nucleus's mass is roughly: M(Z,A) = A*amu
  - 1amu = atomic mass unit = u = 931.494MeV/$c^2$ ≈1.66x$10^{-24}$g
  - The amu is defined such that M($^{12}$C) ≡ 12u
- A nucleus's $_{(charge)}$ radius is roughly: R(Z,A) = (1.2fm)*$A^{1/3}$
  - fm = femtometer (a.k.a. *fermi*) = $10^{-15}$m
  - The radius of a nucleon is often referred to as $r_0$=1.2fm
  - For the RMS radius, multiply by $\sqrt{3/5}$
- Therefore, an estimate for the nuclear density is:
  - $n = \frac{\#}{V} = \frac{\#}{\frac{4}{3}\pi R^3} = \frac{A}{\frac{4}{3}\pi(1.2fm)^3 A} \approx 0.14\ nucleons/fm^3$
  - For fun, in terms of mass-density: $\rho = \frac{(1.66\times10^{-24}g)A}{\frac{4}{3}\pi(1.2fm)^3 A} \approx 2.3 \times \frac{10^{-25}g}{fm^3}$

  …which doesn't sound like much, but this is 2x$10^{14}$ g/cm$^3$ (the Great Pyramid of Giza is only ~$10^{12}$ grams)

*Since A cancels in the ρ expression, the nuclear density is independent of the nuclear size, much as a liquid's density is independent of the size of the drop.*
*Partly inspired by this property, some basic nuclear calculations are based on this liquid drop analogy.*
*(G. Gamow, Proc. Roy. Soc. A 1929,1930)*

# Nuclear Transmutation



- Rules for converting one nuclide (or nuclides) to another (or others)
  - *Charge conservation*: $\sum q_{before} = \sum q_{after}$ (q from protons + positrons + electrons)
  - *Baryon conservation*: $\sum A_{before} = \sum A_{after}$ (A from neutrons + protons)
  - *Lepton number conservation*: $[N_{leptons} - N_{anti-leptons}]_{before} = [N_{leptons} - N_{anti-leptons}]_{after}$

  * Transmutation likelihoods are impacted by energetics and spin/parity selection rules

*The first nuclear reaction intentionally made in the laboratory was $^{14}N(\alpha,p)$ in 1919. (E. Rutherford, Nature 1935).*

*The first measured radioactive decay was α decay from uranium. (H. Becquerel, Comptes Rendus 1896).*

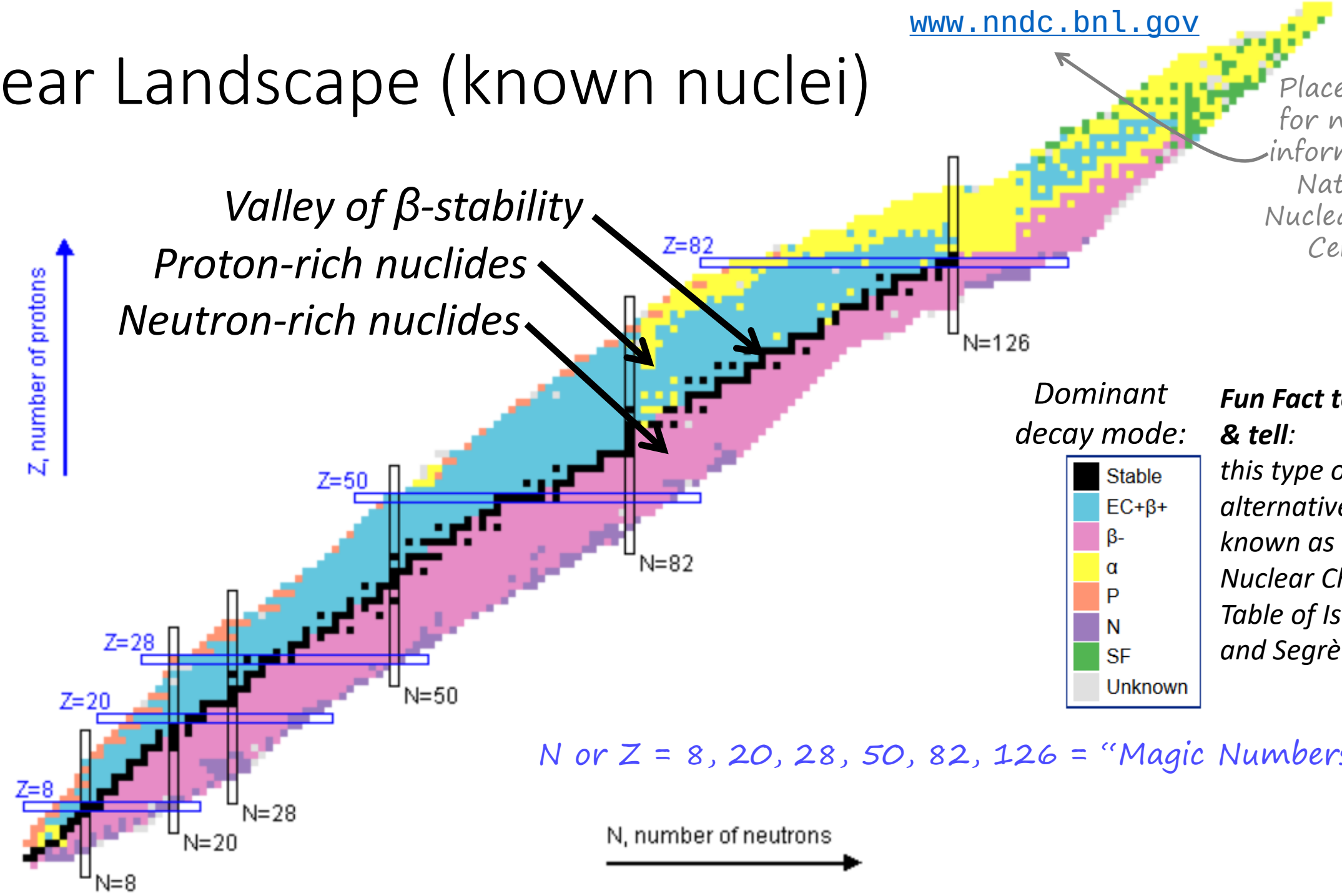
Two Types:

- Reactions
  - Multiple reactants create one or more products
  - Notation: A+b → c+D is written as *A(b,c)D*, where M(b)<M(A) and M(c)<M(D)
    - E.g. $^{12}C+\alpha \rightarrow {}^{16}O+\gamma$ is $^{12}C(\alpha,\gamma)^{16}O$ or even just $^{12}C(\alpha,\gamma)$ and is called "carbon-twelve alpha gamma"
- Decays
  - α, $\beta^+$, $\beta^-$, $e^-$-capture, β-delayed γ/p/α/n emission, fission, cluster emission, prompt γ
  - Lose nucleons for all above except β decay, $e^-$-capture, and prompt γ (following a reaction)

# Nuclear Landscape (known nuclei)

www.nndc.bnl.gov

Place to go for nuclear information: National Nuclear Data Center




Dominant decay mode:

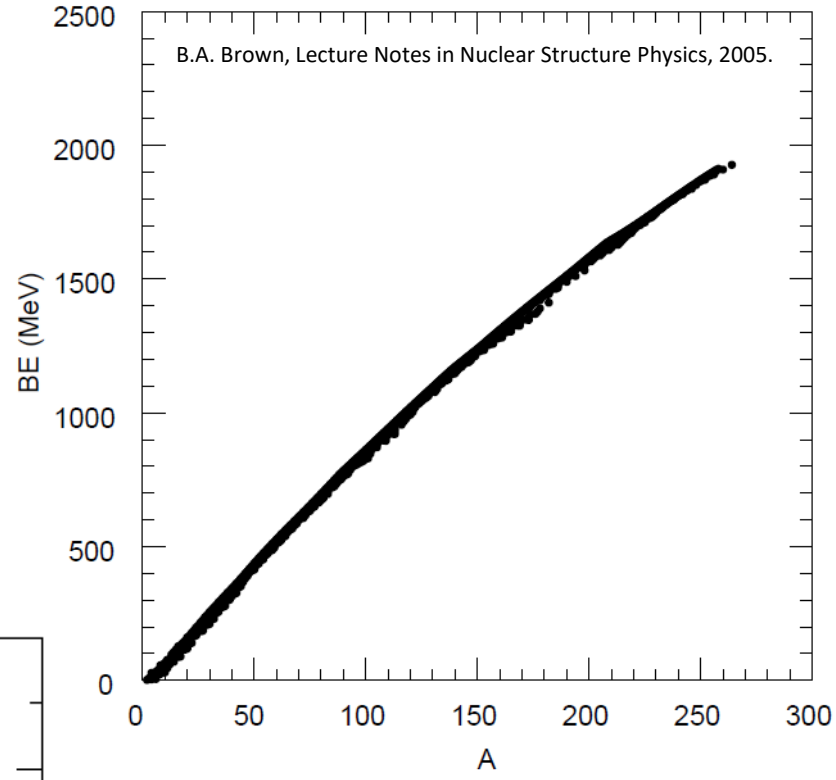
- Stable
- EC+β+
- β-
- α
- P
- N
- SF
- Unknown

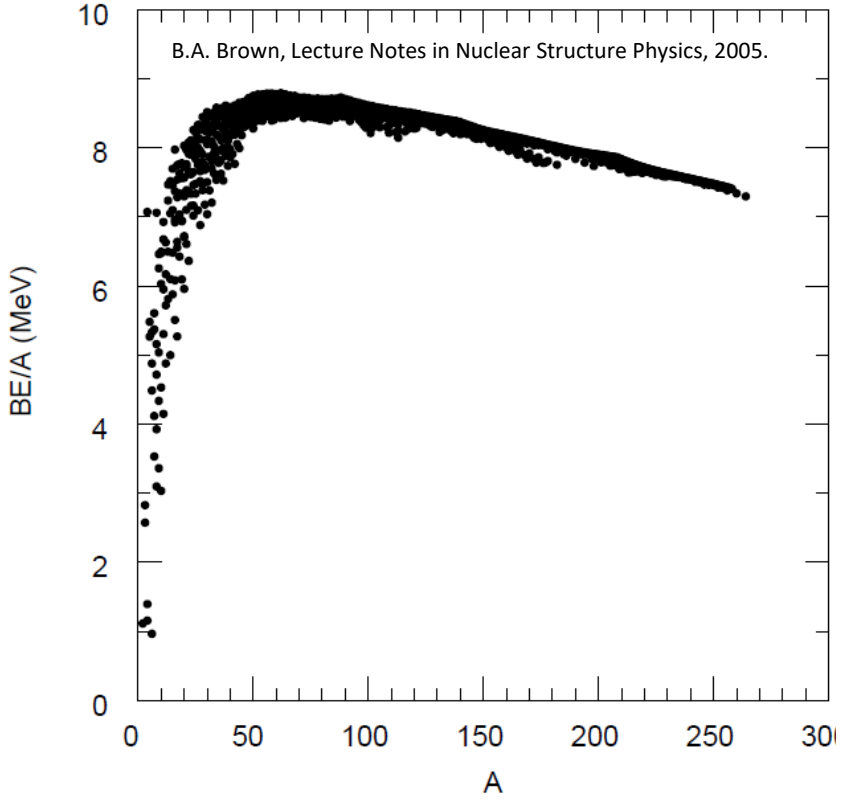
***Fun Fact to know & tell:*** *this type of plot is alternatively known as the Nuclear Chart, Table of Isotopes, and Segrè Plot*

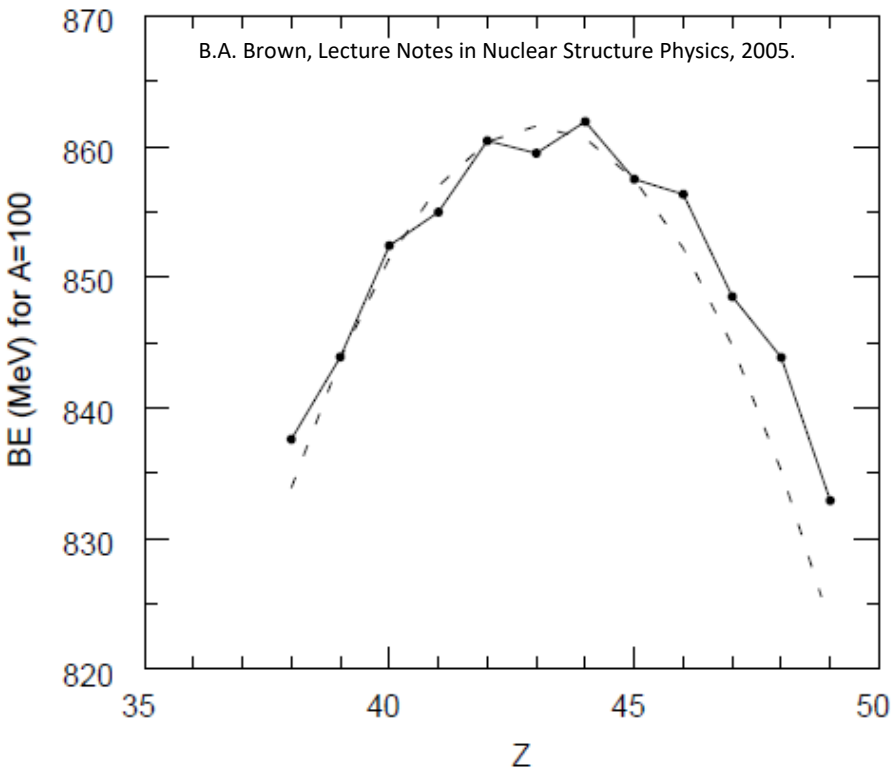
N or Z = 8, 20, 28, 50, 82, 126 = "Magic Numbers"

# Nuclear Binding Energy

- The binding energy increases nearly linearly with A:
- It's not a surprise that more nucleons would lead to more binding, since nucleons will be prone to attraction via the strong force.
- As such, a more interesting quantity is BE/A




What are the key features you notice on the two left plots?

What to they imply about
- stability to decay?
- fusion?

Why do nuclei other than the peak BE/A form at all?

# Aside on nuclear masses, atomic mass excess

- Rather than binding energy, “atomic mass excess” is more commonly used.
- The conversion can be accomplished with the following equations:
  1. $BE(Z,A) \equiv Z*(m_p + m_e) + N*m_n - M(Z,N)$
  2. $ME(Z,A) \equiv M(Z,N) - (Z+N)*m_u$

  here M(Z,N) is the mass of a nucleus with Z protons and N neutrons,

  $m_p = 938.272000\ MeV/c^2$ is the proton mass

  $m_e = 0.51100\ MeV/c^2$ is the electron mass

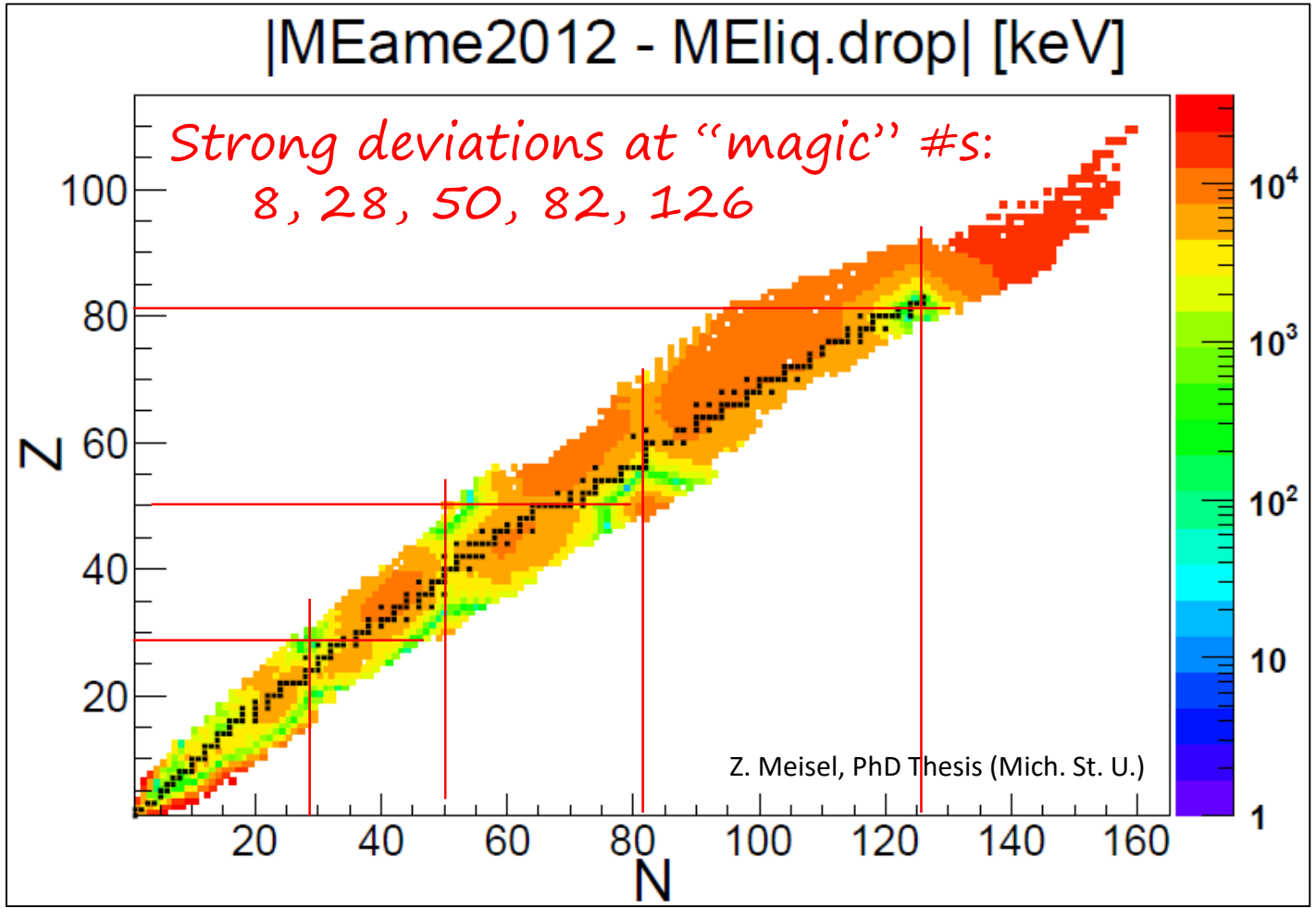
  $m_n = 939.56533\ MeV/c^2$ is the neutron mass

  $m_u = 931.494013\ MeV/c^2$ is the atomic mass unit
- Periodic evaluations are released for all known atomic masses, evaluated from a variety of experimental data by a group of noble souls.
  This is referred to as the “Atomic Mass Evaluation” or “AME”.
  The latest is from 2016 (M. Wang et al. Ch. Phys. C 2017)

# Binding Energy & Magic Numbers

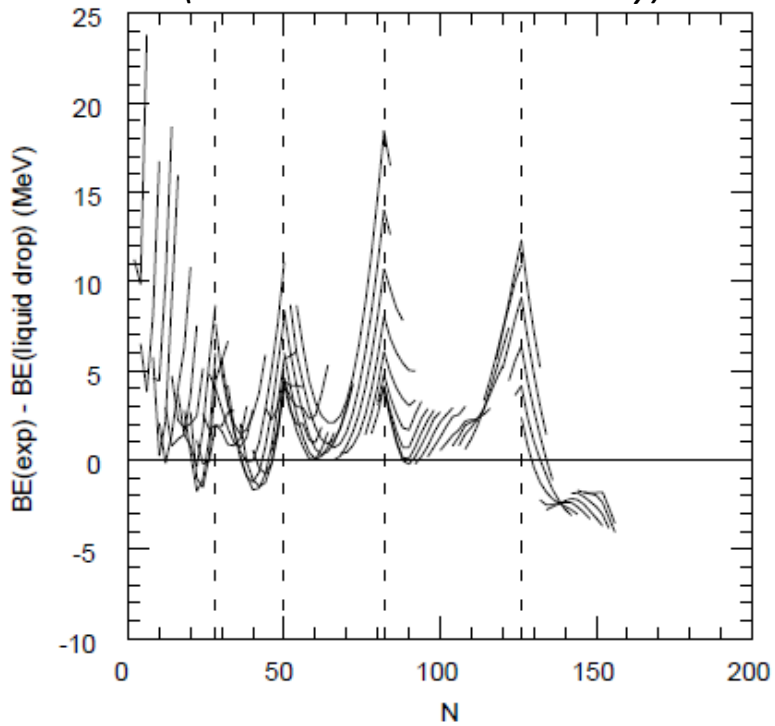
- Binding energy per nucleon looks like a pretty smooth surface, what happens if we fit a smooth function to it & plot the residual? (we'll worry about what this smooth function is next time)




*Even easier to see in 1D:*

*(here even-even nuclei only)*

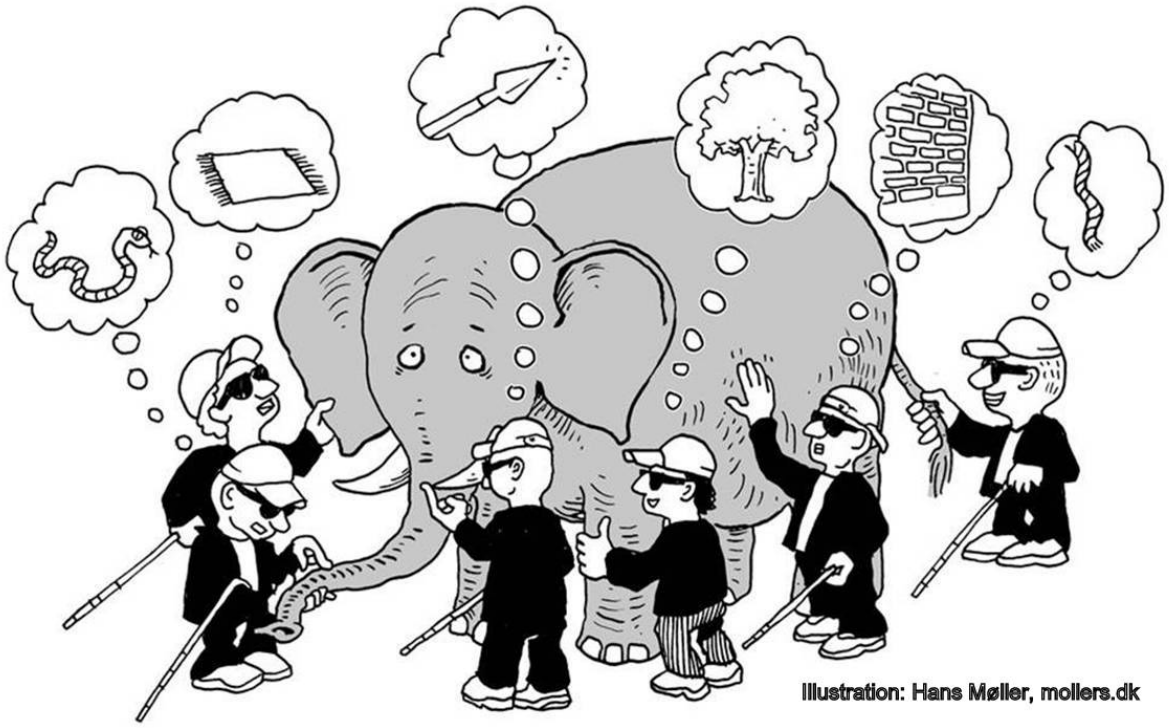



B.A. Brown, Lecture Notes in Nuclear Structure Physics, 2005.

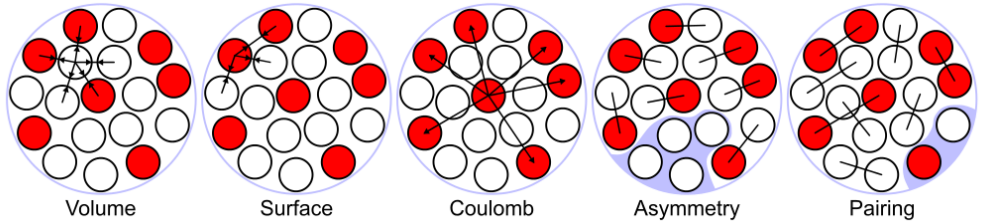
# Lecture 2: *Nuclear Phenomenology*

- Mass models & systematics
- Charge & matter distributions
- Moments & deformation
- Spin, parity, & isospin

Empirically-motivated & focused theory



# The Semi-Empirical Mass Formula




- BE(Z,A) = Volume - Surface - Coulomb - Asymmetry ± Pairing
- One mathematical parameterization* (*of many!*):

*from B. Martin, Nuclear and Particle Physics (2009)

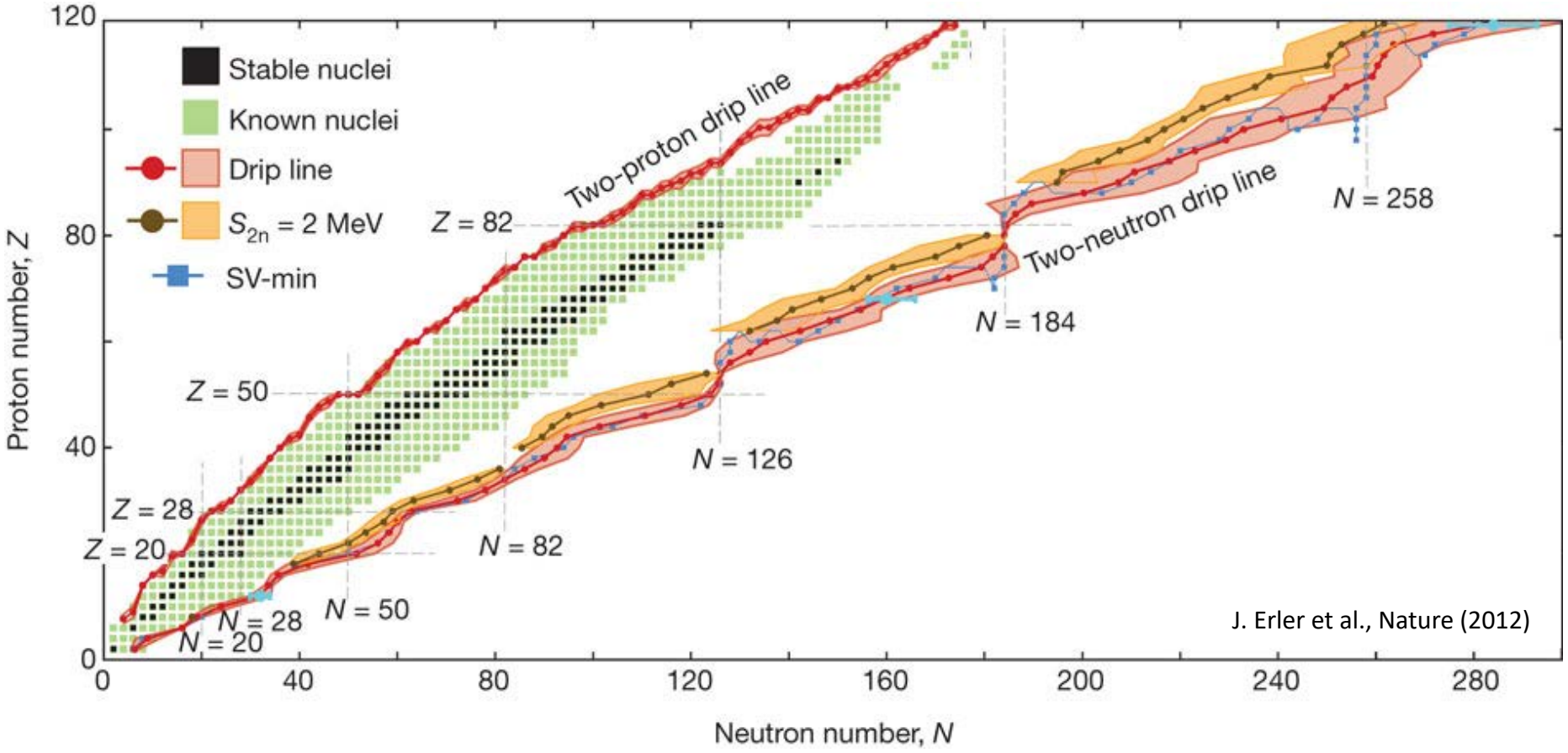
  - $BE(Z,A) = a_v f_v(A) - a_s f_s(A) - a_c f_c(Z,A) - a_a f_a(Z,A) + i a_p f_p(A)$
    - **Volume**: Nucleons cohesively bind, so: $\boldsymbol{f_v(A) = A}$
    - **Surface**: Since radius goes as $R \propto A^{1/3}$ and surface area goes as $SA \propto R^2$, $\boldsymbol{f_s(A) = A^{2/3}}$
    - **Coulomb**: Energy for a charged sphere goes as $\frac{q^2}{R}$ and $R \propto A^{1/3}$, so $\boldsymbol{f_c(Z,A) = \frac{Z(Z-1)}{A^{1/3}}}$
    - **Asymmetry**: Z=N favored (want Z=A/2) but lesser problem for large A, so $\boldsymbol{f_a(Z,A) = \frac{\left(Z-\frac{A}{2}\right)^2}{A}}$
    - **Pairing**: Favor spin-0 nucleon pairs & disfavor unpaired nucleons, empirically $\boldsymbol{f_p(A) = \left(\sqrt{A}\right)^{-1}}$
      - Even-Z, Even-N: $\boldsymbol{i = +1}$
      - Odd-Z, Odd-N: $\boldsymbol{i = -1}$
      - Even-Odd: $\boldsymbol{i = 0}$
    - $a_i$ are fit to data

A mnemonic for remembering SEMF contributions is "VSCAP".

# Nuclear Mass Differences

- The energy released in a nuclear reaction is the "Q-value"
  - $Q = \sum_{reactants} ME(Z,A) - \sum_{products} ME(Z,A)$,
  - For example, $Q_{^{68}Se(p,\gamma)^{69}Br} = ME(^{68}Se) + ME(p) - ME(^{69}Br)$
  - $= (-54.189MeV) + (7.288MeV) - (-46.260MeV)$
  - $= -0.641MeV$
- Considering the case above, we calculated the energy released by adding one proton to $^{68}$Se, which corresponds to the energy it takes to remove one proton from $^{69}$Br, a.k.a. the "proton separation energy", $S_p$
- Similarly, can calculate the energy to remove 1-neutron $S_n$, two-protons $S_{2p}$, or two-neutrons $S_{2n}$
  - $S_p(Z,N) = ME(Z-1,N) + ME(p) - ME(Z,N)$
  - $S_n(Z,N) = ME(Z,N-1) + ME(n) - ME(Z,N)$
  - $S_{2p}(Z,N) = ME(Z-2,N) + 2 * ME(p) - ME(Z,N)$
  - $S_{2n}(Z,N) = ME(Z,N-2) + 2 * ME(n) - ME(Z,N)$

# Nuclear masses define the nuclear landscape




J. Erler et al., Nature (2012)

# Charge & Matter Distributions



- The nucleus can’t be a point object.
  - Nucleons are fermions: Pauli exclusion forbids putting several in the same place.

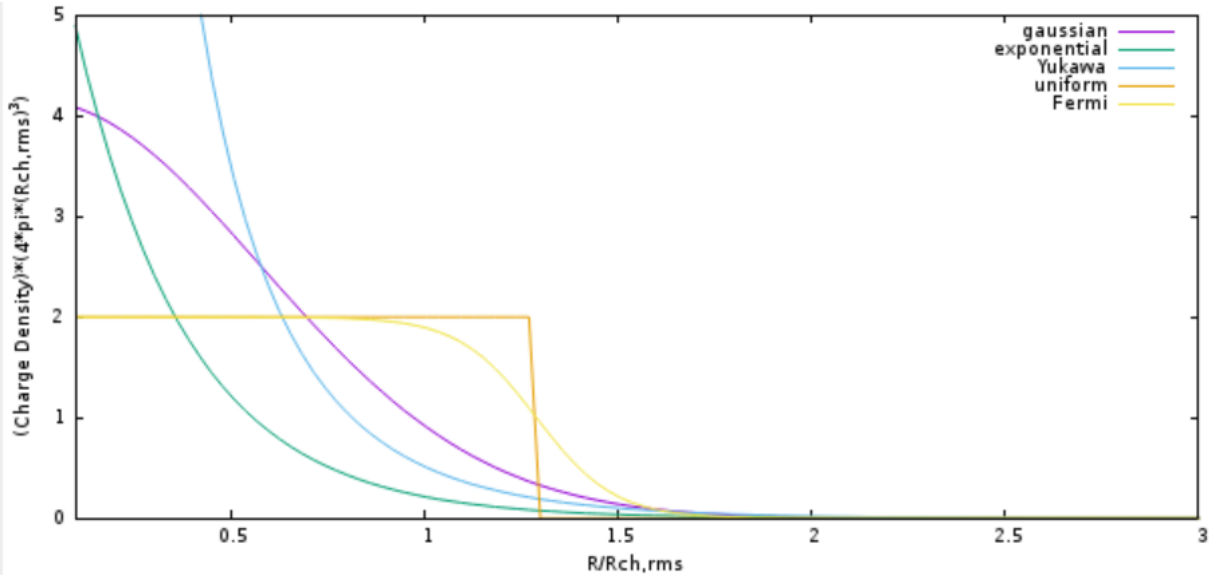
- The radial distribution of charge & matter are probed via scattering.
  - Electrons are best for the charge radius, because structure-less but charged
  - Neutrons are best for the matter radius, because not charged and simple-ish structure

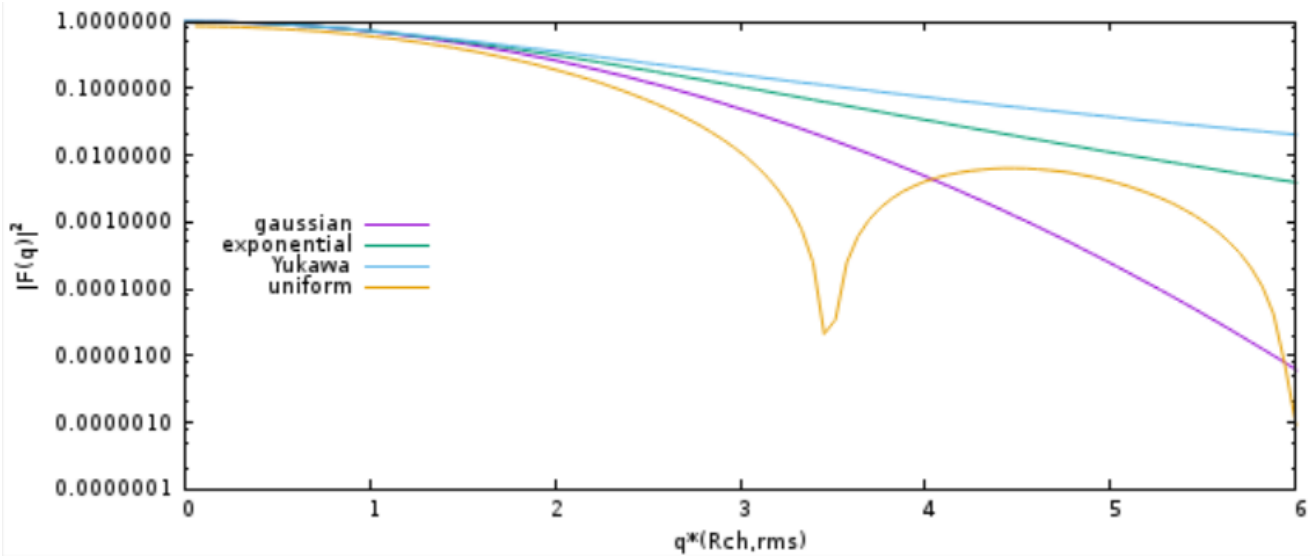
# Nuclear charge distribution

Review: R. Hofstadter, Rev. Mod. Phys. (1956)

- Luckily, sharp people have solved for form factors corresponding to typical charge distributions

  *See, e.g. Table 1 of R. Hofstadter, Rev. Mod. Phys. (1956)*

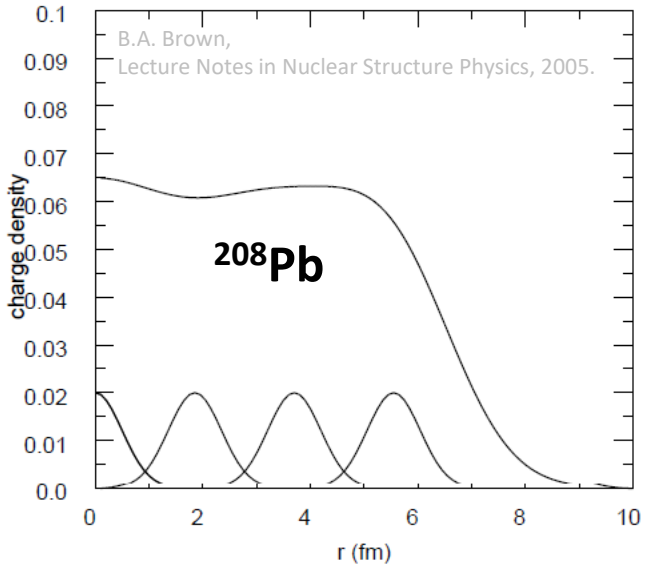



- Comparing measured $\sigma_{scatt}(\theta)$ to calculations reveals a Fermi-like distribution:
  - $\rho_{ch,Fermi}(r) = \frac{\rho_0}{1+\exp\left(\frac{r-R}{a}\right)}$

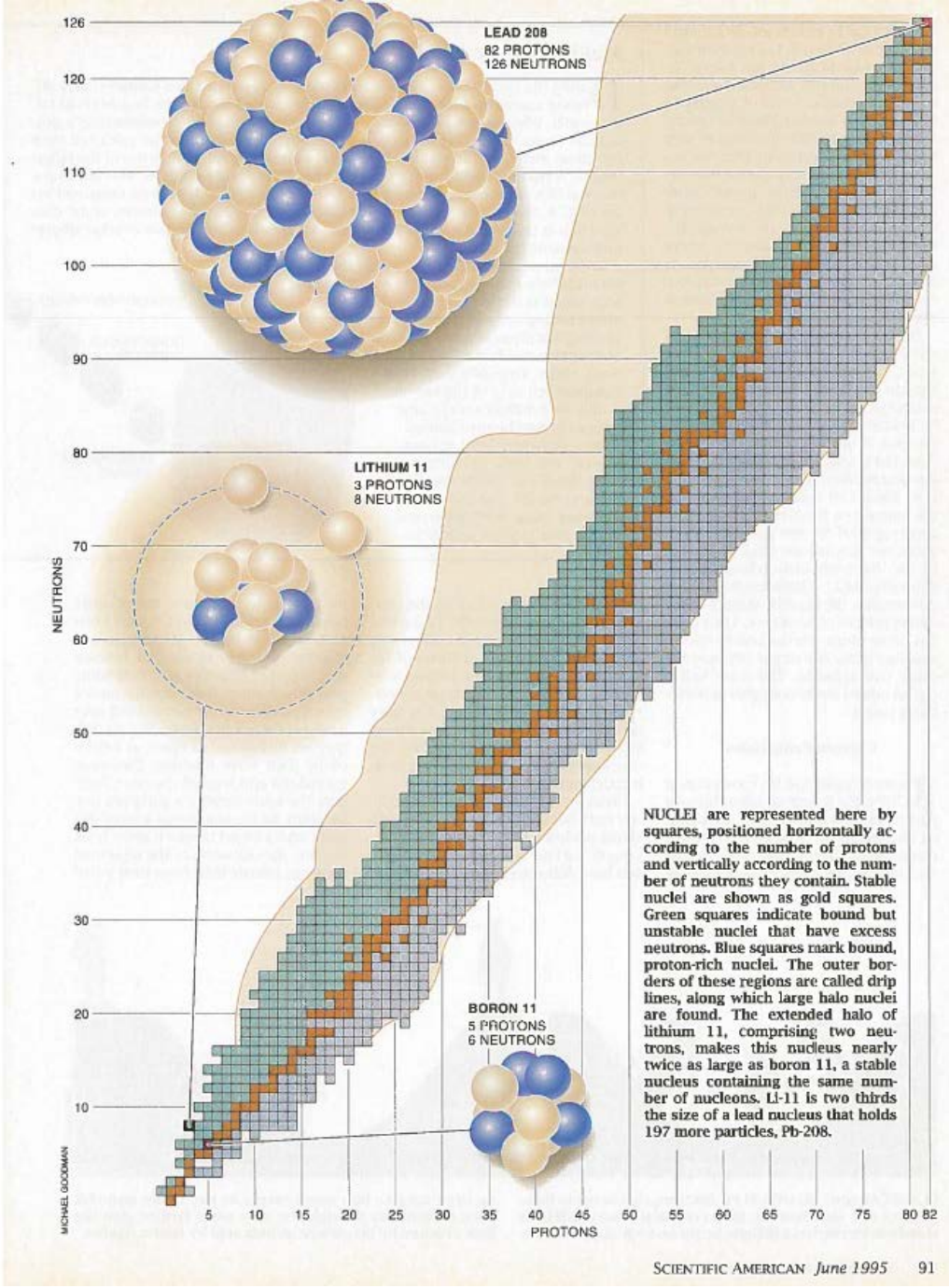
  Typical values are R~1.1*A^1/3fm, a~0.5fm, ρ0~0.07

  - The associated form factor is absurdly cumbersome, so usually a sum of Gaussians is used instead, since they have more manageable $F(q)$:



# *Exceptions to the rule*: Halo nuclei

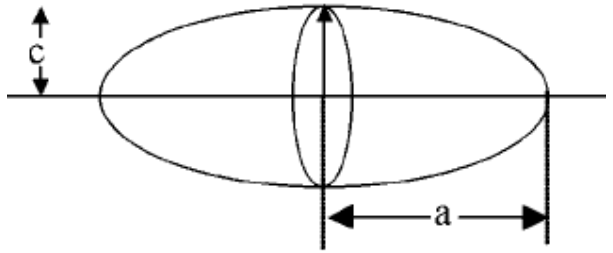
- Some nuclei exhibit radii far larger than expected from the $r_0A^{1/3}$ estimate
- Their large radius is due to 1, 2, or 4 loosely bound nucleons
- The small binding energy of these valence nucleons corresponds to a low tunneling barrier
- From Heisenberg's uncertainty principle, the nucleon(s) can exist in the classically forbidden region beyond the barrier for a rather long time, since $\hbar \propto \Delta E \Delta t$ and $\Delta E$ is small
- Examples:
  - 1n: $^{11}$Be, $^{19}$C
  - 1p: $^{8}$B, $^{26}$P
  - 2n: $^{6}$He, $^{11}$Li, $^{17}$B, $^{22}$C
  - 2p: $^{17}$Ne, $^{27}$S
  - 4n: $^{8}$He, $^{14}$Be, $^{19}$B




S. Austin & G. Bertsch, Scientific American (1995)

# Nuclear electric quadrupole moment, Q

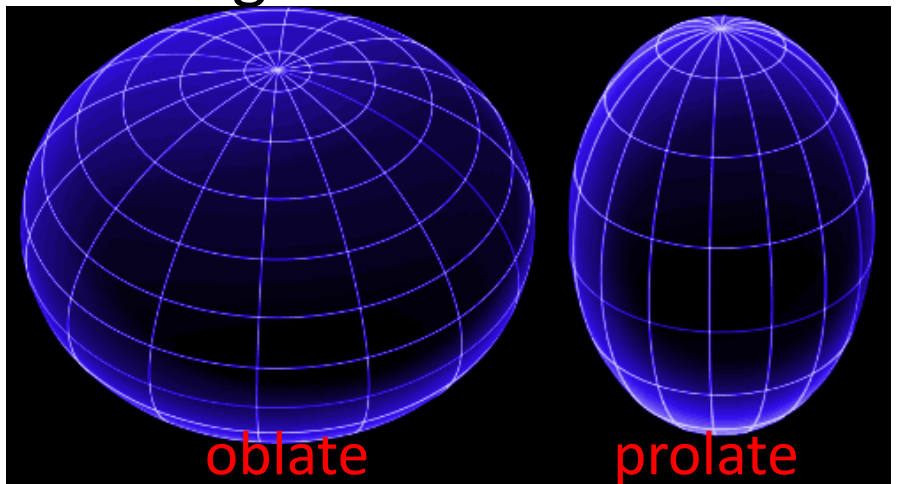
- $Q = \iiint r^2 \rho(r, \theta, \varphi) r^2 dr \sin\theta \, d\theta d\varphi$
- Note that if $\rho$ is spherically symmetric, the integral over $d\theta$ will make $Q = 0$
- So, any non-zero $Q$ indicates a non-spherical nuclear shape
- Choosing an ellipsoid model for the nucleus and evaluating Q above yields (it turns out)
  - $Q = \frac{2}{5} Ze(a^2 - c^2)$




Loveland, Morrissey, Seaborg, Modern Nuclear Chemistry (2006)

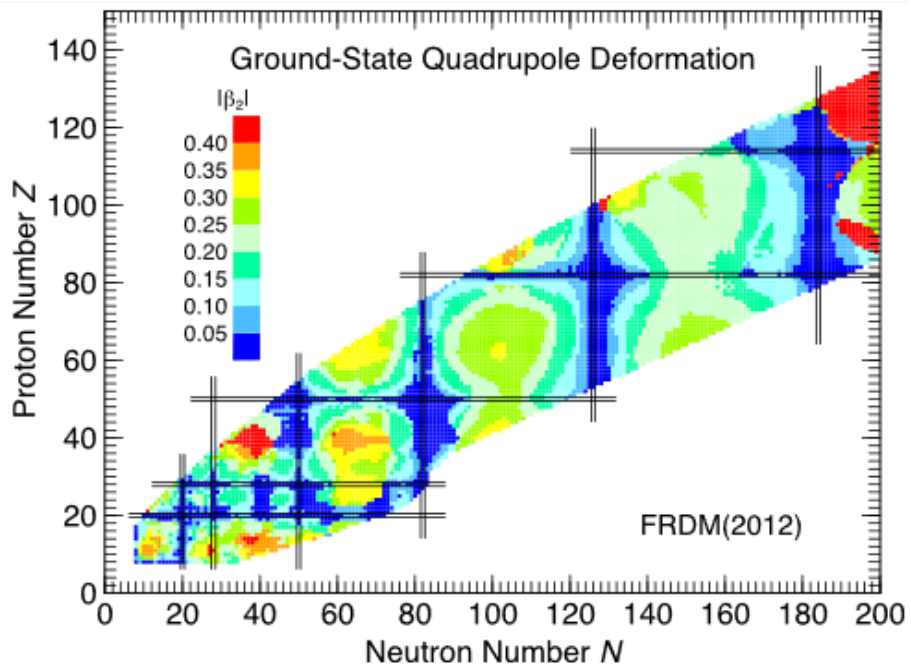
*Q is generally provided in units of e*

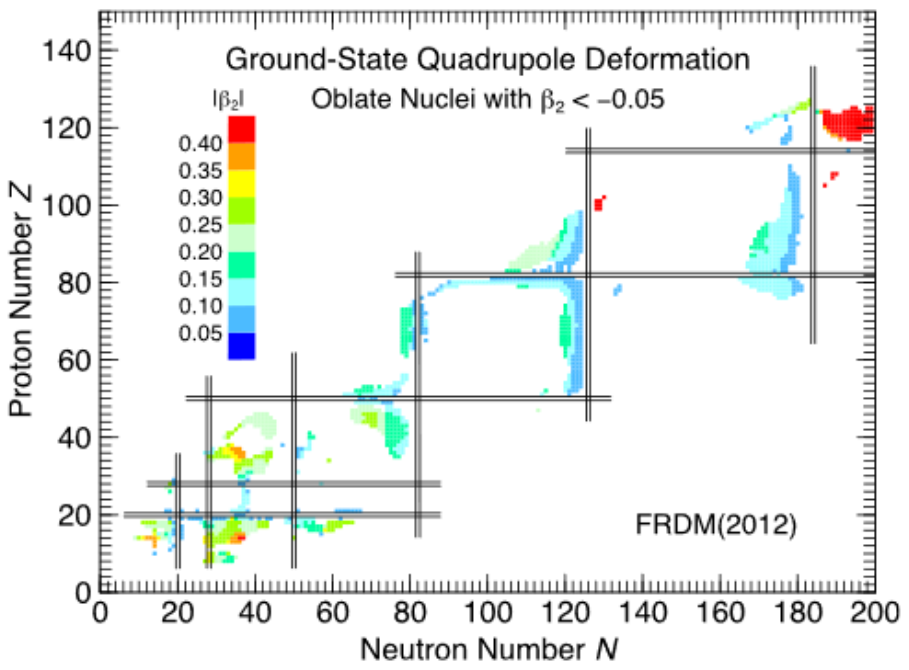
- With a measurement of Q, we can solve for a and c using the formula for the radius of an ellipsoid:
  - $R^2 = \frac{1}{2}(a^2 + c^2) = \left(r_0 A^{1/3}\right)^2$
- If $a > c$, $Q > 0$ and the nucleus is "prolate"
- If $a < c$, $Q < 0$ and the nucleus is "oblate"

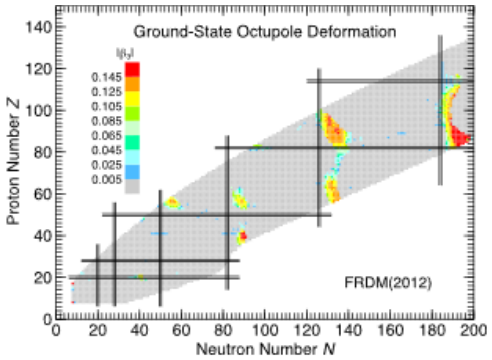


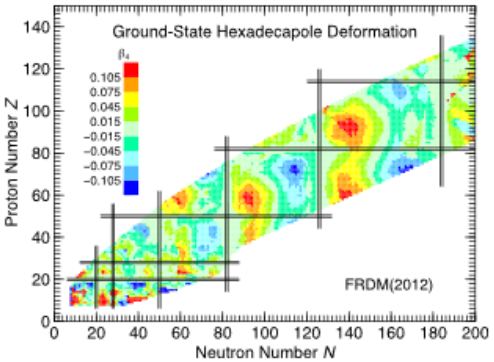
# Q and deformation, β

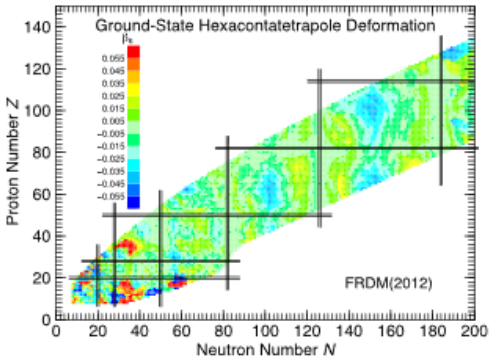
- For an ellipsoid, radius can be represented as an expansion in spherical harmonics
  - $R(\theta,\varphi) = R_{sph}\left(1 + \sum_{\lambda=2}^{\infty}\sum_{\mu=-\lambda}^{\lambda}\alpha_{\lambda,\mu}Y_{\lambda,\mu}(\theta,\varphi)\right)$
  - where $R_{sph}$ is the radius of sphere with the same volume, $r_0A^{1/3}$
  - λ is the multipole (1 is neglected because that's C.O.M. motion), and $\mu$ is the z-projection
  - …for an axially symmetric nucleus (e.g. prolate & oblate), $\mu = 0$
    - $\beta_2 \equiv \alpha_{2,0} = \frac{4}{3}\sqrt{\frac{\pi}{5}}\frac{(a-c)}{R_{sph}}$ …which is a commonly used metric for deformation




P. Möller et al. ADNDT (2016)

# Isospin

- Empirically, the nuclear force between nucleons appears to be charge-independent, i.e. it doesn't matter if you're dealing with neutrons or protons. For instance:
  - Masses of isobars differ only by the different Coulomb energy and the n-p mass difference
  - Nuclear charge radii are well predicted by considering A, not Z and N separately
- Then, one can consider the neutron and proton as two states of the same particle, the nucleon

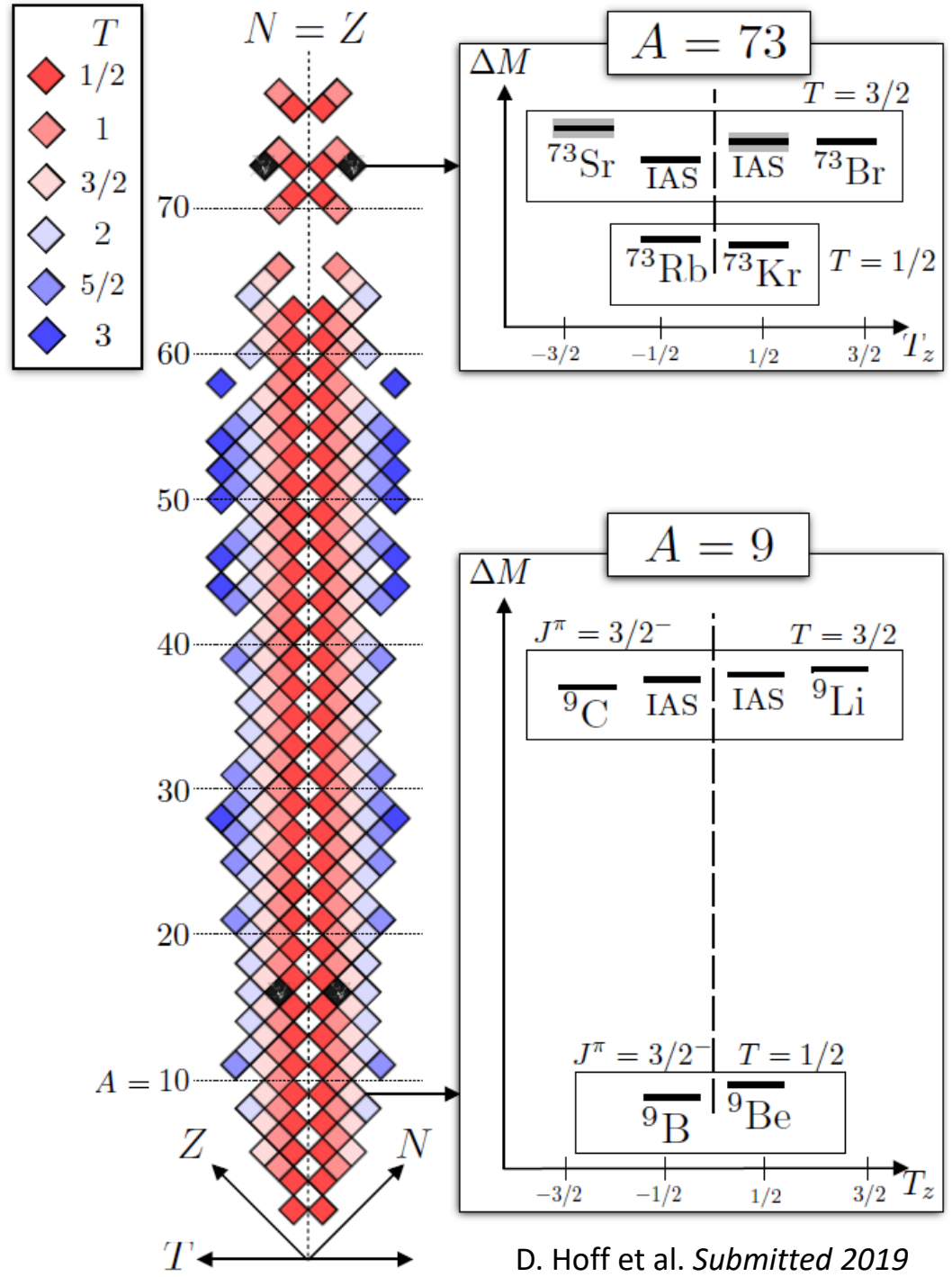
(W. Heisenberg, Z. Phys. A (1932))

- Since the mechanics have already been developed for something in two quantized states (spin-1/2 particles), the associated machinery can be hijacked and assign nuclear states the quantity *isospin*, T
  - The nucleon has $T = 1/2$ with two projections, $T_z = +\frac{1}{2}$ (proton) and $T_z = -\frac{1}{2}$ (neutron)
- For nucleus, $T_z = \frac{Z-N}{2}$ and $|T_z| \leq T \leq \frac{A}{2}$. Usually, for the ground-state $T = T_z$
- Isospin is conserved (approximately) in strong interactions

*just to keep life interesting, sometimes the opposite convention is used, where $T_{z,proton} = -1/2$, $T_{z,neutron} = 1/2$ and $T_{z,nucleus} = (N-Z)/2$

# Isospin symmetry

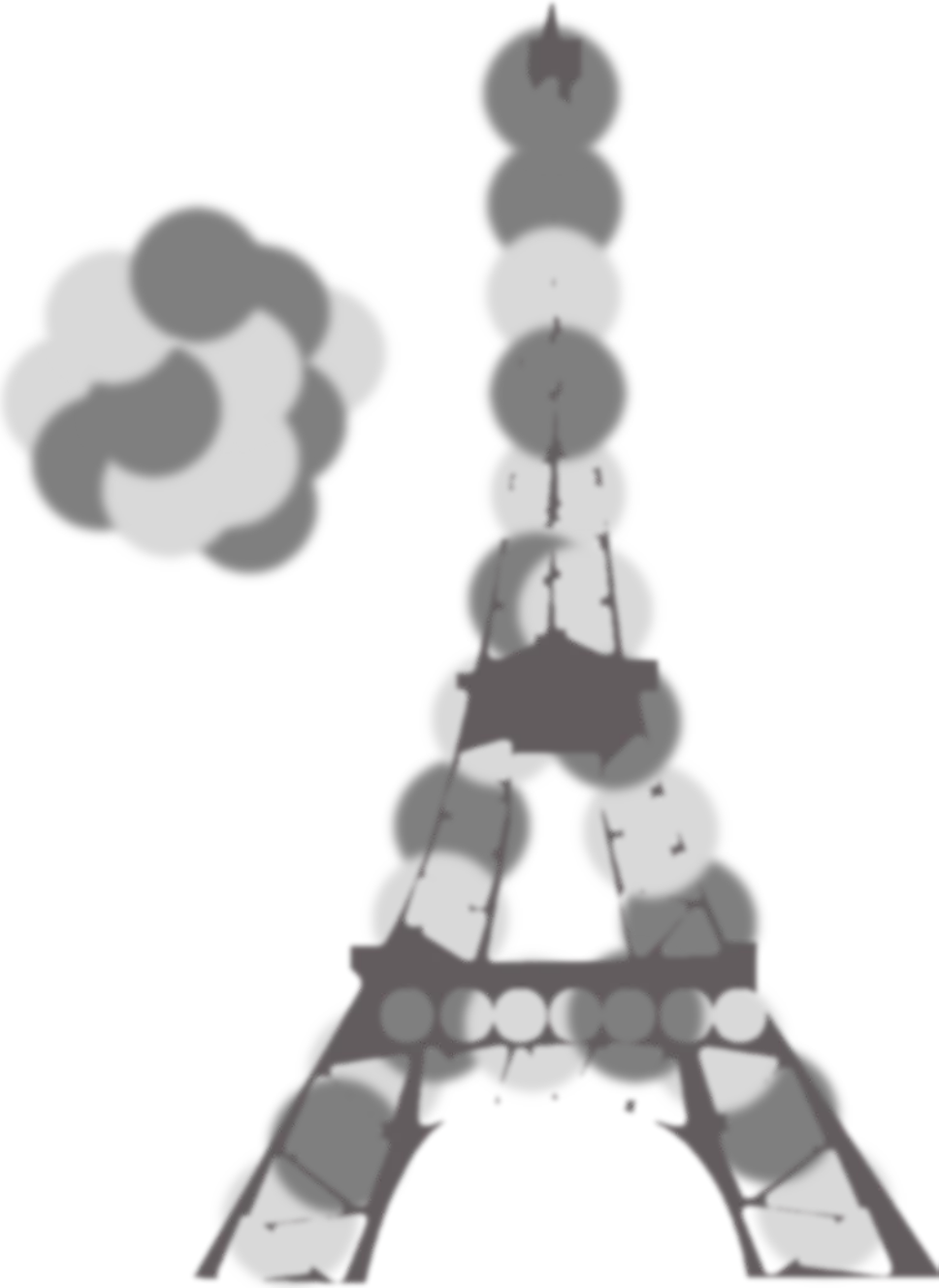
- As an example what isospin gets you, the ground-state spin-parity is the same for “mirror nuclei” (swap *N* & *Z*) except for 2 cases (as of 2019)
- This works for low-ish lying excited states too



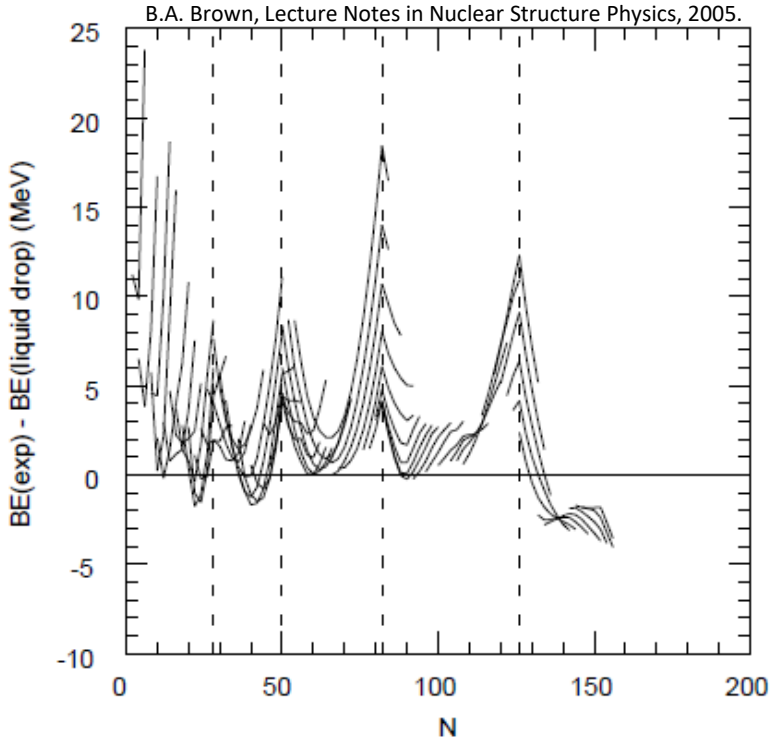

D. Hoff et al. *Submitted 2019*

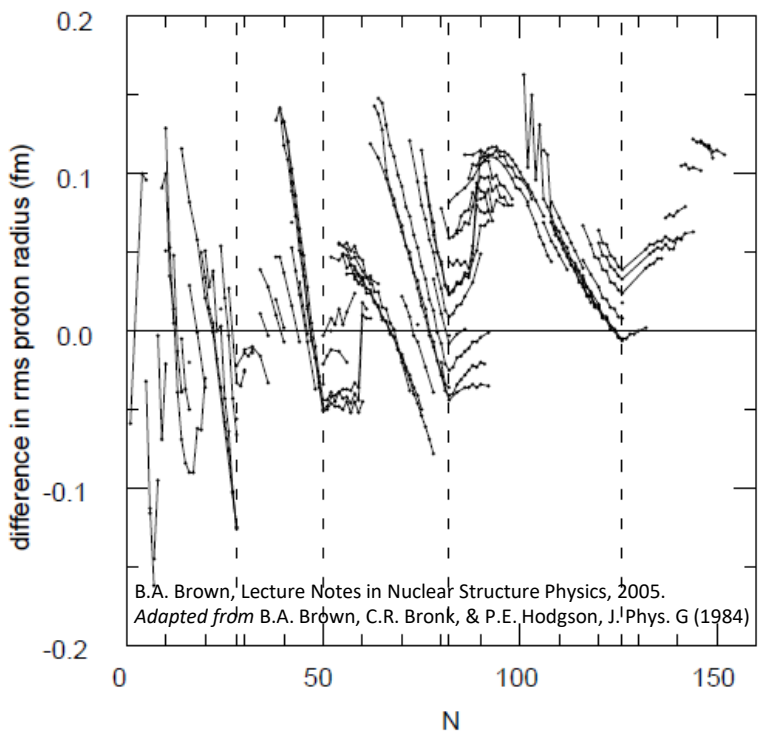
# Lecture 3: *Nuclear Structure 1*

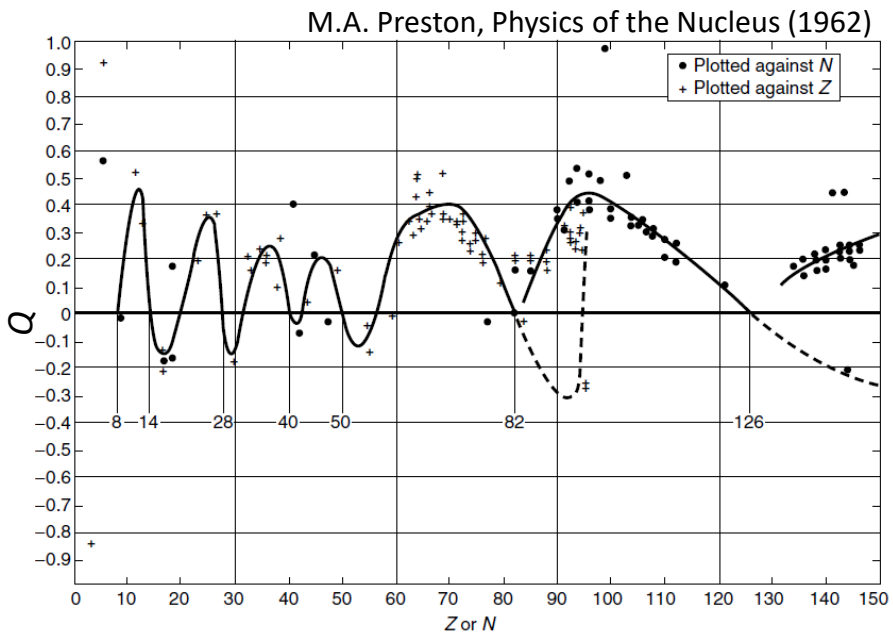
- Why structure?
- The nuclear potential
- Schematic shell model

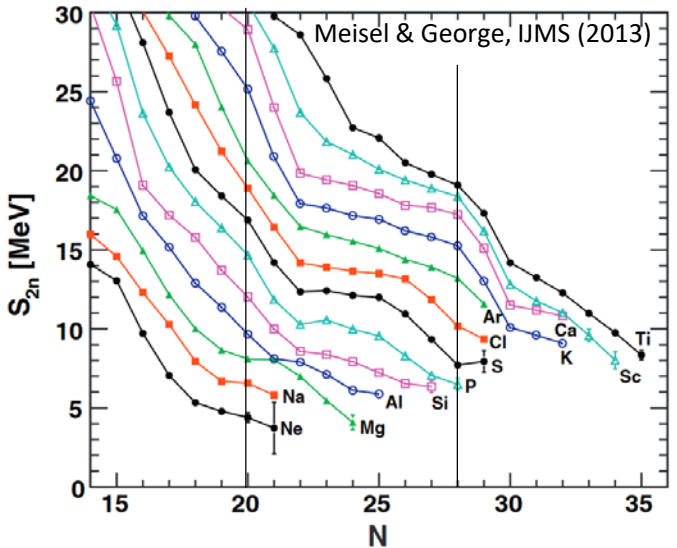


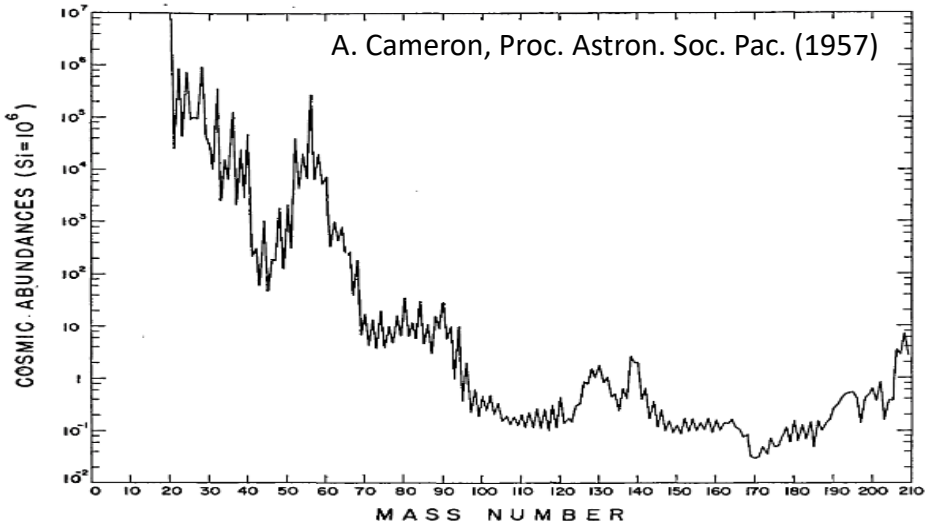
# Empirically, several striking trends related to Z,N. *e.g.*

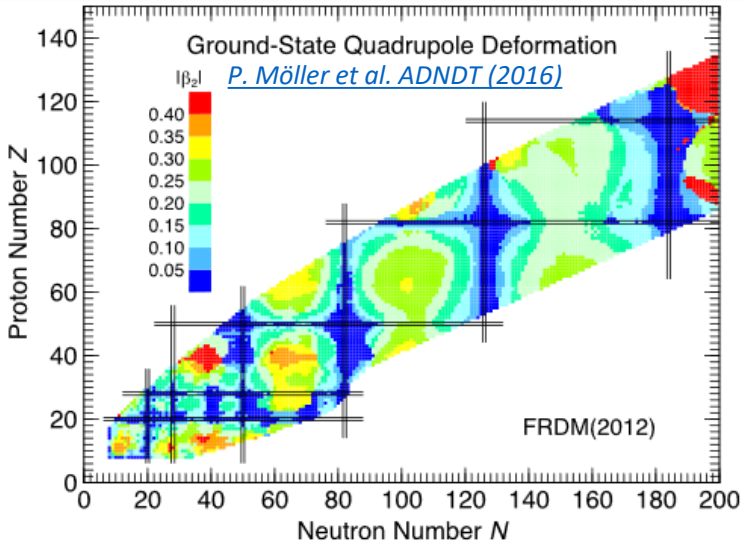


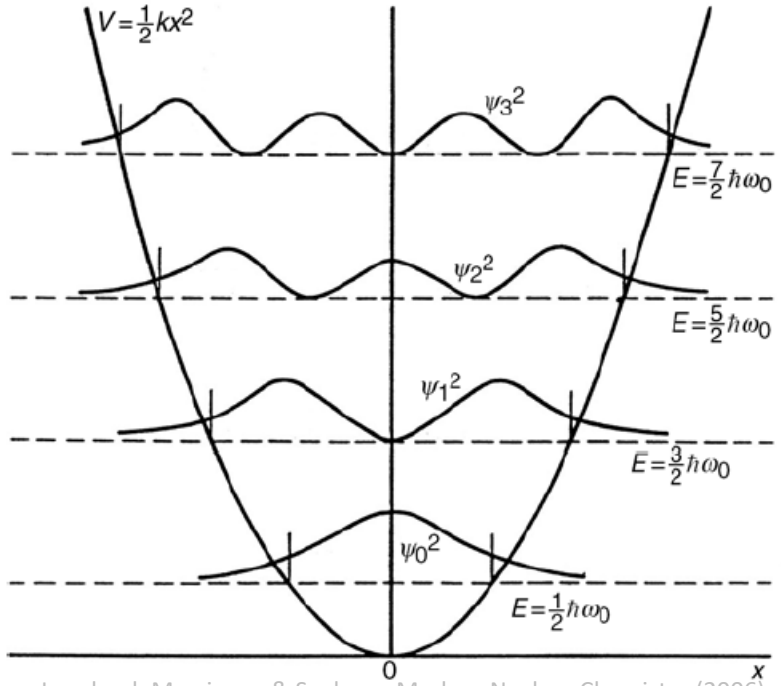

*First magic number evidence compilation by M. Göppert-Mayer Phys. Rev. 1948*

# First stab at the potential, $V$: The Harmonic Oscillator

- Based on some evidence (and logic) that nuclei aren't perfectly constant in density, Heisenberg (Z. Phys. 1935) posited that a parabolic potential could be assumed, conveniently allowing the adoption of the harmonic oscillator solutions
  *(one of the few analytically solved systems!)*
- This provides evenly spaced energy levels $n \geq 1$, with $E_n = \left(n - 1 + \frac{1}{2}\right)\hbar\omega$.
- The corresponding angular momenta are $l = n - 1, n - 3, \ldots \geq 0$.

i.e. only odd or even functions are allowed for each oscillator shell

- The number of particles per angular momentum is $2(2l + 1)$ for $2l + 1$ projections & 2 spins
- So, the number of particles per level is:



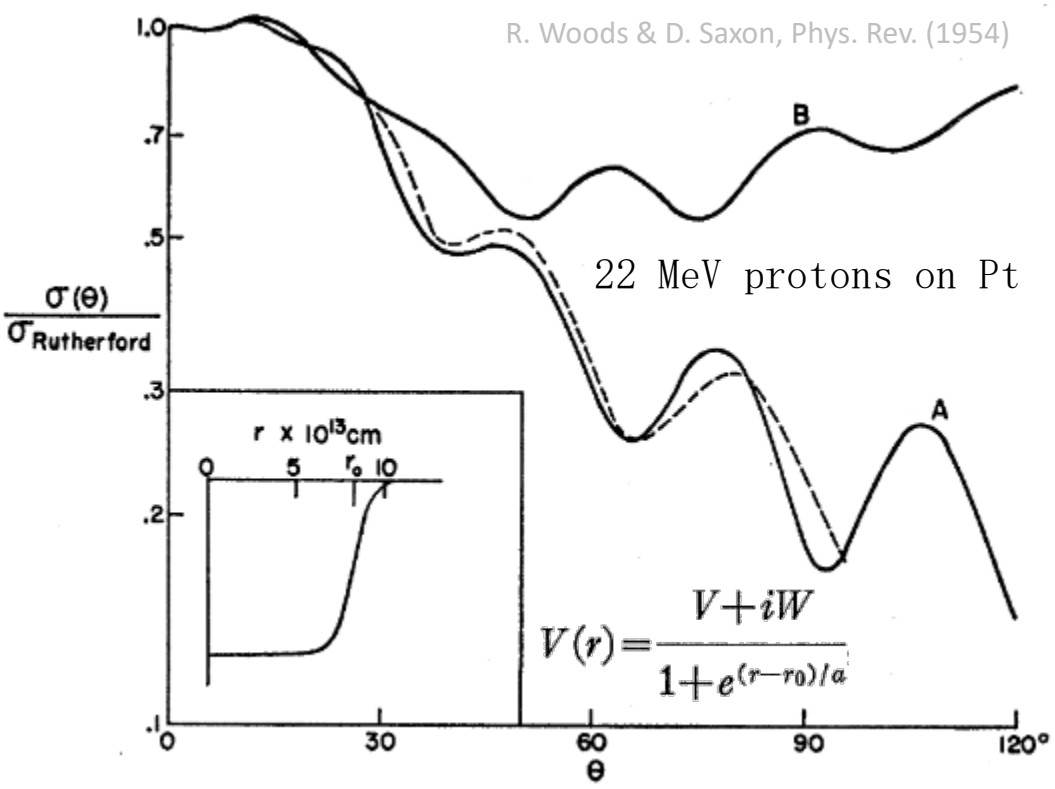
Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

| $n$ | $l$ | # per level | Cumulative |
|---|---|---|---|
| 1 | 0 | 2(2*0+1) = 2 | 2 |
| 2 | 1 | 2(2*1+1) = 6 | 8 |
| 3 | 0,2 | 2(2*0+1) = 2 + 2*(2*2+1) = 10 = 12 | 20 |
| 4 | 1,3 | 2*(2*1+1) = 6 + 2*(2*3+1) = 14 = 20 | 40 |
| 5 | 0,2,4 | 2*(2*0+1) = 2 + 2*(2*2+1) = 10 + 2*(2*4+1) = 18 = 30 | 70 |

Could the HO potential still be useful for some cases?
...can get the job done for light nuclei (e.g. H. Guo et al. PRC 2017)
...but need to be careful, because can impact results (B.Kay et al PRL 2017)

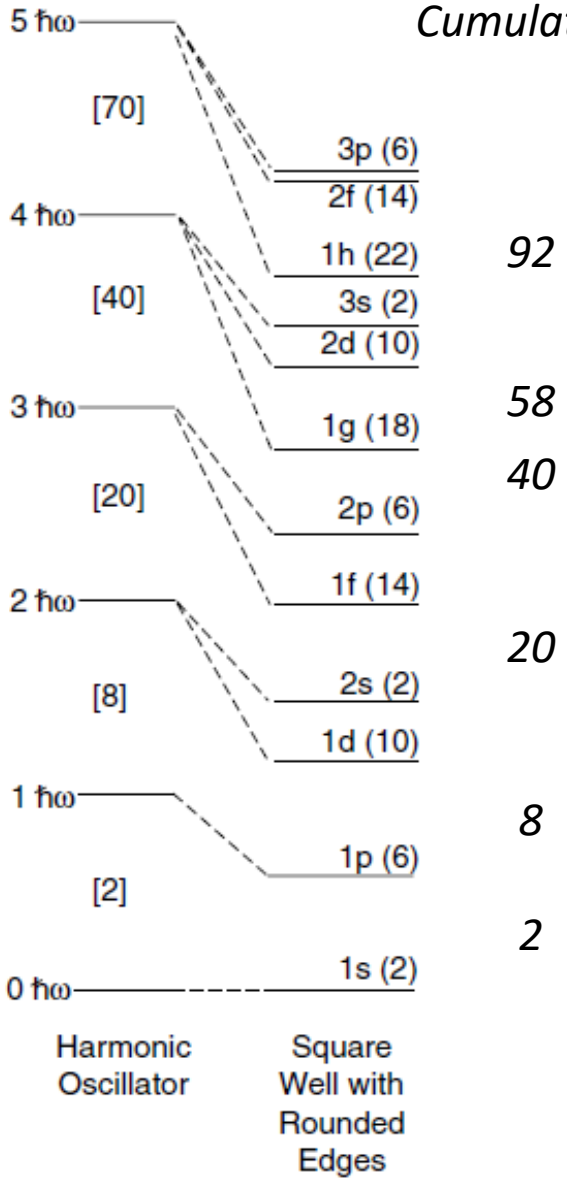
# Move to an empirical potential: **Woods-Saxon**

- Since the nuclear interaction is short-range, a natural improvement would be to adopt a central potential mimicking the empirical density distribution
- This is basically a square well with soft edges, as described by the Woods-Saxon potential:




- Using the Woods-Saxon is a good idea because of commitment to reality… but we're no wiser as to the origin of the magic numbers

*Was this step completely useless? No! It broke the degeneracy in ℓ*

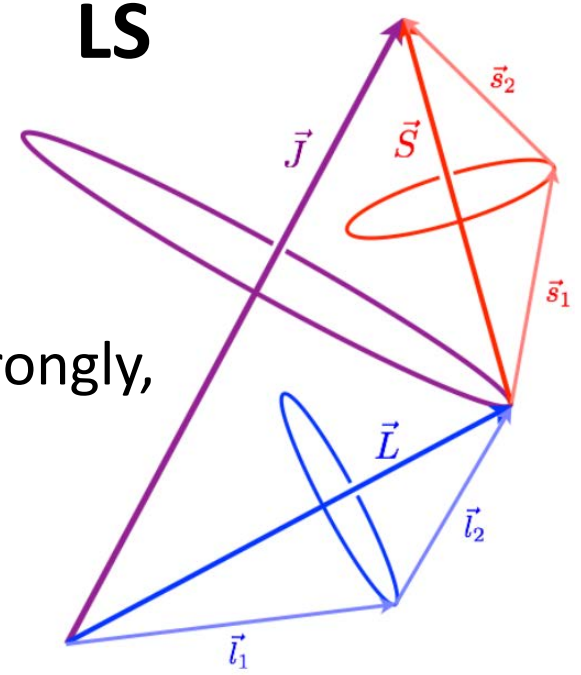



Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

# The missing link: **the spin-orbit interaction**

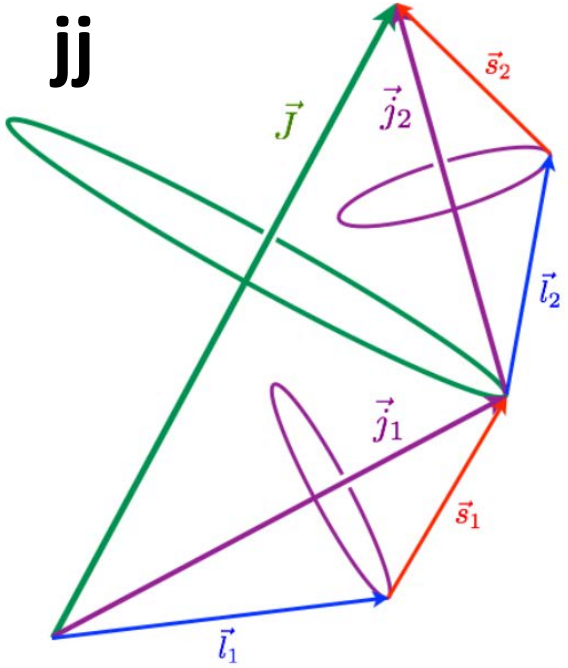
- Due to desperation or genius (or both) Maria Göppert-Mayer
  [Phys. Rev. February 1949] (and nearly simultaneously Haxel, Jensen, & Suess [Phys. Rev. April 1949])
  posited that nucleon spin and orbital angular momentum interacted strongly, making j the good quantum number for a nucleon: $\vec{j} = \vec{l} + \vec{s}$
- Prior to this approach,
  angular momentum was coupled as is typically done for atoms,
  where $\vec{J} = \vec{L} + \vec{S}$ , $\vec{L} = \sum_{nucleons} \vec{l}$ , and $\vec{S} = \sum_{nucleons} \vec{s}$
  - This is "LS coupling"
- Positing that the spin-orbit interaction is stronger than spin-spin or orbit-orbit means that instead, $\vec{J} = \sum_{nucleons} \vec{j}$ and $\vec{j} = \vec{l} + \vec{s}$
  - This is "jj coupling"

**LS**



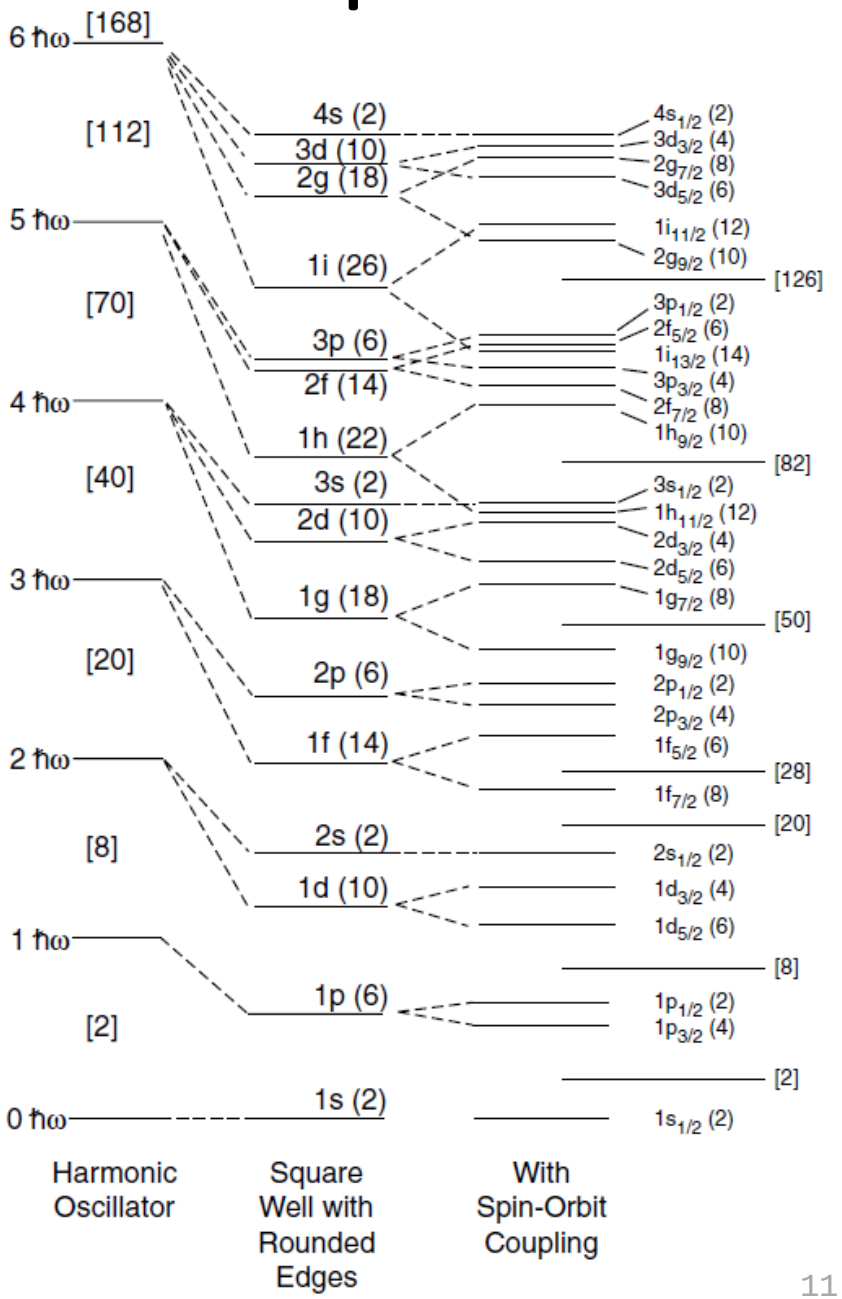

A. Kastberg, Lecture Notes Physique Atomique

**jj**



# Putting it all together: “shells” from the nuclear potential

- Considering the nucleus as nucleons interacting in a mean-field potential, generated by the spatial distribution of all other nucleons, and each nucleon having a strong interaction between its orbital & spin angular momentum, properly predicts the magic numbers.
- Note that **neutrons and protons are considered separately**.
- When adding neutrons or protons to a nucleus, the lowest energy state will (generally) consist of filling each orbital as you go upward.
- The regions between the large gaps in nucleon energy are referred to as “shells”.
  - E.g. Between 8 and 20 neutrons (or protons) is the “sd-shell”, between 28 and 40 neutrons (or protons) is the “fp-shell”.
  - More exotic neutron-rich nuclides exist, so typically people are talking about the neutron shell
  - Nucleons can get excited into higher-lying states, so states above the ground-state are relevant in calculations

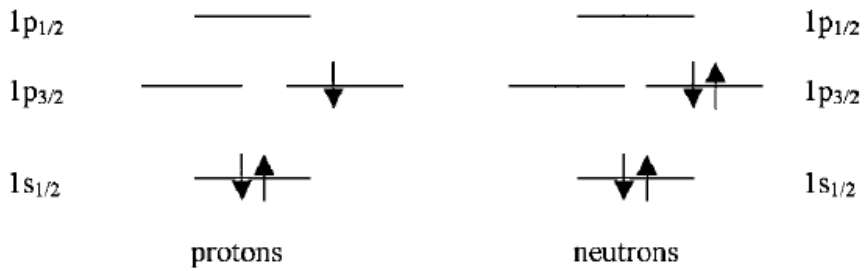



Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

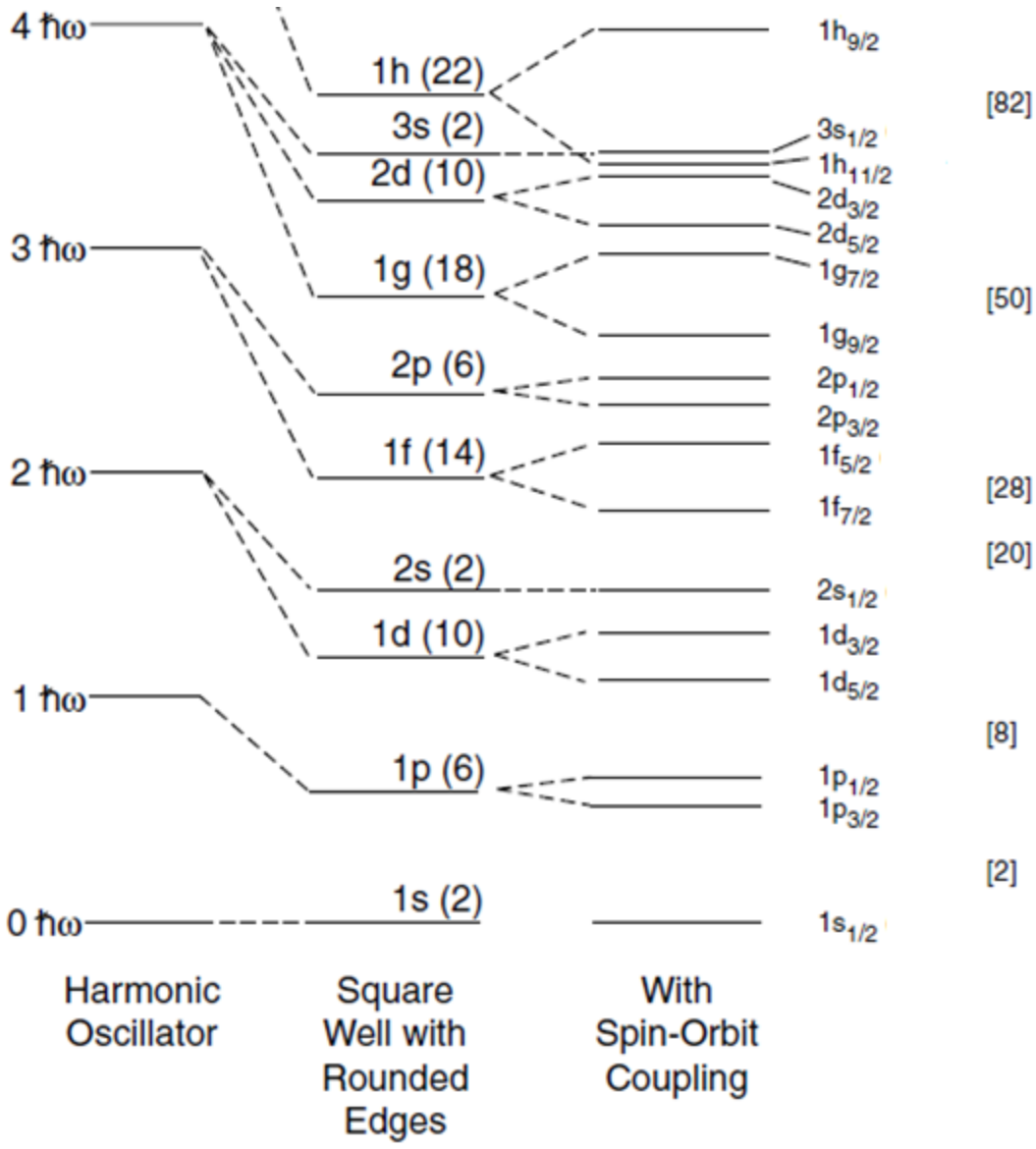
# Filling the shells

- We can construct a nucleus using our "shell model":
  - A nucleon will go in the lowest-energy level which isn't already filled, i.e.
    - the largest angular momentum, $j$
    - for the lowest orbital angular momentum, $l$
    - for the lowest oscillator shell, n
  - $2j + 1$ protons or neutrons are allowed per level
  - Each level is referred to by its $nlj$
    - $n$ by the # for the oscillator shell (convention either starts with 0 or 1)
    - $l$ by spectroscopic notation (s=0,p=1,d=2,f=3,…)
    - $j$ by the half-integer corresponding to the spin

- For example: $^{7}$Li (Z=3, N=4)

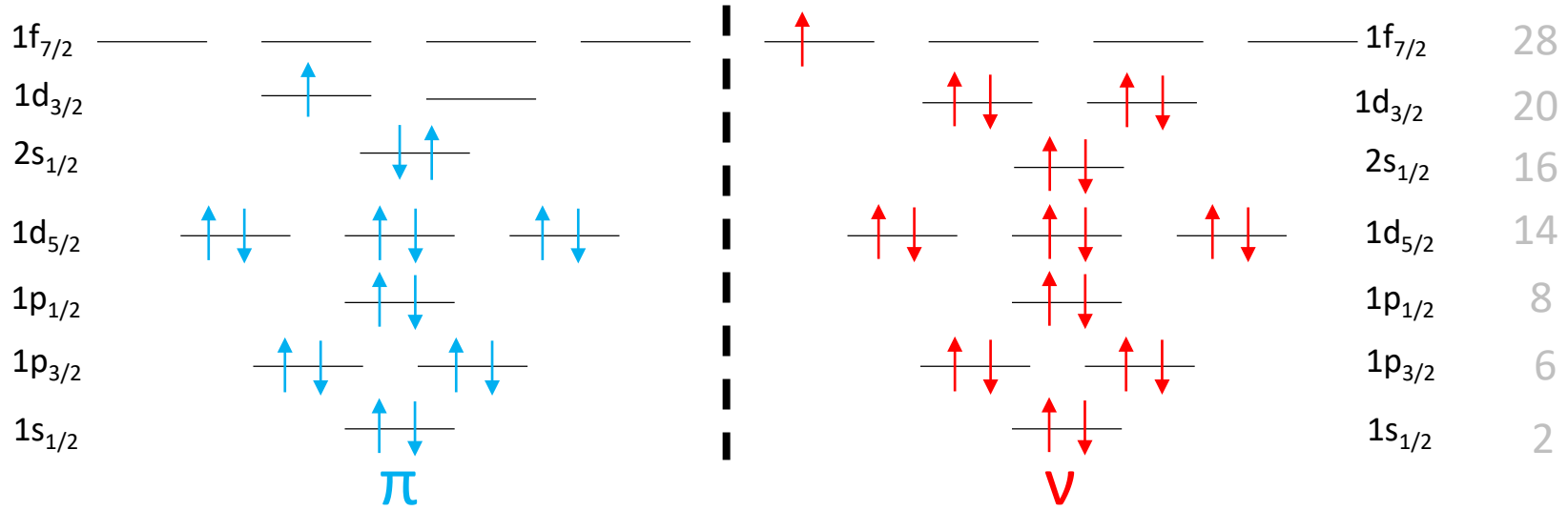



Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)



# Basic properties from the shell model: $J^{\pi}$

- Looking at a more complicated case, $^{38}$Cl (Z=17, N=21)

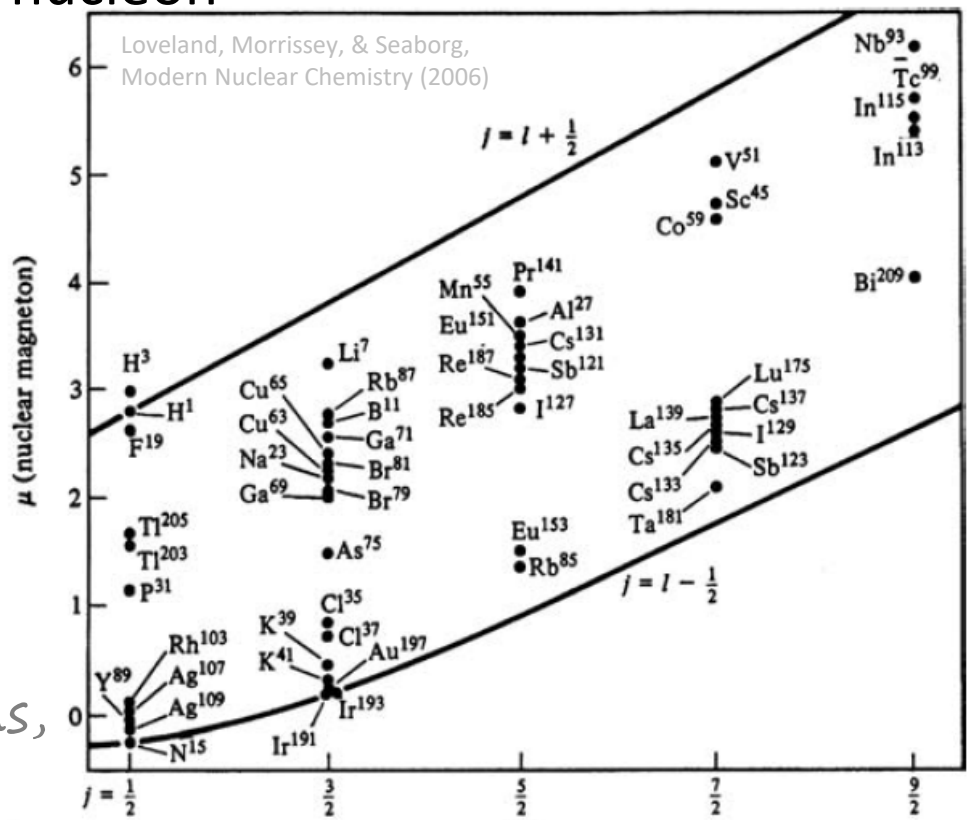



- 2 valence nucleons: one $d_{3/2}$ proton and one $f_{7/2}$ neutron
- Allowed couplings are $|j_1 - j_2| \leq J \leq |j_1 + j_2|$
- So for this case: $J = 2, 3, 4, 5$
- How do we decide which combination has the lowest energy?

Using the descriptively named: "jj coupling rules for Odd-Odd nuclei" from Brennan & Bernstein (Phys. Rev. 1960)

# *What else are $J^\pi$ predictions good for?* Magnetic dipole moments

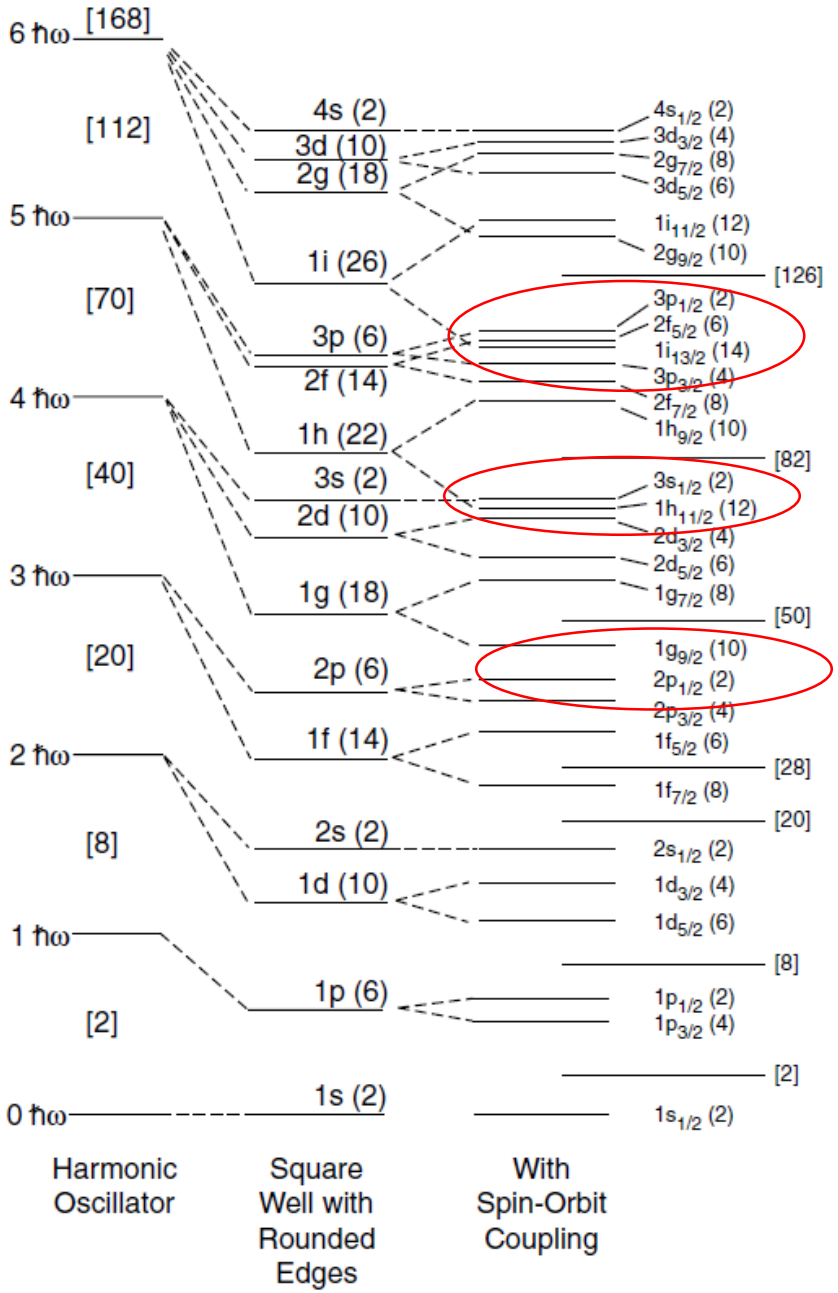
- Recall that for a single particle, the magnetic dipole moment is: $\mu = j g_j \mu_N$
- After some fancy footwork, it can be shown that the Landé g-factor can be expressed as:
  - $g_j = \left(\frac{j(j+1)+l(l+1)-s(s+1)}{2j(j+1)}\right) g_l + \left(\frac{j(j+1)-l(l+1)+s(s+1)}{2j(j+1)}\right) g_s$
- Since spins cancel for paired nucleons, we might expect the magnetic dipole moment of a nucleus with 1-unpaired nucleon to be determined by that nucleon
  Expected values of $\mu$ are therefore:
  - $\mu = j g_j = l g_l + \frac{1}{2} g_s$ for $j = l + \frac{1}{2}$
  - $\mu = j g_j = j\left(1 + \frac{1}{2l+1}\right) g_l - j\left(\frac{1}{2l+1}\right) g_s$ for $j = l - \frac{1}{2}$
  - Protons: $g_l = 1, g_s = 5.6$ , Neutrons: $g_l = 0, g_s = -3.8$
  - These boundaries are the "Schmidt lines"
    (Th. Schmidt, Z.Phys. (1937))
    and nearly all measured $g_j$ fall between these




*As with excited state Jπ's, deviation between experiment & shell model predictions tell us something interesting is going on. E.g. mixing between single-particle occupations, polarization of the core, corrections to meson-exchange mediating the nuclear force* (G. Neyens, Rep. Prog. Phys. (2003), J. Booten et al. Phys.Rev.C (1991))

# *What else are $J^\pi$ predictions good for?* **Isomers** (long-lived x.s.)

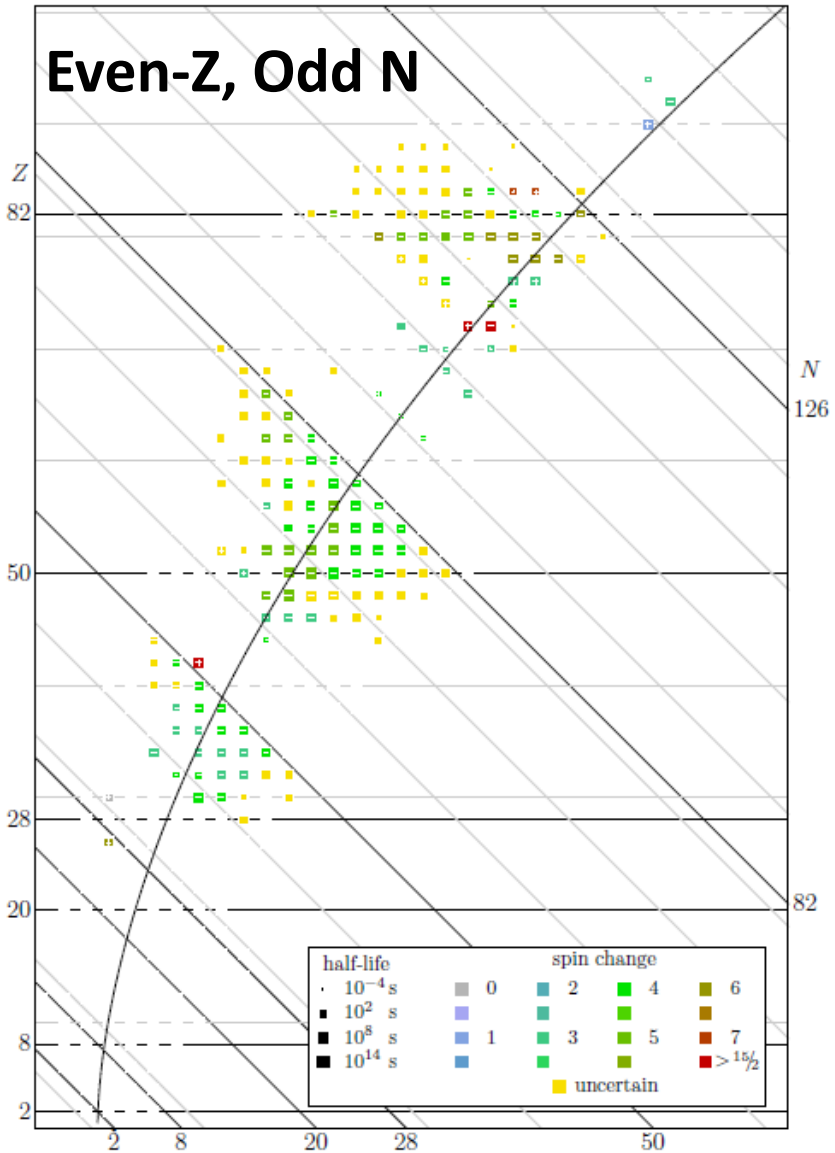
- Most excited states decay via $\gamma$-emission in a matter of femto-seconds, but some stick around for many nanoseconds, milliseconds, seconds, or even universe lifetimes.
- These are meta-stable states, a.k.a. isomers
- The reason is $\gamma$-emission is suppressed, since it would require large angular momentum transfer
- So, where do we expect low-lying high-$j$ excited states?
  - Where a large $\Delta j$ exists between neighboring levels
    (thanks to the spin-orbit interaction)
    that are near the last single particle orbit.
    - Namely, below magic #'s 50, 82, 126
    - For these cases, we expect a parity change
  - Where multiple $j$ are possible for the ground-state
    (but one is favored by the Brennan-Bernstein rules)
    and high-$j$ single-particle levels are involved
    - Namely, odd-odd nuclei
    - For these cases, we don't expect a parity change

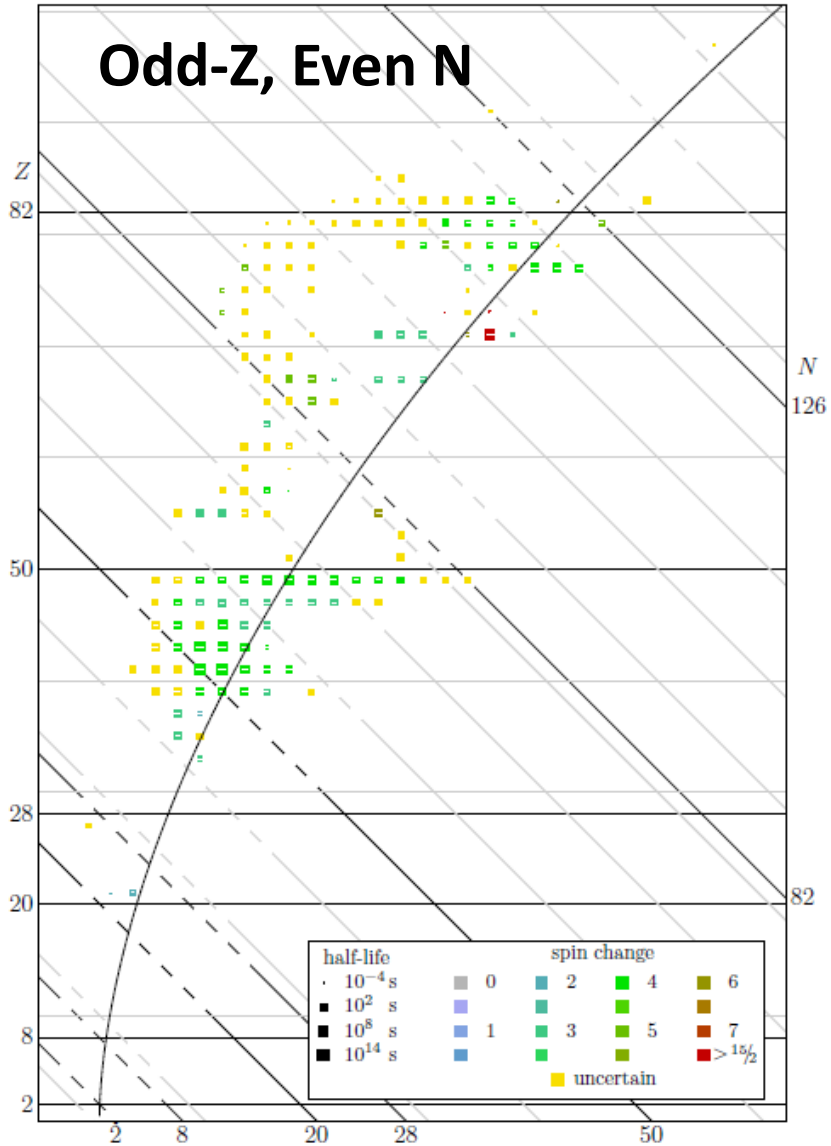


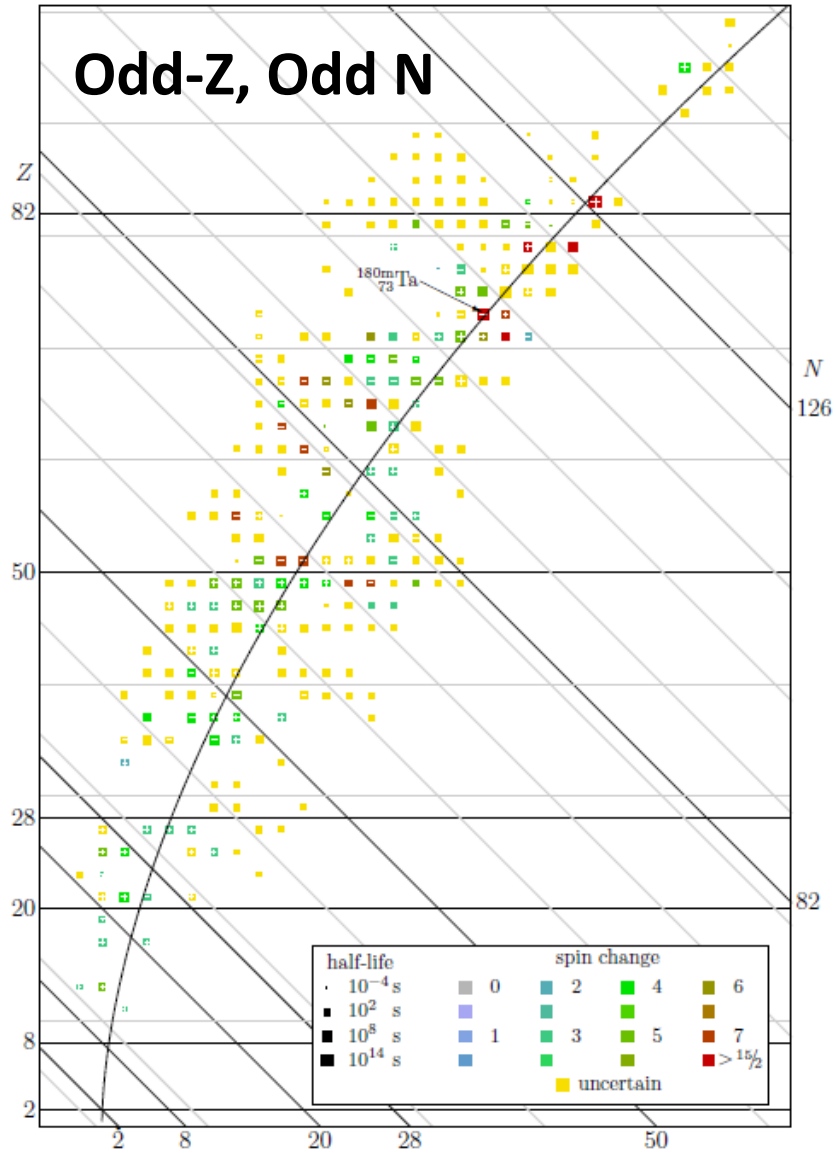

Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

# Isomers on the Nuclear Chart

L. van Dommelen, Quantum Mechanics for Engineers (2012)



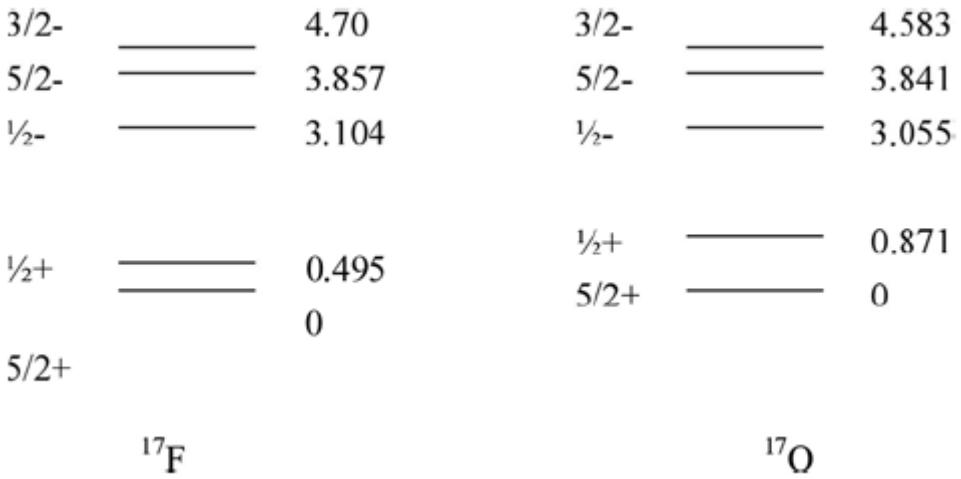



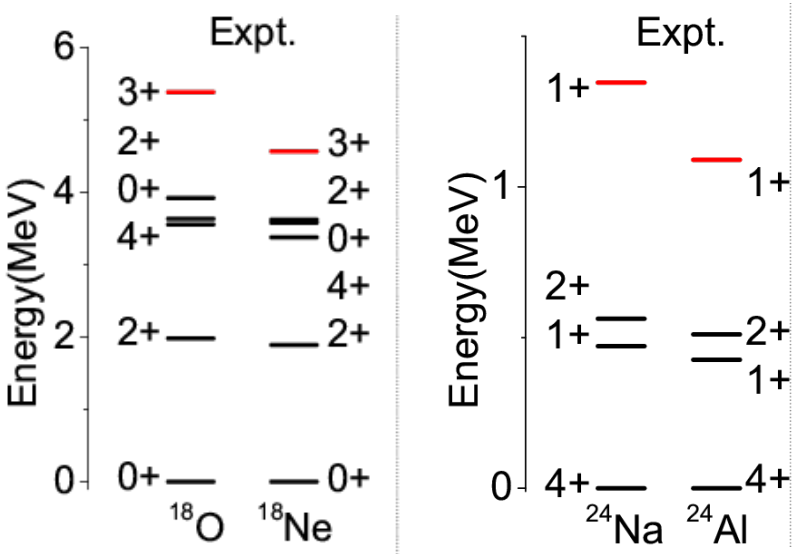

*Special cases exist (mostly for higher-A nuclides) where even-even nuclei have isomers*
*(e.g. M. Müller-Veggian et al., Z.Phys.A (1979))*

# *What else are $J^\pi$ predictions good for?* Mirror Nuclei

- Note that our methods to determine $J^\pi$ didn't depend on whether we were working with protons or neutrons
- If we interchange N for Z, we will get the same answer
- Such pairs are called "mirror nuclei"
- When we examine the levels for mirror nuclei, correcting for the different Coulomb energy, we see a remarkable similarity
- E.g.




Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)




C. Yuan et al., Phys. Rev. C (2014)

- Such mirror symmetry is evidence for charge-independence of the nuclear force and a justification for the concept of isospin

*It's also handy when you need estimates for a nucleus you can't access when you can access its mirror (e.g. C.Akers et al. PRC 2016)*

# Lecture 4: *Nuclear Structure 2*

- Independent vs collective models
- Collective Model
  - Rotation
  - Vibration
  - Coupled excitations
- Nilsson model



# Nuclear Models

"There's no small choice in rotten apples." Shakespeare, *The Taming of the Shrew*

- No useful fundamental & universal model exists for nuclei
  - E.g. based on the nuclear interaction, how do we describe all nuclear properties?
  - Promising approaches include "ab initio" methods, such as Greens Function Monte Carlo, No-core shell model, Coupled cluster model, density functional theories
- Generally one of two classes of models is used instead
  - Independent particle models:
    - A nucleon exists within a mean-field (maybe has a few interactions)
    - E.g. Shell model, Fermi gas model
  - Collective models:
    - Groups of nucleons act together (maybe involves shell-model aspects)
    - E.g. Liquid drop model, Collective model

# Collective Model

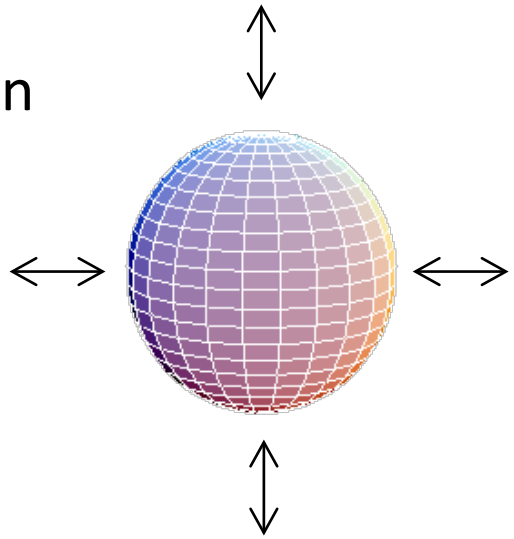
- There are compelling reasons to think that our nucleus isn't a rigid sphere
  - The liquid drop model gives a pretty successful description of some nuclear properties.

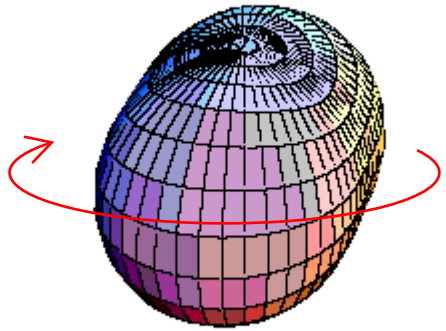
    *…can't liquids slosh around?*
  - Many nuclei have non-zero electric quadrupole moments (charge distributions)

    *…this means there's a non-spherical shape.*

    *…can't non-spherical things rotate?*
- Then, we expect nuclei to be able to be excited rotationally & vibrationally
  - We should (and do) see the signature in the nuclear excited states

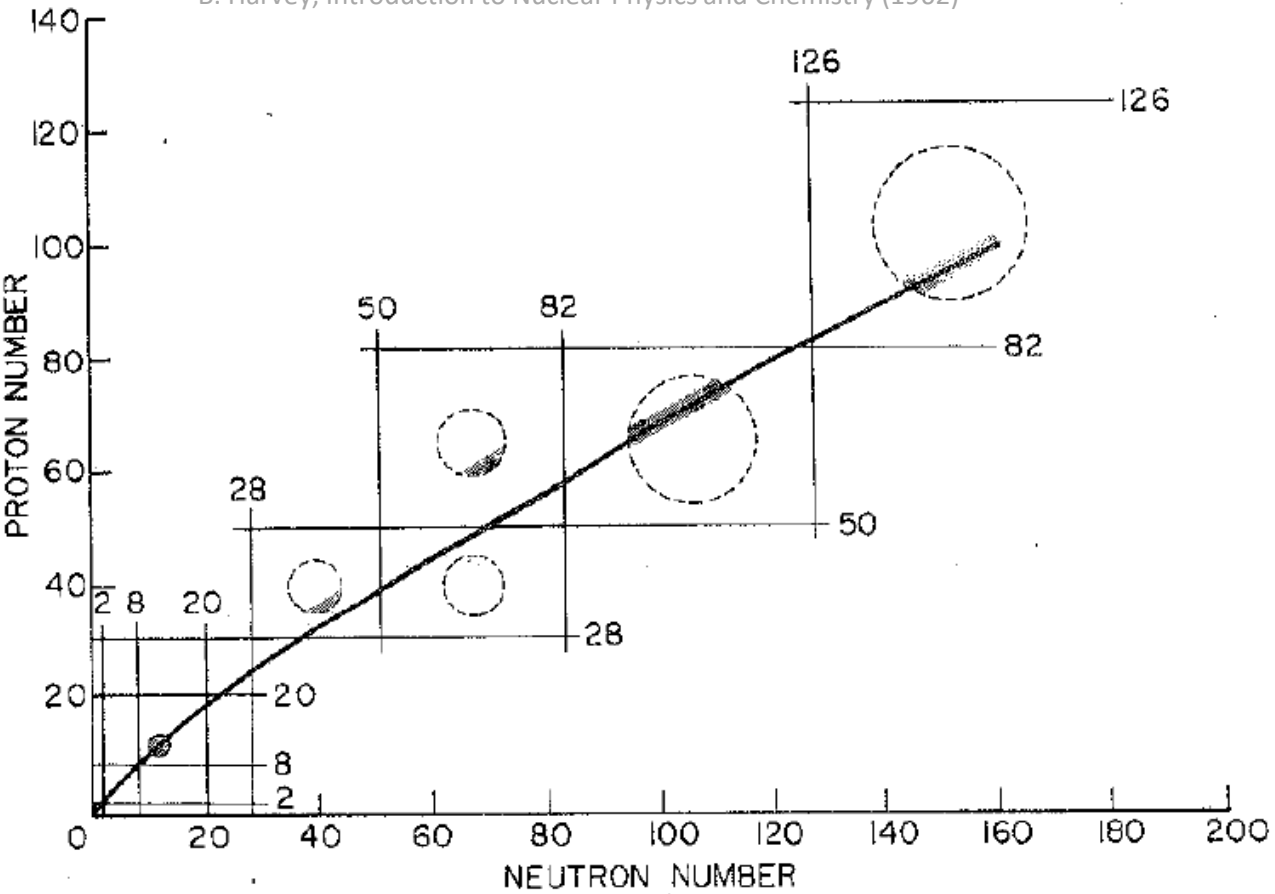
- The relative energetics of rotation vs vibration can be inferred from geometry
  - The rotational frequency should go as $\omega_r \propto \frac{1}{R^2}$ (because $I \equiv \frac{L}{\omega}$ and $I \propto MR^2$)
  - The vibrational frequency should go as $\omega_v \propto \frac{1}{(\Delta R)^2}$ (because it's like an oscillator)
  - So $\omega_r \ll \omega_v$



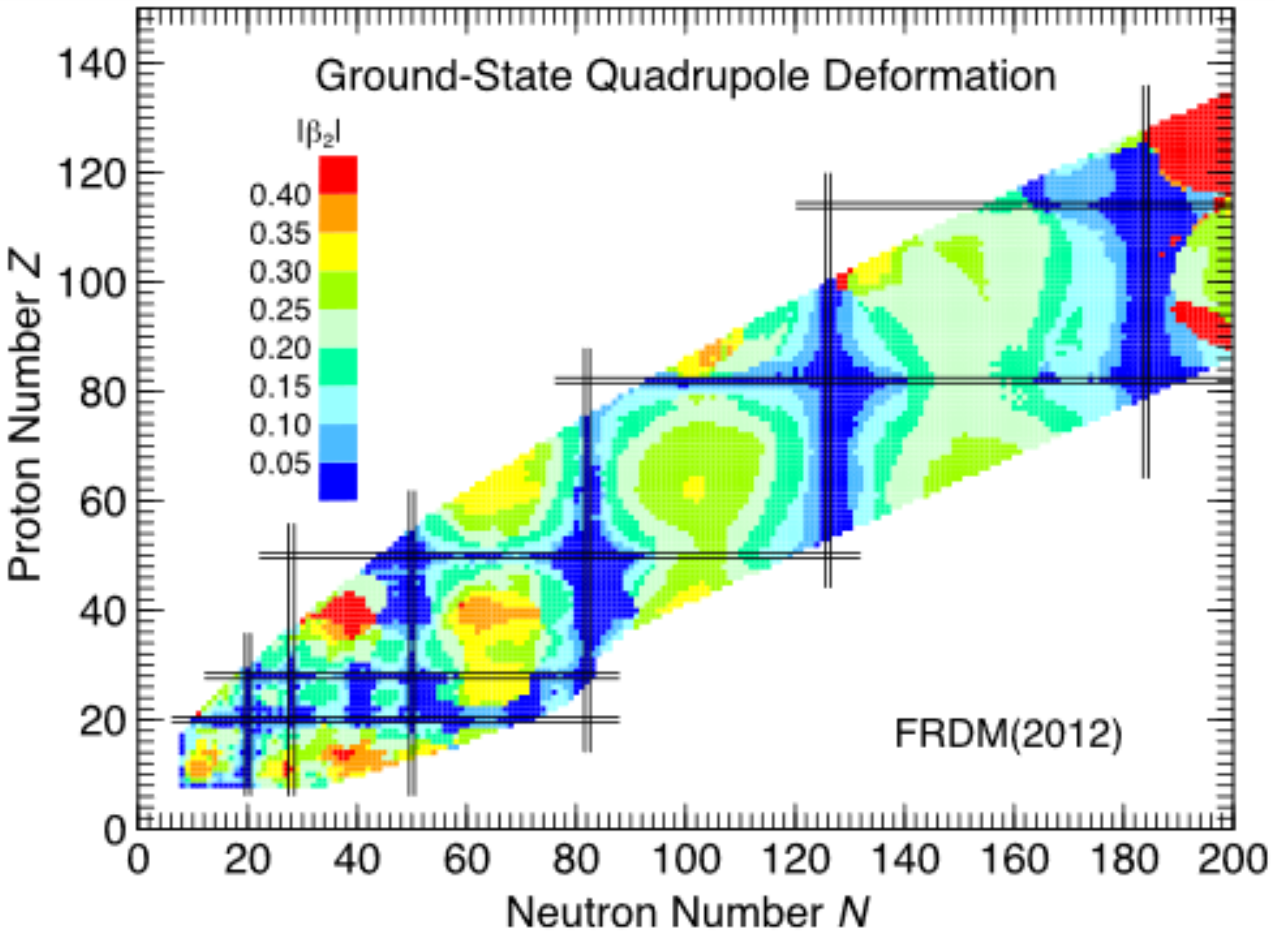


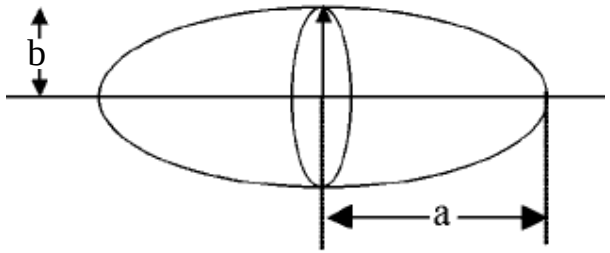
# **Predicted** regions of deformation

B. Harvey, Introduction to Nuclear Physics and Chemistry (1962)




[P. Möller et al. ADNDT (2016)](#)




Loveland, Morrissey, Seaborg, Modern Nuclear Chemistry (2006)

$$\beta_2 \equiv \alpha_{2,0} = \frac{4}{3}\sqrt{\frac{\pi}{5}}\frac{(a-b)}{R_{sph}}$$

# Rotation: *Rigid rotor*

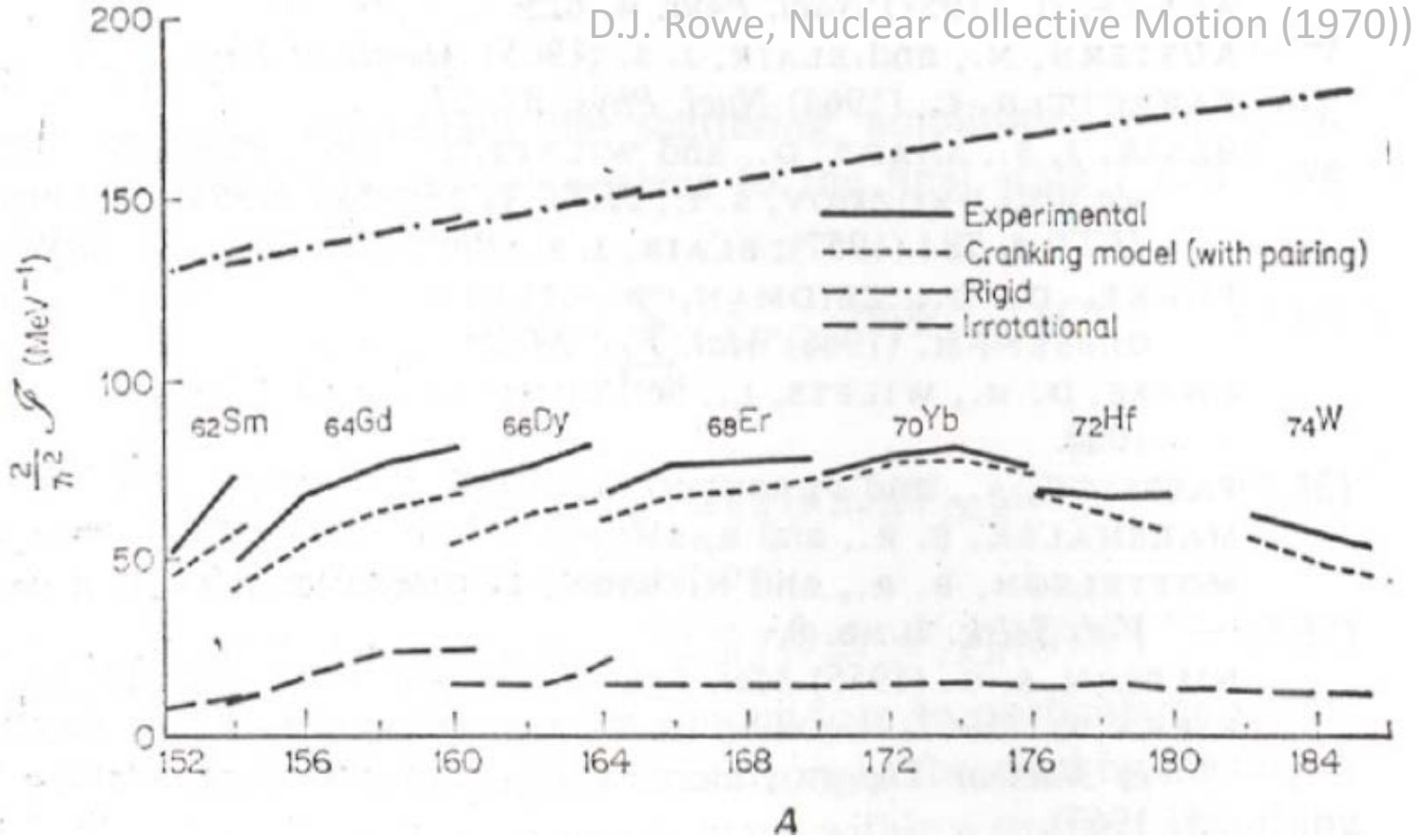
- The energy associated with a rotating object is: $E_{rot} = \frac{1}{2} I \omega^2$
- We're working with quantum stuff, so we need angular momentum instead where $J = I\omega$
  - So, $E_{rot} = \frac{1}{2}\frac{J^2}{I}$
  - …and $J$ is quantized, so $E_{rot} = \frac{\hbar^2 j(j+1)}{2I}$
- Thus our rotating nucleus will have excited states spaced as $j(j+1)$ corresponding to rotation

- For a solid constant-density ellipsoid, $I_{rigid} = \frac{2}{5} MR^2(1 + 0.31\beta + 0.44\beta^2 + \cdots)$

  where $\beta = \frac{4}{3}\sqrt{\frac{\pi}{5}}\frac{(a-b)}{R_{sph}}$ (A.Bohr & B.Mottelson, Dan. Matematisk-fysiske Meddelelser (1955))

# Empirical moment of inertia

- $E_{rot} = \frac{\hbar^2 j(j+1)}{2I}$, so measuring $\Delta E$ between levels should give us $I$
- It turns out, generally: $I_{irro} < I_{expt} < I_{rigid}$



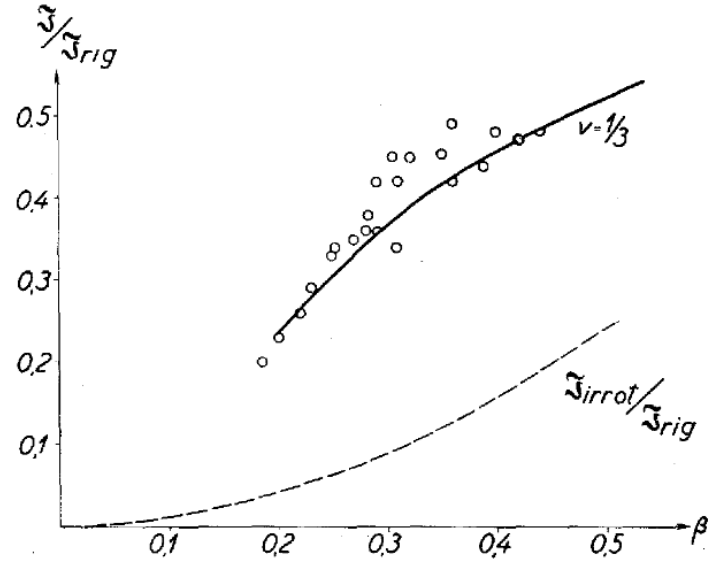
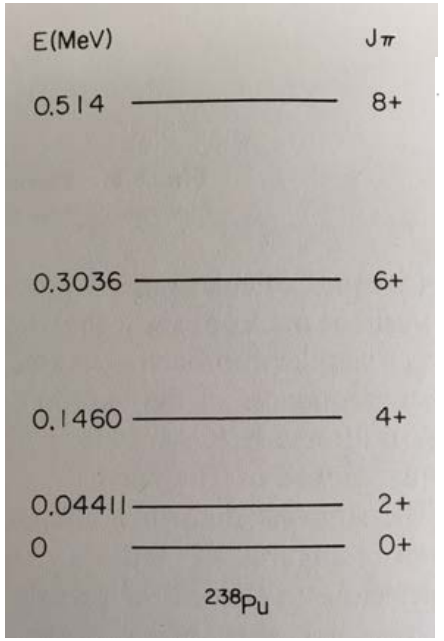




*Fig. 2. Dependence of Nuclear Moments of Inertia on the Nuclear Deformation.* The empirical moments of inertia for even-even nuclei in the region $150 < A < 188$ are plotted as a function of the nuclear deformation. The moments of inertia, obtained from the data in Table I, are given in units of the rigid moment (18), while the deformation parameters $\beta$ are obtained from the $Q_0$-values in Table I by means of (19). The nuclear radius has been taken to be $R_0 = 1.2\, A^{1/3}\, 10^{-13}$ cm. The full-drawn curve represents a theoretical estimate, based on the two-nucleon model with an interaction parameter $v = 1/3$ (cf. Fig. 1). For comparison, the moment of inertia corresponding to irrotational flow is shown by the dotted curve.

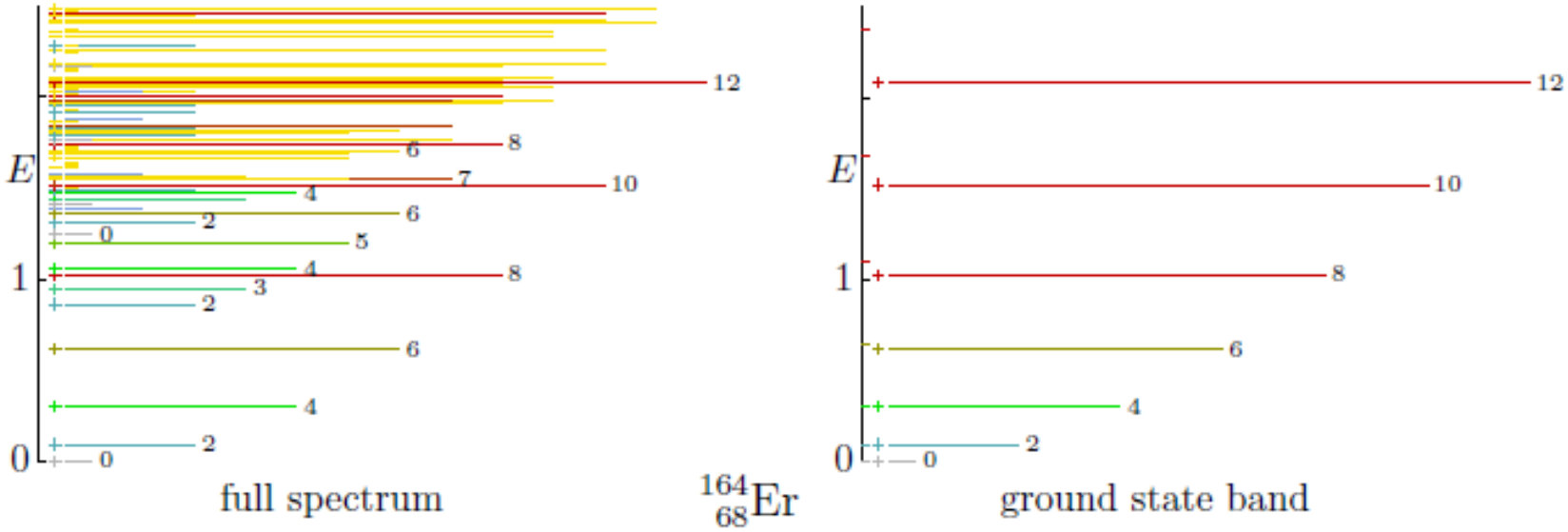
A.Bohr & B.Mottelson, Dan. Matematisk-fysiske Meddelelser (1955))

# Rotational bands: sequences of excited states

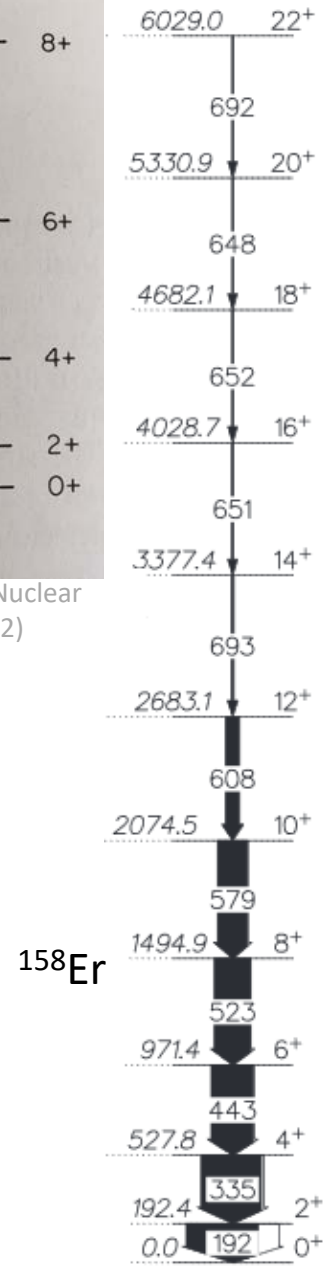
- $E_{rot} = \frac{\hbar^2 j(j+1)}{2I}$, so for a given $I$, $\Delta E \propto j(j+1)$
- Note that parity needs to be maintained because rotation is symmetric upon reflection and so $0^+$ ground-states can only have j=0,2,4,… (because $\pi = (-1)^J$)
- Without observing the decay scheme, picking-out associated rotational states could be pretty difficult
- Experimentally, coincidence measurements allow schemes to be mapped




B. Harvey, Introduction to Nuclear Physics and Chemistry (1962)



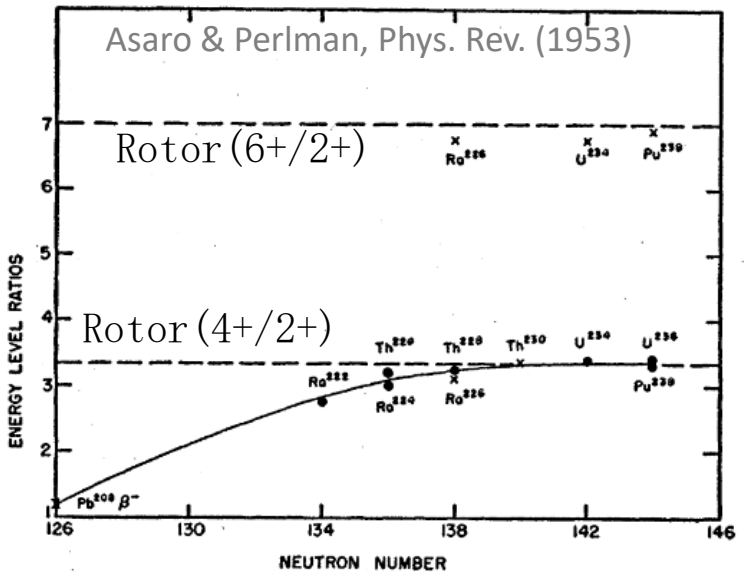

L. van Dommelen, Quantum Mechanics for Engineers (2012)

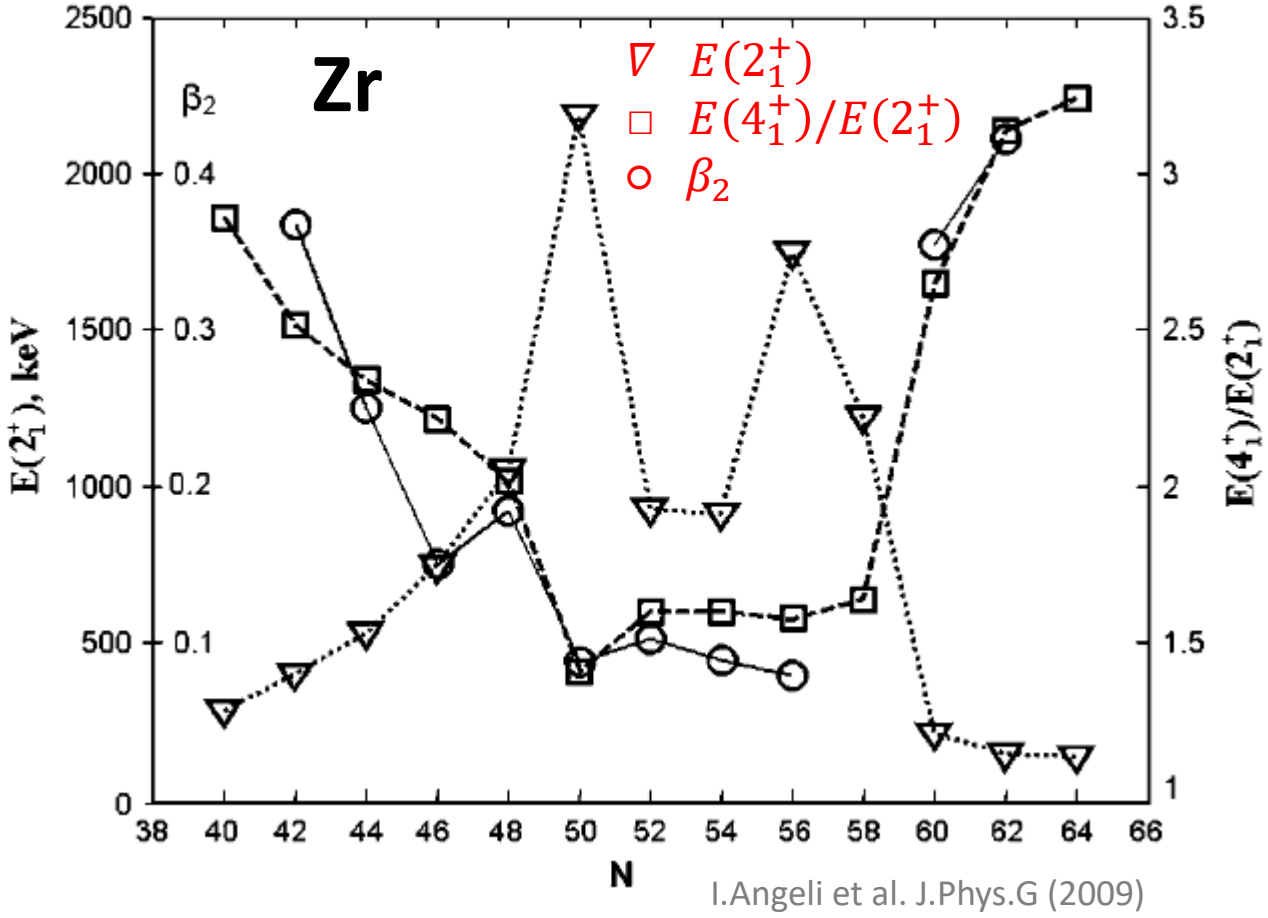



T.Dinoko et al. EPJ Web Conf. (2013)

# Inferring structure from rotational bands

- Since $E_{rot} = \frac{\hbar^2 j(j+1)}{2I}$ and $I \propto \beta$ , we can use rotational bands to probe deformation
- More deformed nuclei have larger $\beta$, so excited state energies for the band should be low
  - The 1st 2+ excited state energy is often used to probe this
- The rotor model for rotational bands is validated by the comparison of band excited state energy ratios to the rotor prediction
- The ratio of the yrast $4^+$ and $2^+$ excited state energies is generally close to the rotor prediction for nuclei far from closed shells



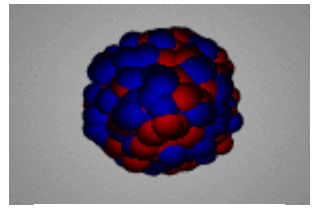
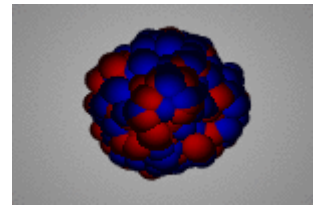
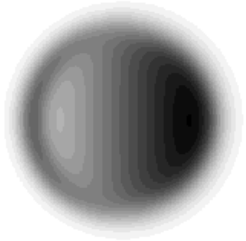
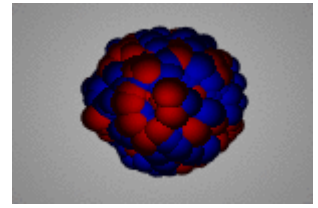
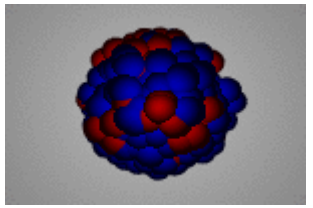
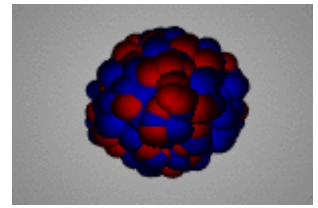
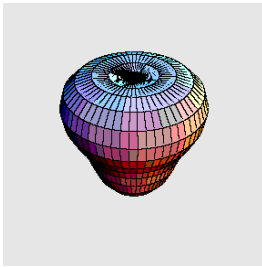
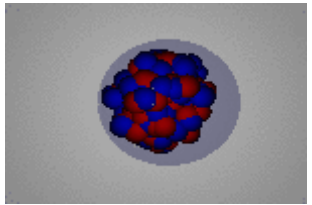



# Vibrational modes

- Considering the nucleus as a liquid drop, the nuclear volume should be able to vibrate H.J. Wollersheim
- Several multipoles are possible
  - Monopole: in & out motion ($\lambda = 0$)
    - a.k.a the breathing mode
  - Dipole: sloshing back & forth ($\lambda = 1$)
    - If all nucleons are moving together, this is just CM motion
  - Quadrupole: alternately compressing & stretching ($\lambda = 2$)
  - Octupole: alternately pinching on one end & then the other ($\lambda = 3$)
  - + Higher
- Protons and neutrons can oscillate separately ("isovector" vibrations)
- All nucleons need not move together
  - e.g. the "pygmy dipole" is the neutron skin oscillation

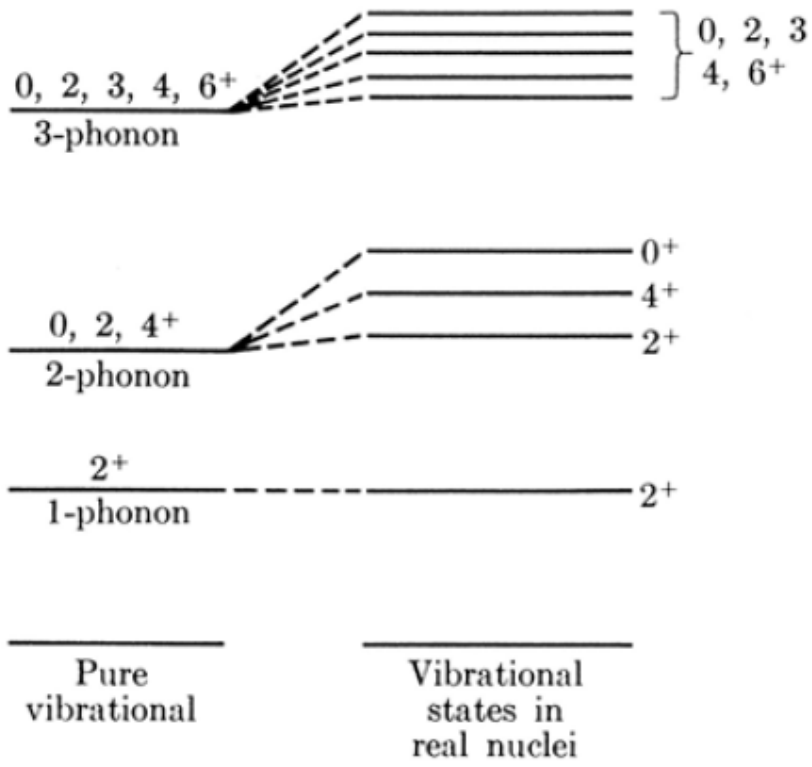
The gifs to the right are for giant resonances, when all protons/neutrons act collectively

***Isoscalar*** ***Isovector***

# Vibrational energy levels

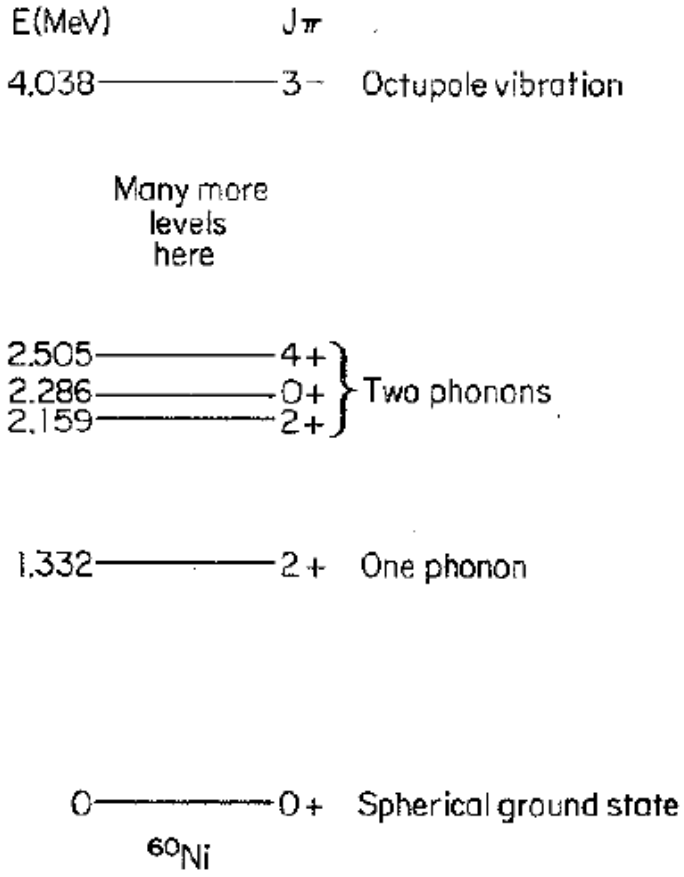
- Just as the quantum harmonic oscillator eigenvalues are quantized, so too will the energy levels for different quanta (*phonons*) of a vibrational mode.
- Similarly, the energy levels have an even spacing, $E_n = (n + \frac{1}{2})\hbar\omega$
- Even-even nuclides have 0+ ground states, and thus, for a $\lambda = 2$ vibration, $n = 2$ excitations will maintain the symmetry of the wave-function (i.e. $n = 1$ excitations would violate parity)
- Therefore, the 1st vibrational state will be $2^+$



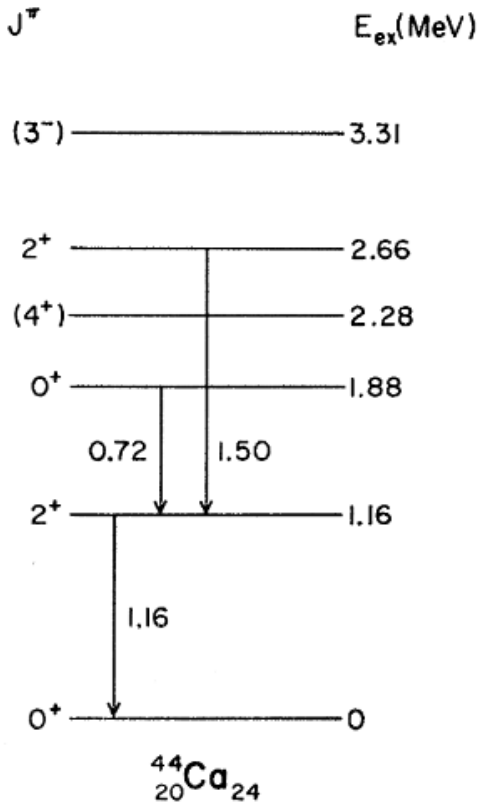

Loveland, Morrissey, Seaborg, Modern Nuclear Chemistry (2006)

- We can excite an independent quadrupole vibration by adding a second phonon
- The second phonon will build excitations on the first, coupling to either $0^+$,$2^+$, or $4^+$
- Employing a nuclear potential instead winds up breaking the degeneracy for states associated with a given number of phonons

# Vibrational energy levels are pretty obvious for spherical nuclei

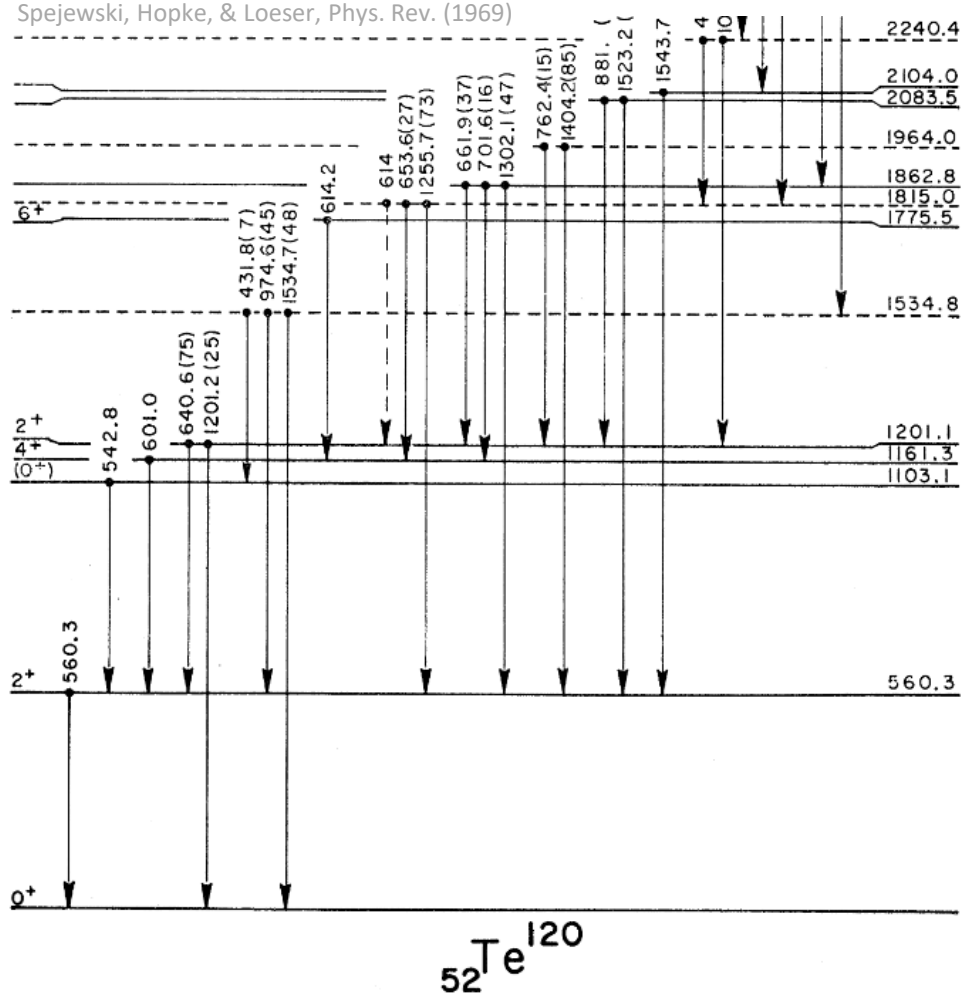



G. Harvey, Introduction to Nuclear Physics and Chemistry (1962)




Matin, Church, & Mitchell Phys. Rev. (1966)

Spejewski, Hopke, & Loeser, Phys. Rev. (1969)




- The typical signature for vibrating spherical nucleus is $E(4_1^+)/E(2_1^+) \approx 2$
  …though that obviously won't be the case for a deformed nucleus

# Recap of basic structure models discussed thus far

- Schematic shell model
  - Great job for ground-state $J^\pi$
  - Decent job of low-lying excited states for spherical nuclei, particularly near closed shells
  - Miss collective behavior that arises away from shell closures
- Collective model
  - Rotational excitations explain several $J^\pi$ for deformed, **even-even** nuclei
    *These are "mid shell" nuclei, because they're not near a shell closure*
  - Vibrational excitations explain several $J^\pi$ for spherical, **even-even** nuclei
    *These are "near shell" nuclei, because they're near a shell closure*
  - Miss single-particle behavior that can couple to collective excitations

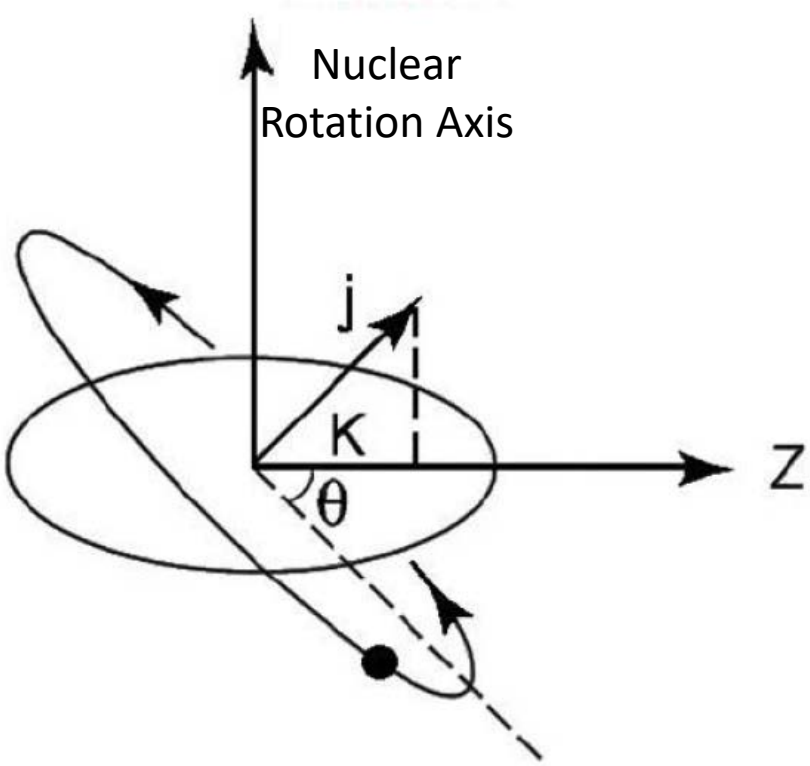
*What do we do for collective behavior for odd-A nuclei?*
the Nilsson model (a.k.a. deformed shell model)

# Nilsson Model: combining collective & single-particle approaches

- Our schematic shell model was working perfectly fine until you threw it away like a cheap suit because of a little deformation!

  Luckily, Mottelson & Nilsson (Phys. Rev. 1955) weren't so hasty.

- Consider a deformed nucleus with axial symmetry that has a single unpaired nucleon orbiting the nucleus
- *We're sticking to axial symmetry because*
  - *Most deformed nuclei have this property (mostly prolate)*
  - *The math, diagrams, and arguments are easier*
- The nucleon has some spin, $j$, which has a projection onto the axis of symmetry, $K$
- $j$ has $\frac{2j+1}{2}$ possible projections $K$
- Therefore, each single particle state from our shell model now splits into multiple states, identified by their $K$, each of which can contain two particles (spin up and spin down) for that given projection

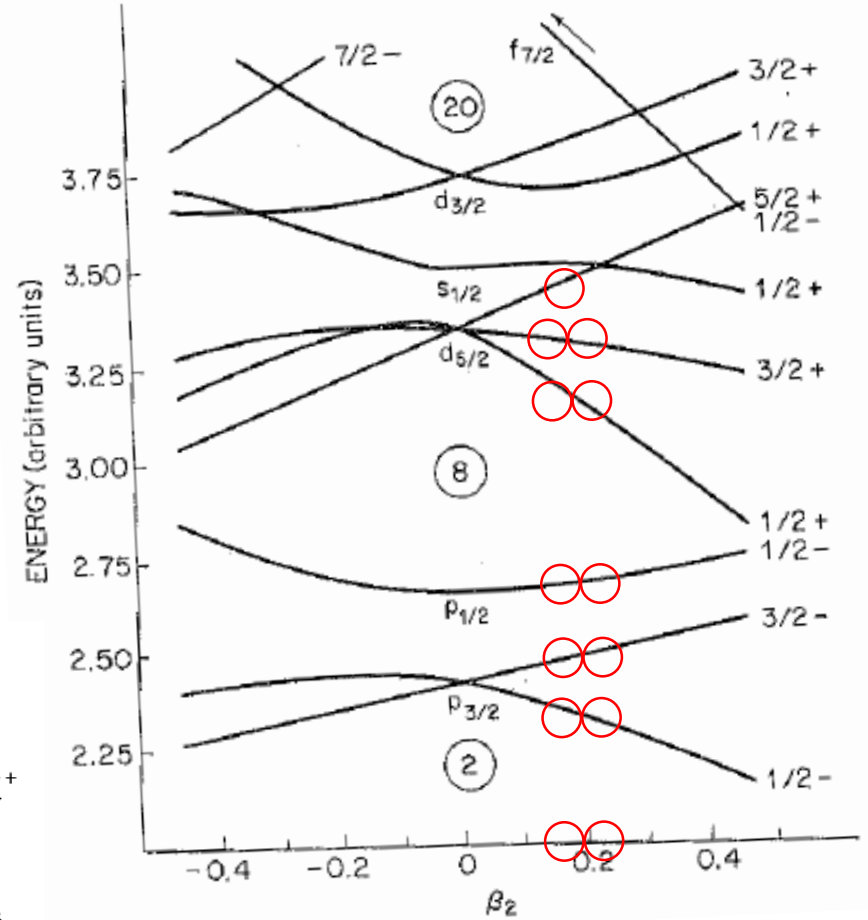


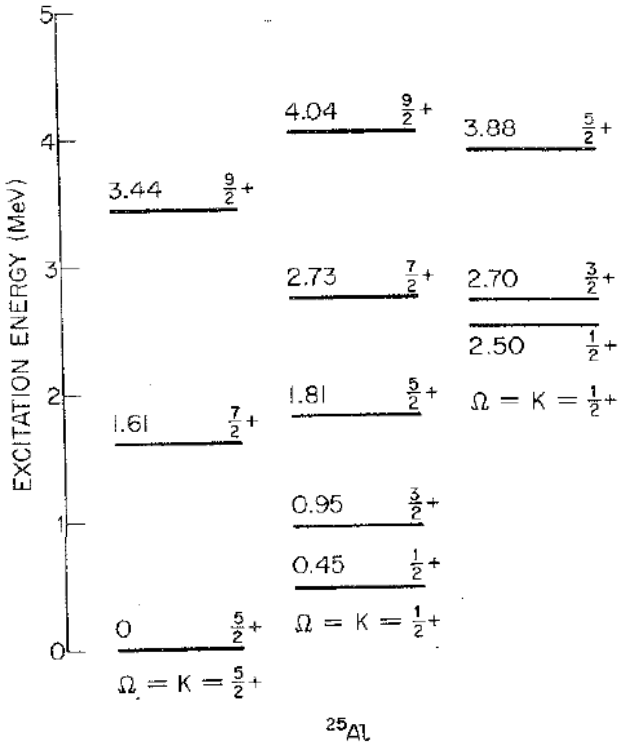

R.Casten, Nuclear Structure from a Simple Perspective (1990)

# Nilsson Model: Example

- Consider $^{25}$Al, for which we expect $\beta_2 \approx 0.2$, like $^{27}$Al
- There are 13 protons and 12 neutrons, so the unpaired proton will be responsible for $J^\pi$
- Filling the single-particle levels,
  - We place two protons in the $1s_{1/2}$ level, which isn't shown
  - Then two more in $1/2^-$, two more in $3/2^-$, two more in $1/2^-$, two more in the $1/2^+$, two more in the $3/2^+$
  - And the last one winds up in the $5/2^+$ level
- So, we predict $J^\pi_{g.s} = {}^5/_2{}^+$
- For the first excited sate, it seems likely the proton will hop up to the nearby $1/2^+$ level
- Agrees with data
- Since $^{25}$Al is deformed, we should see rotational bands with states that have (integer)+$j$ and $\propto j(j+1)$ spacing

B. Harvey, Introduction to Nuclear Physics and Chemistry (1962)

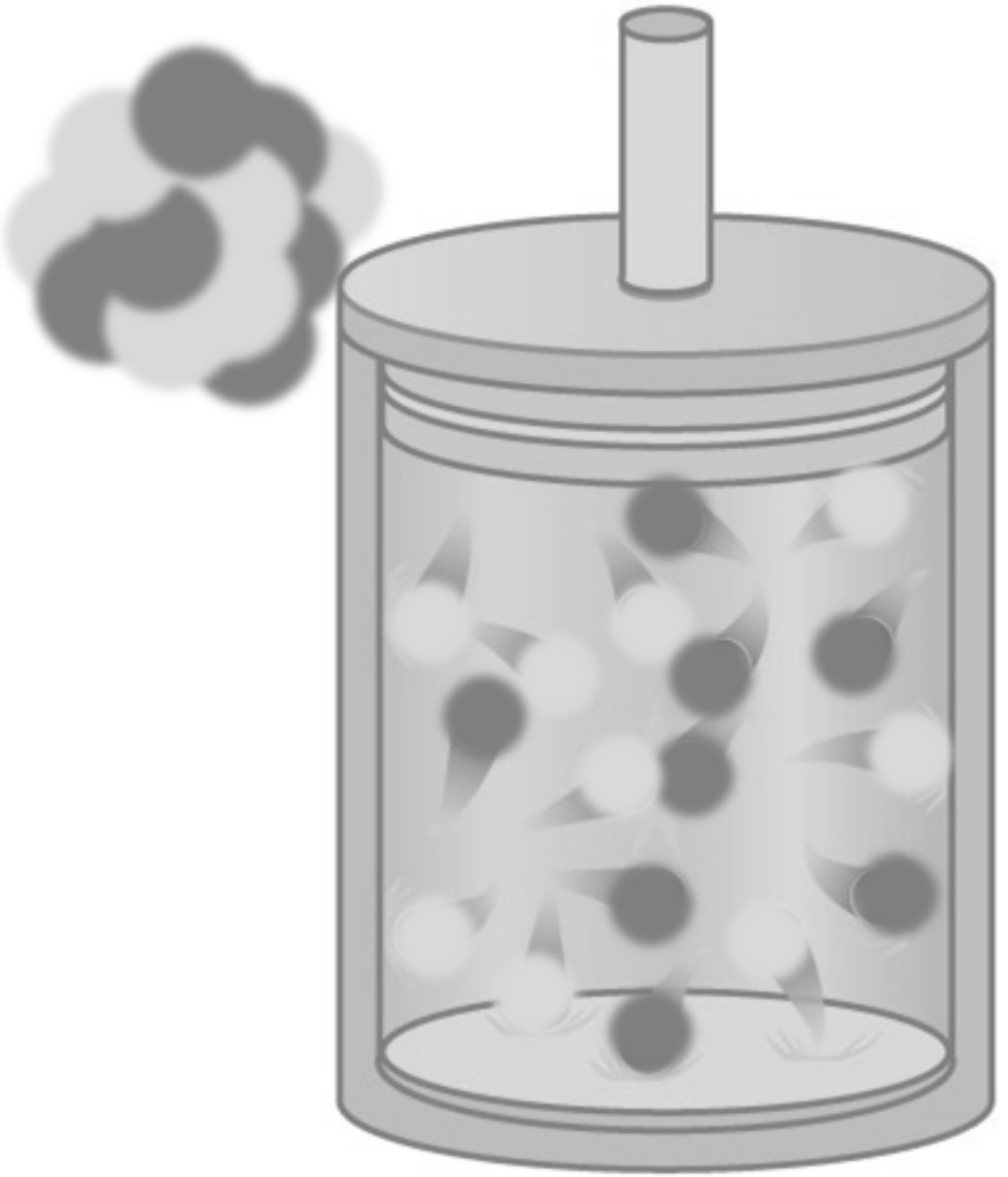





# Lecture 5: *Nuclear Structure 3*

- Fermi Gas Model
  - Microscopic approach
  - Thermodynamic approach
- Nuclear level density



# The nucleus as a Fermi gas

- To now we've focused on nuclear properties for cases where a few degrees of freedom dominate
  - E.g. a single uncoupled nucleon in a spherical well (shell model)
  - E.g. a collective rotation or vibration of nucleons in the nucleus (collective model)
  - E.g. a single uncoupled nucleon in a deformed well along with a collective rotation (Nilsson model)
- However, a typical nucleus has many nucleons and therefore many degrees of freedom
- These will become important in particular for highly-excited nuclei, since many degrees of freedom will then be relevant
- As such, we can understand some nuclear properties from the viewpoints of statistical mechanics and thermodynamics
- Since our neutrons and protons are spin-$^1/_2$ particles, they're fermions and so Fermi-Dirac statistics will apply
- Protons and neutrons are treated as two independent systems of fermions
- The approach of treating a nucleus as a fluid of fermions is known as the Fermi gas model

# Estimating nuclear properties from microscopic system properties

- Consider the average kinetic energy of our nucleons, $\langle E_{KE} \rangle \approx 30.6 \left(\frac{Z \text{ or } N}{A}\right)^{2/3} MeV$
  - The de Broglie wavelength $\lambda = \frac{h}{p} = \frac{2\pi\hbar}{\sqrt{2mE}}$
  - For $m \approx 931.5 MeV/c^2$ and $\hbar c \approx 197 MeV\ fm$, $\lambda \approx \frac{5.15}{(Z \text{ or } N)^{1/3}} A^{\frac{1}{3}}$
    - so for $Z = N = \frac{A}{2}$, $\lambda \approx 4.1 fm$
  - Compare to the heaviest stable $Z = N$ isotope, $^{40}$Ca
    - $R\left(^{40}Ca\right) \approx 1.2(40)^{1/3} fm = 4.1 fm$



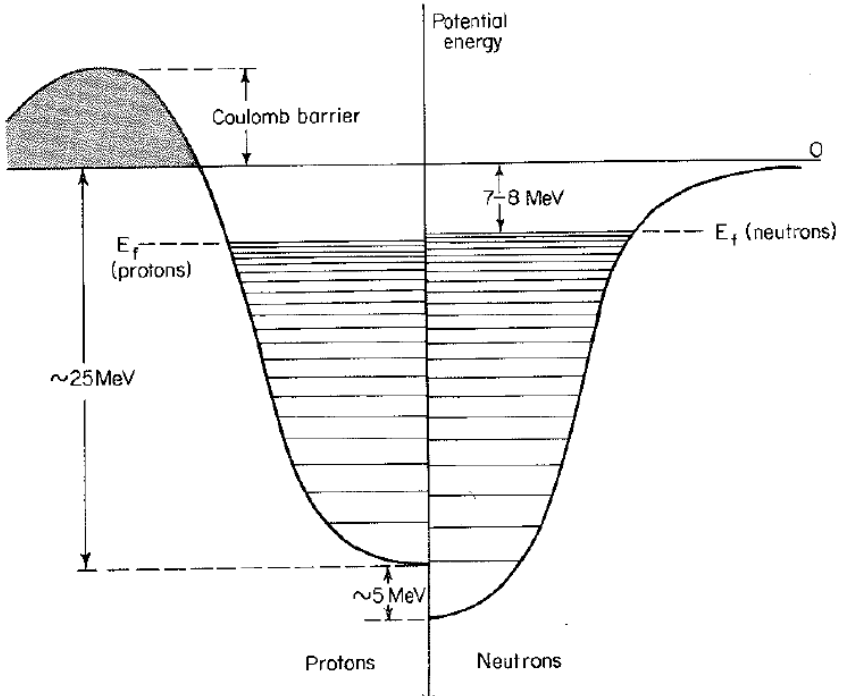

- Now consider the total kinetic energy of the nucleons
  - We can estimate the total kinetic energy as: $E_{KE} = Z\langle E_{KE,Z} \rangle + N\langle E_{KE,N} \rangle$
  - From the virial theorem (2*KE=n*PE, where V(r)~r$^n$. For a potential $V(r) \propto r^2$, like a harmonic oscillator), $E_{PE} = E_{KE}$
  - So an estimate for the binding energy per nucleon is $\frac{\text{BE}}{\text{A}} = \frac{E_{PE}}{A} = 30.6 \frac{Z^{5/3} + N^{5/3}}{A^{5/3}}$
    - For $Z, N$ corresponding to most stable nuclides, $\frac{BE}{A} \approx 19 MeV$
    - Compare to the bulk-binding term of the SEMF, $a_v A$, where $a_v \approx 15 MeV$

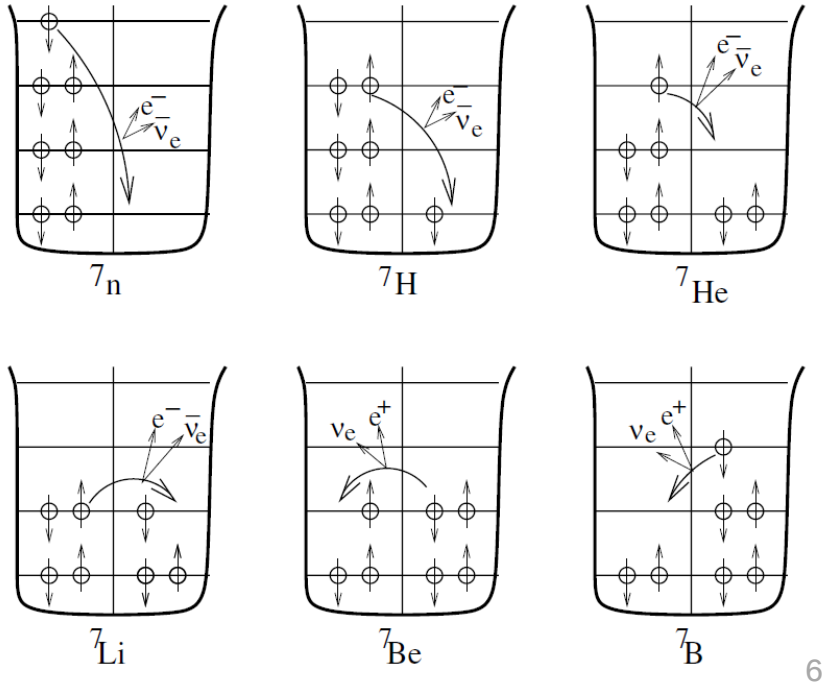


# Implications of the Fermi levels

- Protons and neutrons each fill available levels up to their respective Fermi levels
- For each case, the Fermi level must be $\sim 8\ MeV$ below the top of the potential well, since $\mathrm{BE/A} \sim 8 MeV$ for stable nuclides
- For $N = Z, E_F \approx 32 MeV$, so the potential well depth is $V_0 \approx E_F + BE/A \approx 40 MeV$
- However, protons suffer from Coulomb repulsion, and so their potential well depth is somewhat more shallow
- As such, it takes fewer protons to fill single-particle levels up to the same binding energy
- If there is a mismatch in the neutron and proton Fermi levels (which is greater than $M_n - (M_p + M_e)$), then it is energetically favorable for a neutron to transmute into a proton *or vice versa*
- By expanding our estimate on the previous slide for $BE$ in terms of $N - Z$, it turns out $BE \approx \frac{3}{5}E_F + \frac{4}{3}E_F \frac{\left(Z - \frac{A}{2}\right)^2}{A}$, revealing the asymmetry term from the SEMF
- This estimates $a_a \approx 43$, where the SEMF fit yields $a_a \approx 90$

G. Harvey, Introduction to Nuclear Physics and Chemistry (1962)



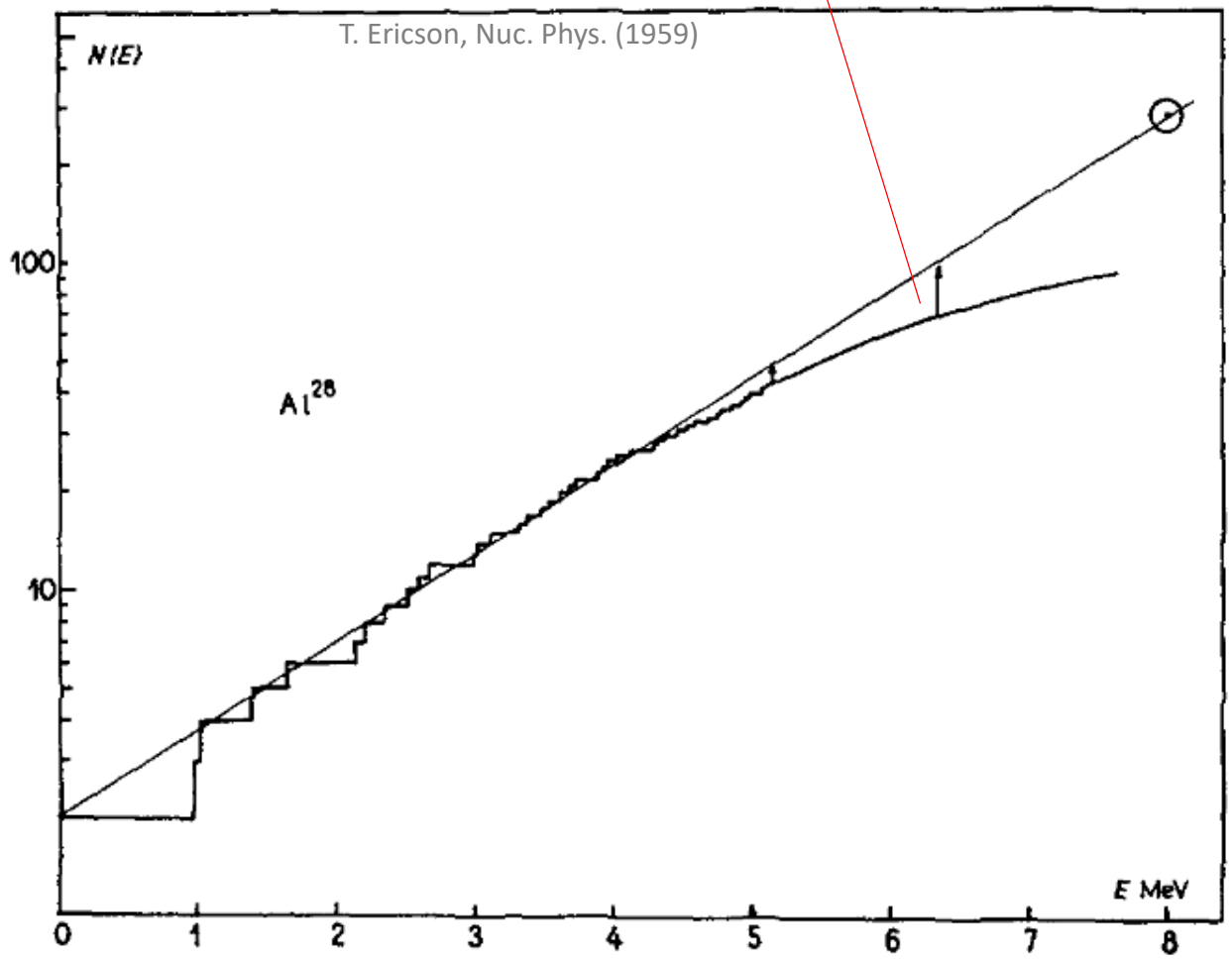

Basdevant, Rich, & Spiro, Fundamentals in Nuclear Physics (2005)

# Connection to thermodynamics

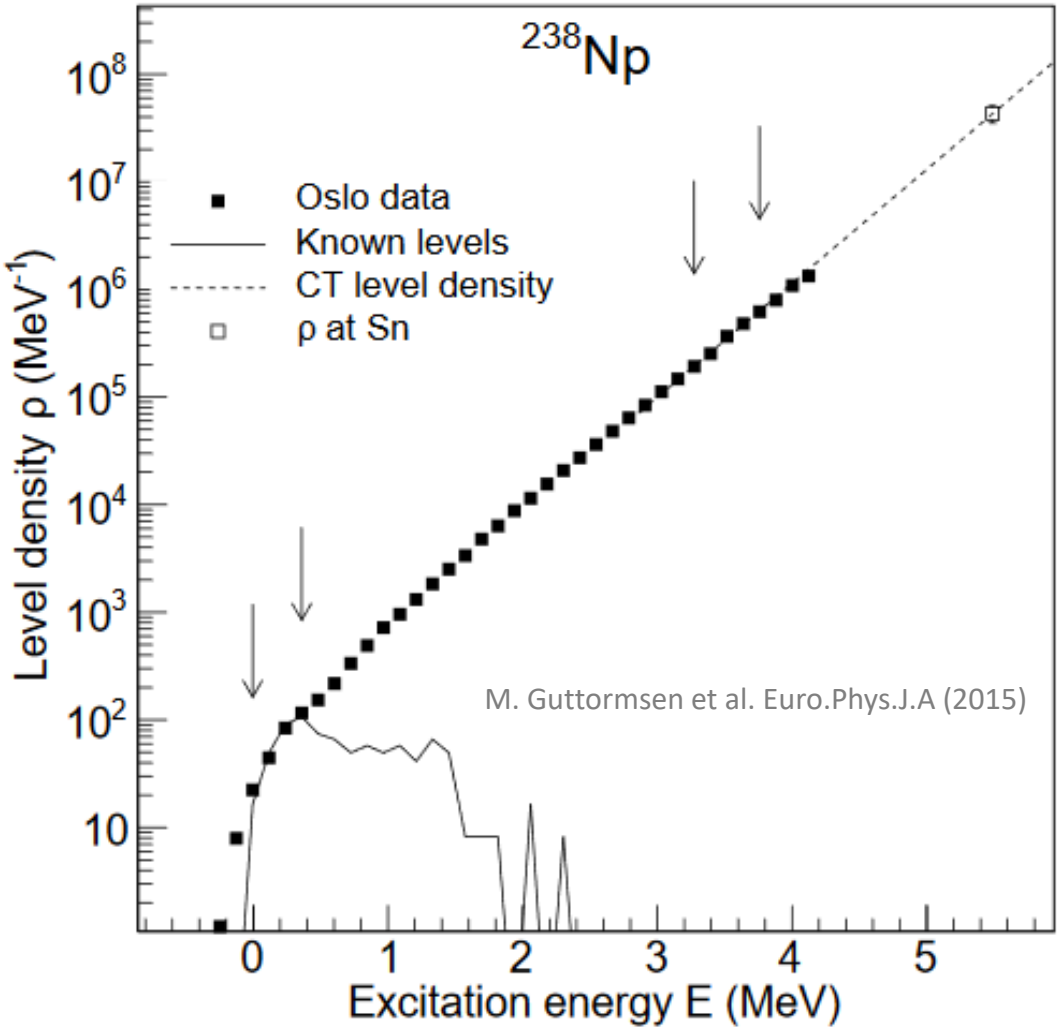
- Now we can solve for our entropy: $S(E) = \int \frac{dE}{T(E^*)} + constant \approx \int k_B \sqrt{\frac{a}{E^*}} dE + constant$
- Since a zero-temperature system has zero entropy, $S(E) \approx k_B 2\sqrt{aE^*}$
- Recall from the microscopic picture, $S = k_B \ln(g)$
- So, the number of accessible configurations (a.k.a. nuclear states) for our system is $g \approx \exp\left(2\sqrt{aE^*}\right)$
- The density of states is going to be proportional to the total number of states $\left(\frac{d}{dx} e^x = e^x\right)$
- So, the state density $\rho(E^*) = C \exp\left(2\sqrt{aE^*}\right)$, where $C$ is a constant
- A more careful treatment using partition functions and other statistical mechanics tools yields: $\rho(E^*) = \frac{\sqrt{\pi}}{12 a^{1/4} E^{*5/4}} \exp\left(2\sqrt{aE^*}\right)$ H. Bethe, Phys. Rev. (1936)
- Going back to our estimate for $^{238}$U and using $a = 1/d$ and $E^* = S_n$, we get $\rho \approx 3 \times 10^4 MeV^{-1}$
- In practice, $C$ and $a$ are usually fit to data
- $C$ in particular isn't so relevant, since we can normalize $\rho(E^*)$ to the region at low excitation energy where individual levels can be counted and ideally also to $\rho(E^* = S_n)$

# Experimental results confirm the exponential behavior of $\rho(E^*)$

*One challenge in comparing to counts of discrete states is knowing if your measurement missed any levels*




*Techniques which are sensitive to the integrated number of levels can overcome this …though with assistance from models*



# From state density to level density

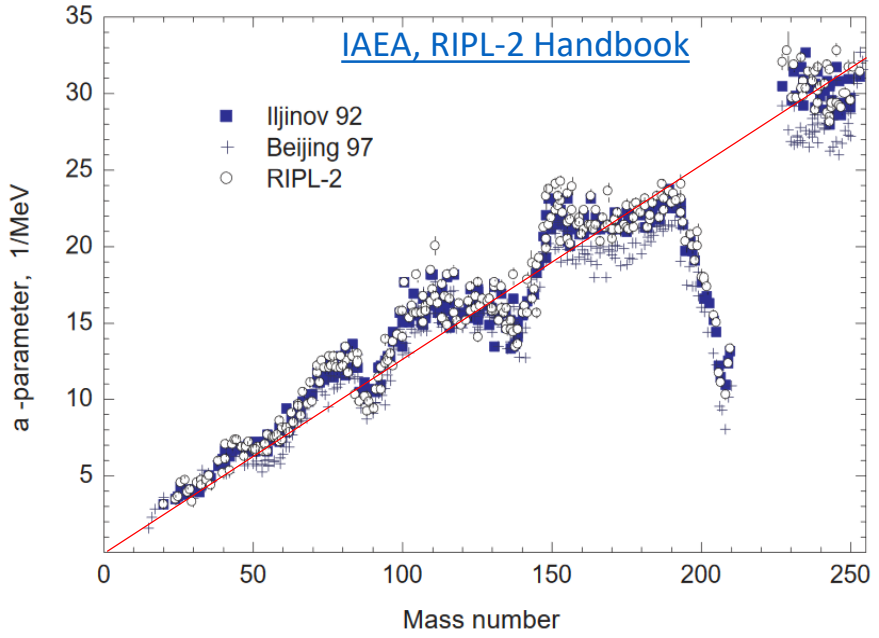
- As such, our level density $\rho(E^*,J) = \rho(E^*,J=M) - \rho(E^*,J=M+1)$ is
$$\rho(E^*,J) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{-J^2}{2\sigma^2}\right)\rho(E^*) - \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{-(J+1)^2}{2\sigma^2}\right)\rho(E^*)$$

*an alternative approach approximates ρ(E*,J=M) – ρ(E*,J=M+1) ≈ ∂ρ(E*,M)/∂M and gets the same result

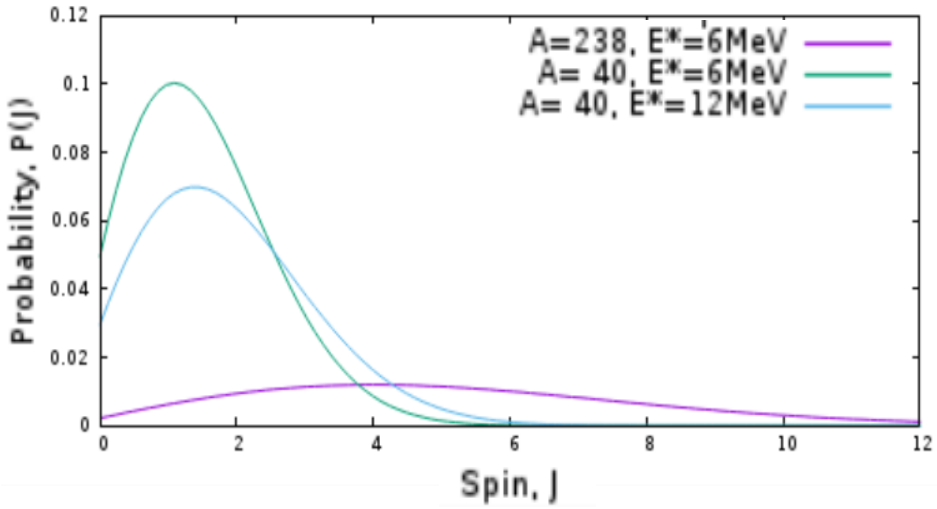
- It turns out (See, e.g. Chapter 2: Statistical Models for Nuclear Decay (A.L. Cole)), this is approximately equal to:
$$\rho(E^*,J) \approx \rho(E^*)\frac{2J+1}{2\sigma^2}\exp\left(\frac{-\left(J+\frac{1}{2}\right)^2}{2\sigma^2}\right) = \rho(E^*)P(J)$$
- The level-density is the state-density adjusted for the distribution of nuclear spin states, the "spin-distribution"
- As an aside, an important and often overlooked point is that we assumed states with a given $J, M$ were degenerate in energy and therefore we're assuming spherical nuclei.
- The variance of the spin distribution, $\sigma^2$ is known as the *spin-cutoff parameter* and it obviously impacts our results a great deal *[Often just σ is referred to as the spin-cutoff parameter, so beware]*
- A common approach to estimating $\sigma^2$ is to consider the nucleus as a rigid rotating body
- If we assume an ensemble of nuclei each have some energy $E$ due to the nuclear temperature $T \approx \frac{1}{k_B}\sqrt{E^*/a}$ and that $E$ expressed rotationally, then $P(E) \propto \exp\left(\frac{E_{rot}}{k_B T}\right) \propto P(J)$
- Recalling from last time that $E_{rot} = \frac{\hbar^2 J(J+1)}{2I}$, we see that in this approximation $\sigma^2 = \frac{I k_B T}{\hbar^2}$

# $\sigma^2$ and the spin distribution

- Having found $\sigma^2 = \frac{Ik_BT}{\hbar^2}$, now we need to estimate the moment of inertia $I$
- Since we assumed a spherical nucleus earlier to justify the degeneracy of $E^*$ in $M$, we'll double-down and use $I$ for a rigid sphere: $I = \frac{2}{5}MR^2$
- Using this and the previously derived formula for the nuclear temperature $T \approx \frac{1}{k_B}\sqrt{E^*/a}$, and the standard estimate for the nuclear radius $R = r_0 A^{1/3}$: $\sigma^2 = \frac{2Mr_0^2 A^{2/3}}{5\hbar^2}\sqrt{\frac{E^*}{a}}$
- Using $M = Am_u \approx \left(931.5\,^{MeV}/_{c^2}\right)\mathrm{A}$, $\hbar c \approx 197 MeV\ fm, r_0 \approx 1.2 fm$: $\sigma^2 \approx 0.014 A^{5/3}\sqrt{\frac{E^*}{a}}$
- It turns out, fits to neutron resonances yield $a \approx {}^{A}/_{8}\,MeV^{-1}$
- Therefore, $\sigma^2 \approx 0.0509 A^{7/6}\sqrt{E^*}$




***Implied spin distributions***



# Lecture 6: *Radioactive Decay*

- Decay Equations
- Multiple Species
- Equilibrium
- Natural Radioactivity & Dating
- Radiotracers



# Decay Equations for One Nuclear Species

- Solving for $N(t)$ from the seemingly innocuous equation $\frac{dN}{dt} = -\lambda N(t)$, is actually pretty hard, unless one employs the Laplace transform, which turns our differential equation into an algebraic one
- For the LHS, we assume $N(t)$ to be an exponential function (empirically a safe bet) and use the derivative property of Laplace transform [with complex variable $s$]:
  - $\frac{dN}{dt} \rightarrow sN(s) - N(0)$
- For the RHS, simply swap-in $s$ for time: $-\lambda N(t) \rightarrow -\lambda N(s)$
- Therefore, $sN(s) - N(0) = -\lambda N(s)$
- Which is re-written as: $N(s) = \frac{N(0)}{(s+\lambda)}$
- Using one of several different methods (See E.g. D.Pressyanov, Am.J.Phys. 2002, or a Math Methods book), the inverse Laplace transform can be employed, yielding the familiar relation:
  - $N(t) = N(0)e^{-\lambda t}$, the number of nuclei existing at time $t$
- Since the Activity (decays/second) $A = \lambda N$,
  - $A(t) = A(0)e^{-\lambda t}$

# Common Descriptors for Decay

- Aside from the decay constant $\lambda$, other more intuitive quantities are often used
- It is common to state the time at which half of the nuclei in a radioactive sample will have undergone decay, a.k.a. the half-life: $t_{1/2}$
  - By definition: $\frac{N(t_{1/2})}{N(0)} = \frac{1}{2}$ … so, $\frac{1}{2} = e^{-\lambda t_{1/2}}$
  - Re-write as $2 = e^{\lambda t_{1/2}}$, which makes it apparent that $t_{1/2} = \frac{\ln(2)}{\lambda}$
- An alternative piece of trivia is the mean lifetime for the nuclei in the sample, $\tau$
  - By definition: $\tau = \frac{\int_0^\infty tN(t)dt}{\int_0^\infty N(t)dt} = \frac{\int_0^\infty tN(t)dt}{N(0)/\lambda} = \frac{N(0)/\lambda^2}{N(0)/\lambda} = \frac{1}{\lambda}$
  - Therefore $t_{1/2} = \ln(2)\tau \approx 0.693\tau$
- We can re-state the lifetime in terms of the equivalent energy width using the Heisenberg uncertainty principle:
  - $\Delta E \cdot \Delta t \geq \hbar$
  - Taking the mean lifetime as the time uncertainty, $\Delta E = \frac{\hbar}{\tau} \approx \frac{6.582\times10^{-22} MeV}{\tau\ (in\ seconds)}$

# Radioactive Decay Units

- For samples in the lab, we usually care about how many decay products a sample is emitting, i.e. the Activity
- A sensible unit is the Becquerel: 1 Bq ≡ 1 decay/second
- A historical unit, based on the decay rate from a gram of radium (E. Rutherford, Nature (1910)), is the Curie: 1 Ci ≡ 37 GBq
- For context:
  - Soil most places on earth has $^{40}$K, $^{238}$U, $^{226}$Ra, and $^{232}$Th for a total of a few-hundred Bq/kg (IAEA Technical Report #1162)
  - The glassy residue from the July 1945 Trinity nuclear weapon test has an activity of ~100Bq/g (P.Parekh et al. J. Enviro. Rad. (2006))
  - A typical household smoke detector contains ~1 μCi (a.k.a. 37 kBq) of $^{241}$Am
- The activity per unit mass is called the "specific activity"

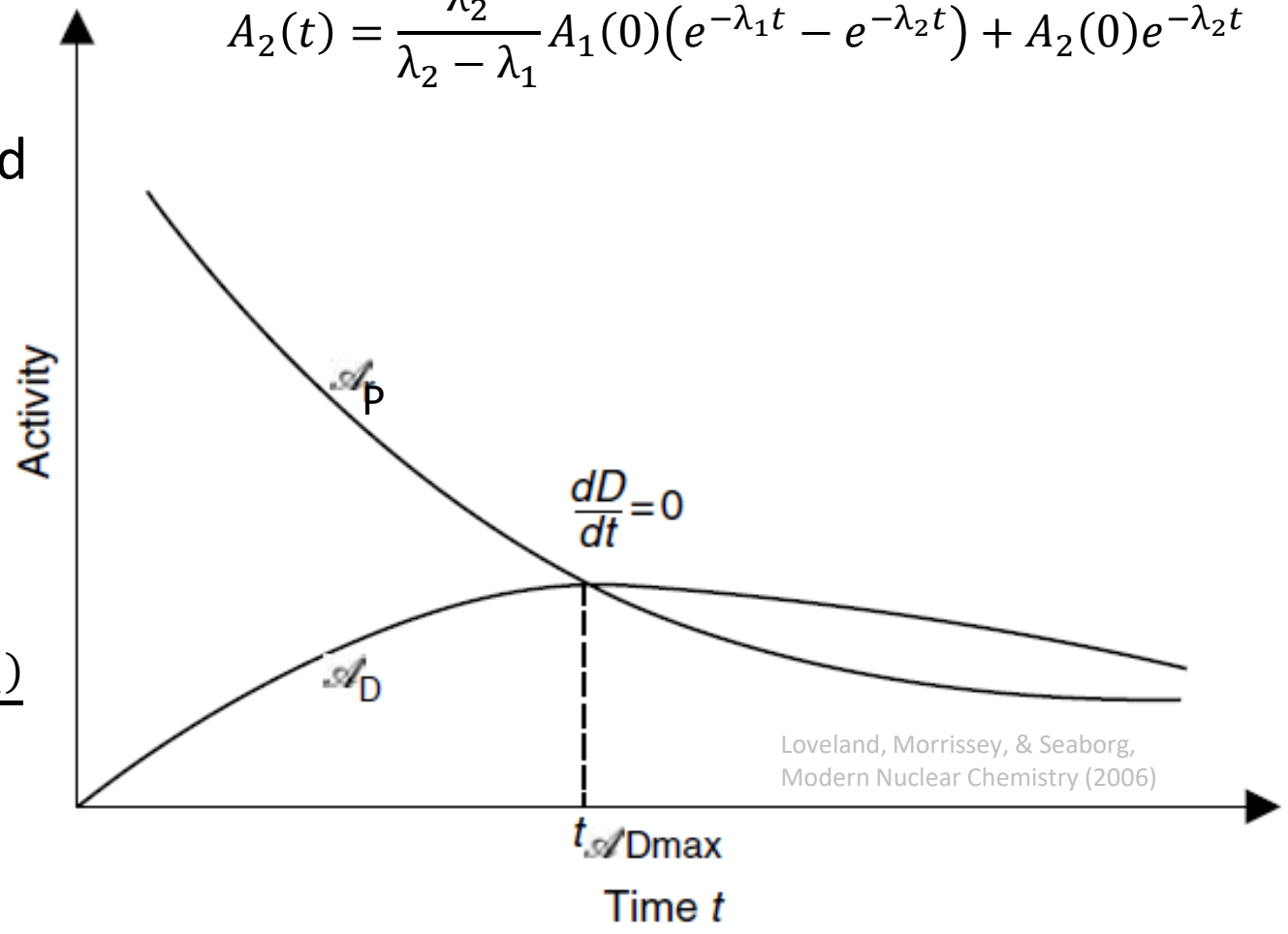
# “Daughter” Activity



- The total activity is some combination of the parent and daughter activity, where the exact time dependence will depend on the relationship between $\lambda_1$ and $\lambda_2$
- The relative magnitudes of $\lambda_1$ and $\lambda_2$ determine which type of “equilibrium” best describes the situation (covered in a moment)
- The time at which the daughter nucleus peaks in abundance (and activity) will be at $\frac{dA_2(t)}{dt} = 0$, which occurs at $t_{max} = \frac{\ln(\lambda_2/\lambda_1)}{\lambda_2 - \lambda_1}$
- For the case of a stable daughter, $\lambda_2 = 0$, with zero initial abundance

$$N_2(t) = N_1(0)\left(1 - e^{-\lambda_1 t}\right)$$

$$A_2(t) = \frac{\lambda_2}{\lambda_2 - \lambda_1} A_1(0)\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) + A_2(0)e^{-\lambda_2 t}$$




Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

# Decay "Equilibrium"

*Consider 4 limiting cases:*

a) $\tau_{parent} < \tau_{daughter}$
- The daughter nucleus will buildup faster than it decays
- "No Equilibrium"

b) $\tau_{parent} > \tau_{daughter}$
- Decay of the daughter is slightly faster than buildup
- Doesn't get a name

c) $\tau_{parent} \gg \tau_{daughter}$
- The daughter decays almost immediately after being made
- "Transient Equilibrium"

d) $\tau_{parent} \gg t \gg \tau_{daughter}$

(i.e. parent super-long-lived or infinite [beam])
- Parent is constantly replenished
- "Secular equilibrium"



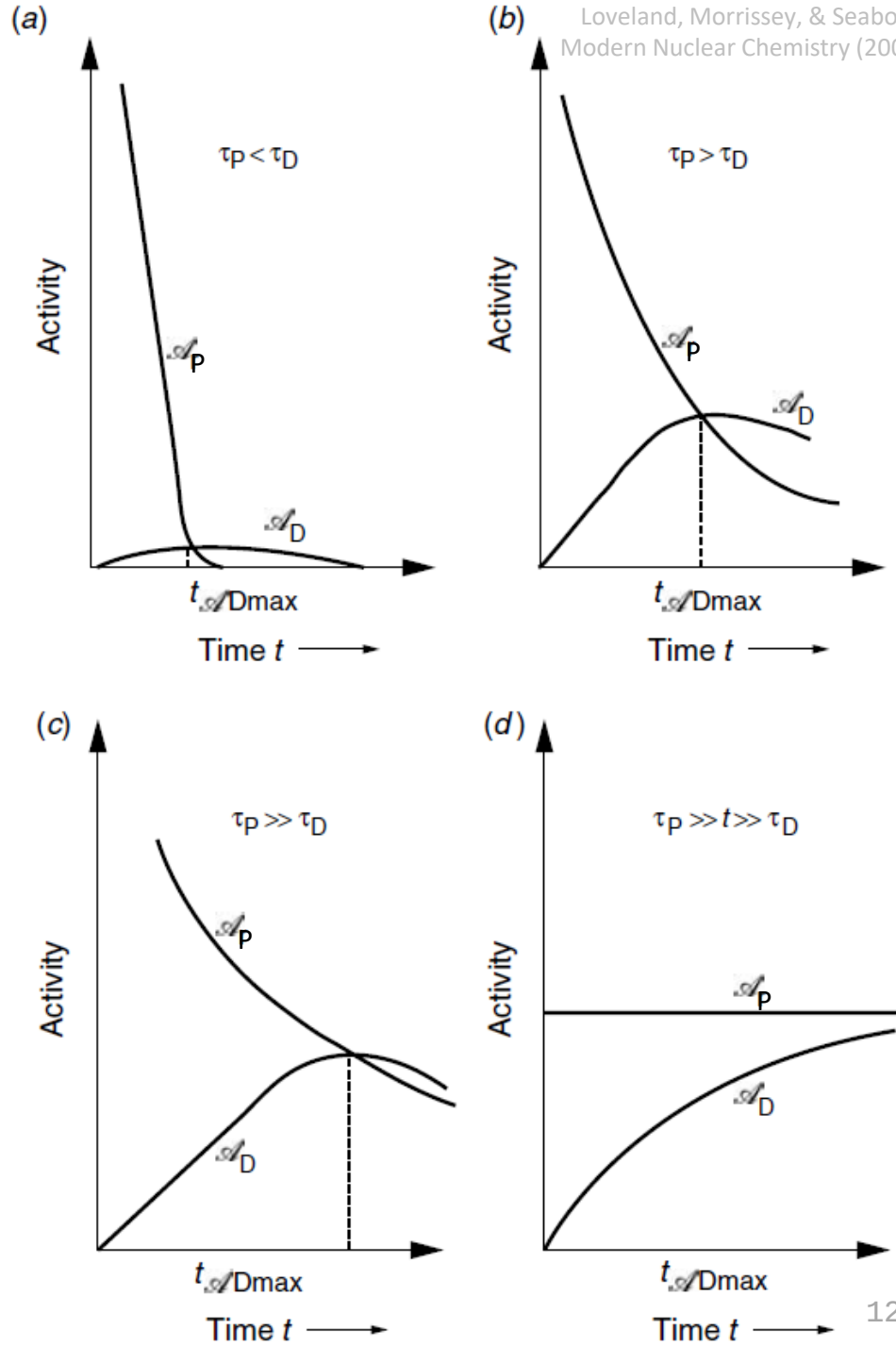






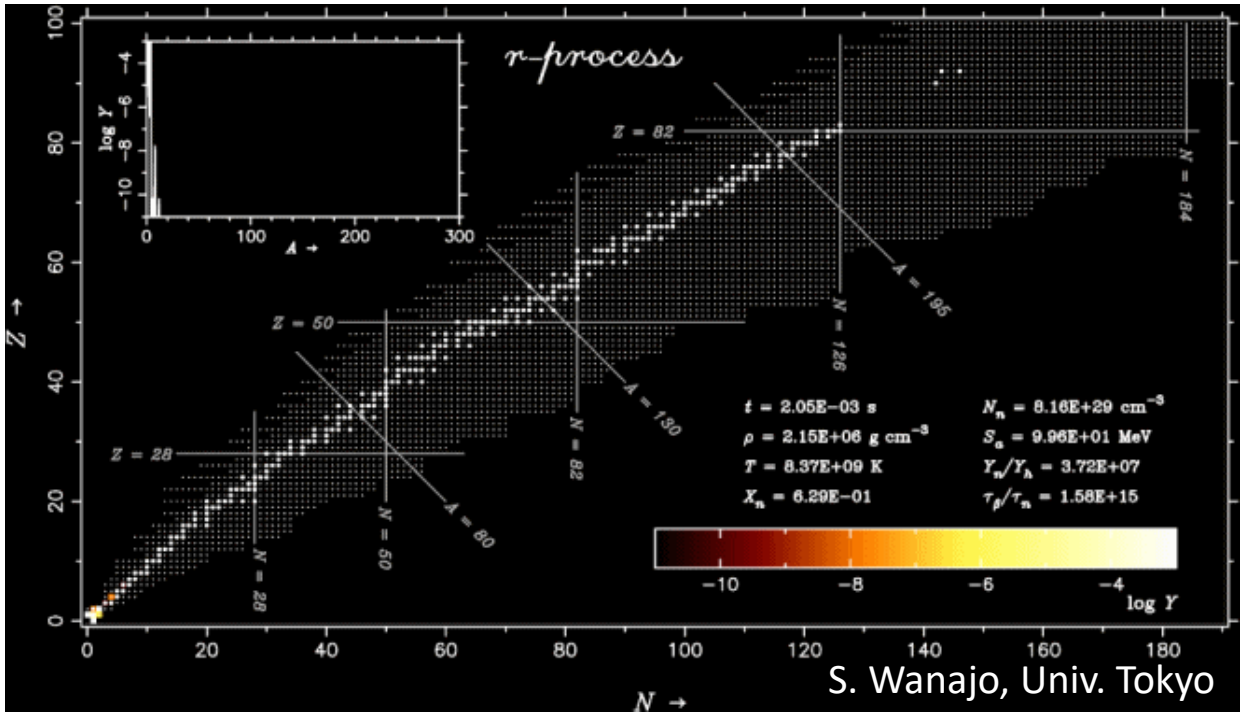


Loveland, Morrissey, & Seaborg
Modern Nuclear Chemistry (2006)

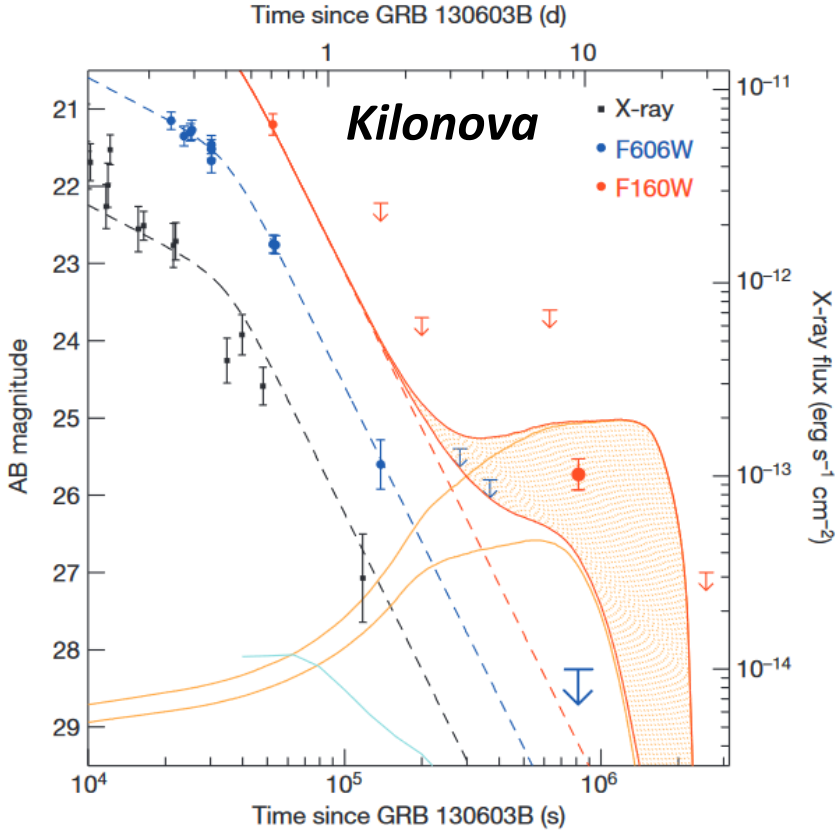


# No Equilibrium: $\tau_{parent} < \tau_{daughter}$

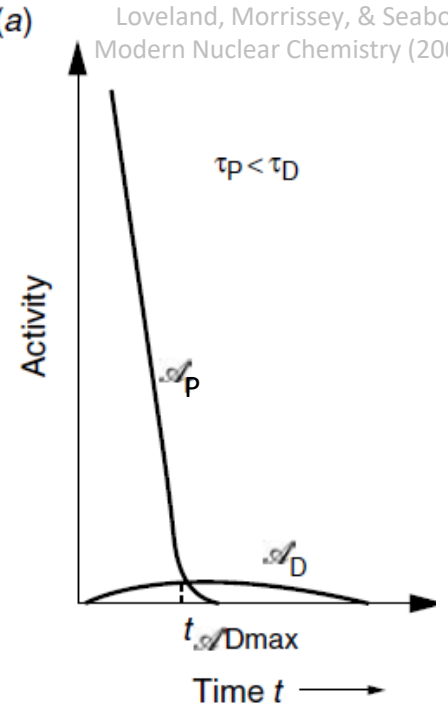
- This case will apply when exotic nuclides β-decay back to stability
- The decay can just be modeled as the daughter decay
- For example, this would be useful for modeling radioactivity due to core collapse supernova or neutron-star merger nucleosynthesis products




*radioactivity isn't really consequential for the CCSN light-curve



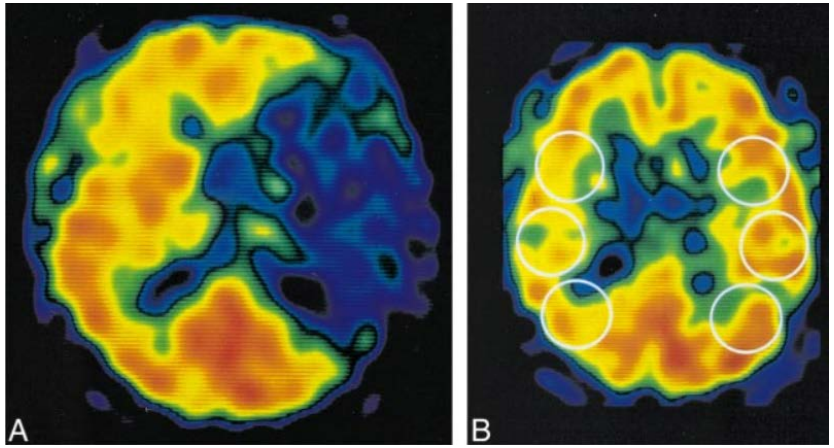

N. Tanvir et al. Nature (2013)



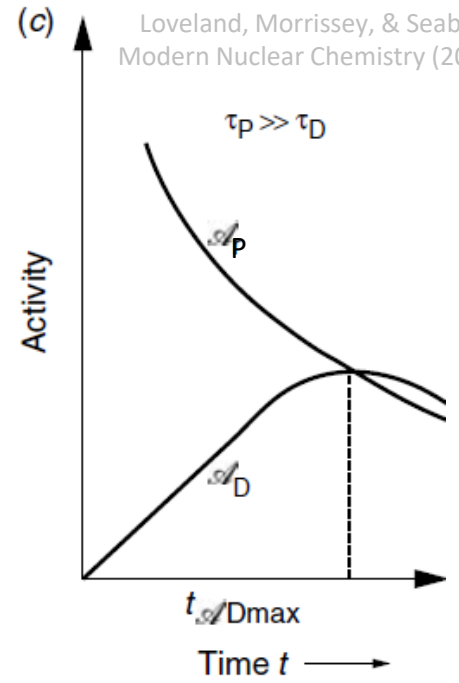

Loveland, Morrissey, & Seaborg Modern Nuclear Chemistry (2006)

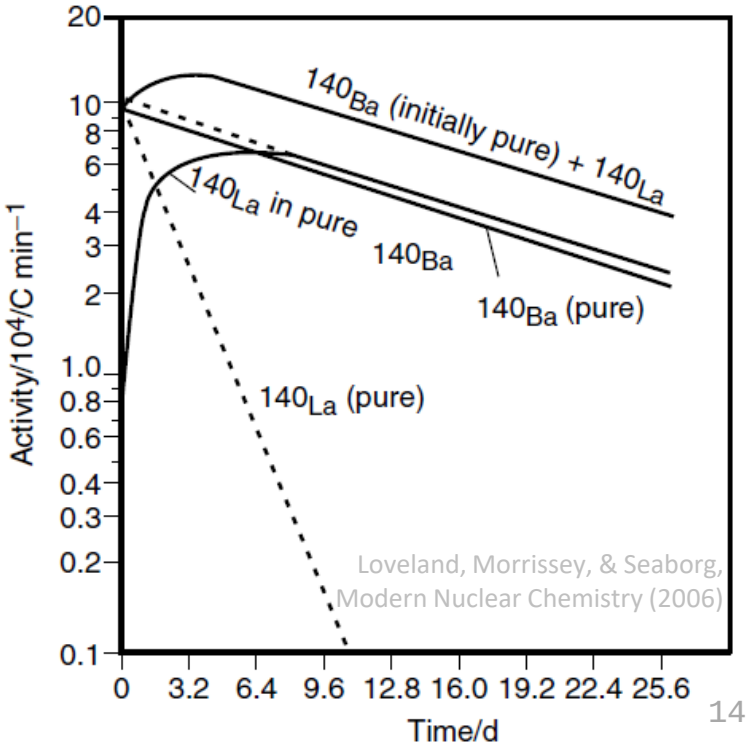
# Transient Equilibrium: $\tau_{parent} \gg \tau_{daughter}$

- When the parent is more than ~10x longer-lived than the daughter nuclide, the daughter activity will effectively be that of the parent
- In this case, $\lambda_2 > \lambda_1$. So, at large $t$, $e^{-\lambda_2 t}$ will be much smaller than $e^{-\lambda_1 t}$
- As such, $A_2(t) = \frac{\lambda_2}{\lambda_2 - \lambda_1} A_1(0)\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) + A_2(0)e^{-\lambda_2 t}$ becomes: $A_2(t) \approx \frac{\lambda_2}{\lambda_2 - \lambda_1} A_1(0)e^{-\lambda_1 t} = \frac{\lambda_2}{\lambda_2 - \lambda_1} A_1(t)$
- Therefore, in transient equilibrium at long times, the parent and daughter activity are related by a constant multiple, $\frac{\lambda_2}{\lambda_2 - \lambda_1}$
- Example: $^{99m}$Tc generator
  - $^{99}$Mo ($t_{½} \approx 66hrs$) is produced off-site (from $^{235}$U or n-irradiation of $^{98}$Mo)
  - On-site chemistry techniques are used to extract $^{99m}$Tc ($t_{½} \approx 6hrs$), which can then be used for SPECT



K. Ogasawara et al. American Journal of Neuroradiology (2001)



# Secular Equilibrium: $\tau_{parent} \gg t_{long} \gg \tau_{daughter}$

- Consider our previous relation

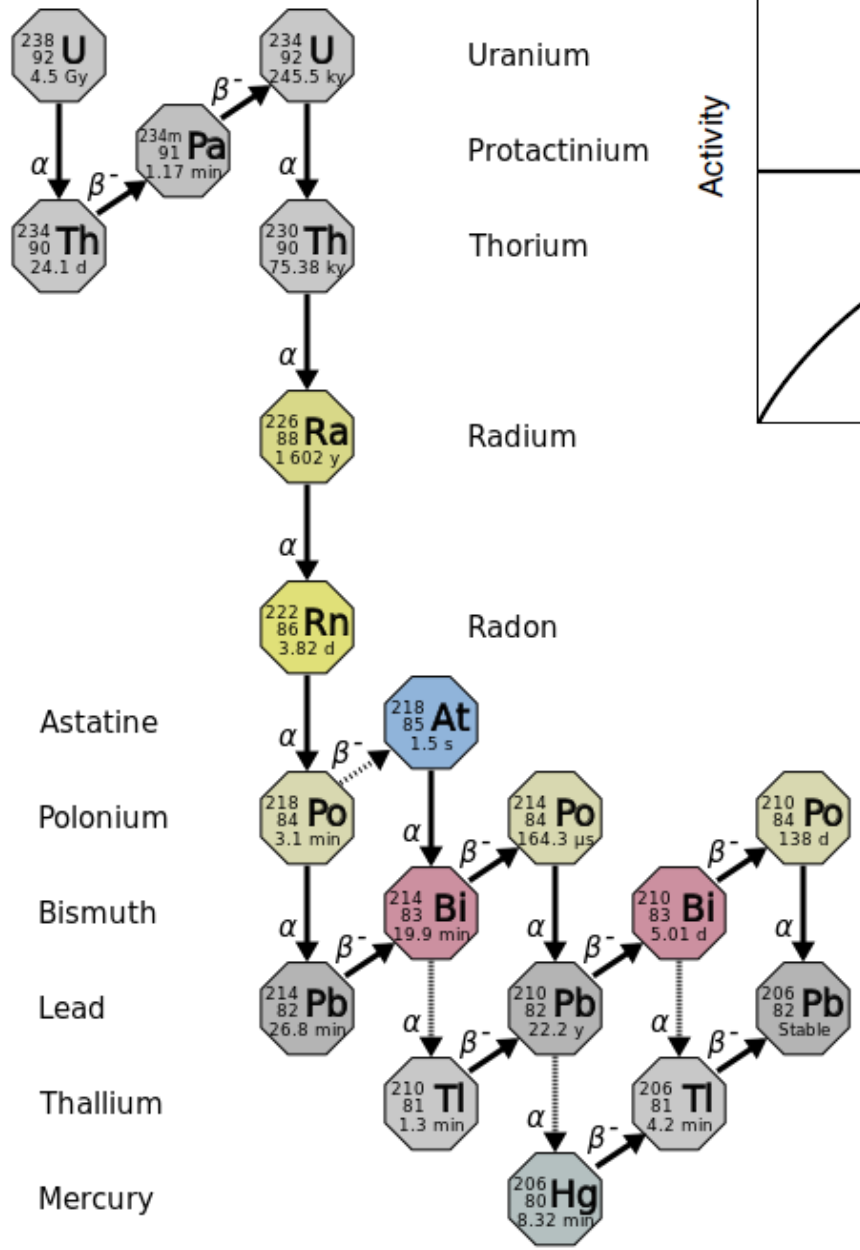
  $A_2(t) \approx \frac{\lambda_2}{\lambda_2 - \lambda_1} A_1(t)$ when $\lambda_2 \gg \lambda_1$

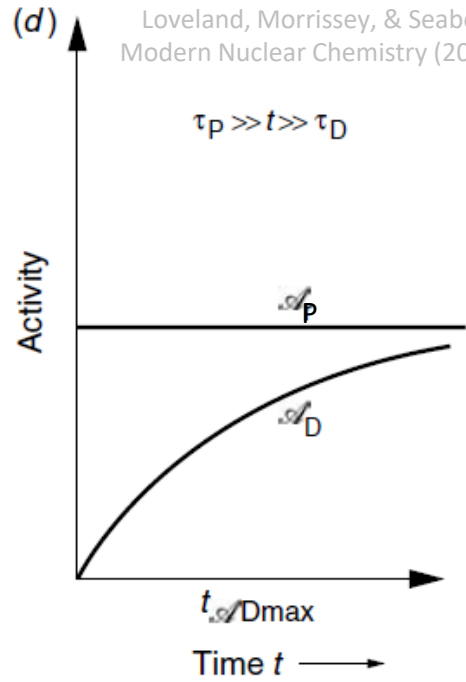
- For very large half-life disparities,

  $\lambda_2 \gg \lambda_1$, $(\lambda_2 - \lambda_1) \approx \lambda_2$, so $A_2(t) \approx A_1(t)$

- This applies for the decay chain of some transuranic nuclides

- For example: Radon from Granite
  - Granite has relatively large concentrations of $^{238}$U ($t_{½} \approx 4.5\text{Gyr}$), which wouldn't really be a problem, since it's trapped in the rock
  - However, $^{238}$U decay leads to $^{222}$Rn ($t_{½} \approx 4\text{days}$), which is a gas that easily escapes the rock and gets the chance to α-decay in your lungs
  - So, fancy countertops and state capitol buildings will be hazardous for perpetuity

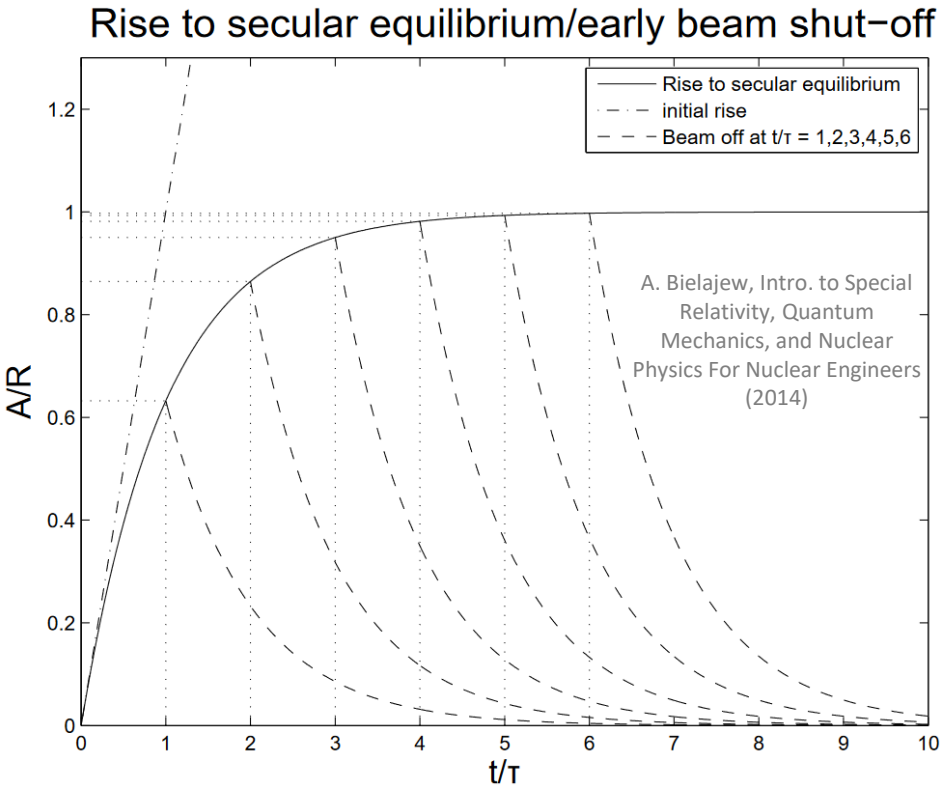



Loveland, Morrissey, & Seaborg
Modern Nuclear Chemistry (2006)

# Secular Equilibrium & Radioactivity from Nuclear Reactions

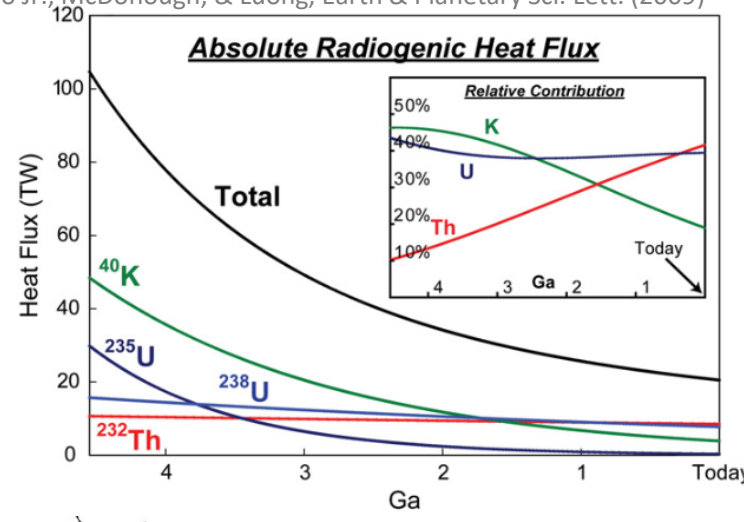
- If a radioactive nucleus is made in a nuclear reaction (e.g. in a star, from cosmic rays, or using an accelerator), the constant replenishment of the daughter is similar to the case of a very long-lived parent
- In particular this is true of the reaction rate is much slower than the daughter decay …which is pretty much always the case
- Starting with $N_2(0) = 0$ and recasting "decay" of $1 \rightarrow 2$ as a reaction, $R_{12} = \lambda_1 N_1(0) = A_1(0)$, $A_2(t) = \frac{\lambda_2}{\lambda_2 - \lambda_1} A_1(0)\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) + A_2(0)e^{-\lambda_2 t}$ becomes $A_2(t) = \frac{\lambda_2}{\lambda_2 - \lambda_1} R_{12}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$
- Since $\lambda_1 \approx 0, \quad A_2(t) \approx R_{12}\left(1 - e^{-\lambda_2 t}\right)$, which is known as the "activation equation"
- Linear growth of the daughter at short times, but have diminishing returns as time increases
- If you're making a custom radioactive source on-site, this lets you know how long to bother performing the production reaction
- This relation allows nuclear reaction cross sections to be measured using the daughter decay, and it lets you know how long to perform the reaction

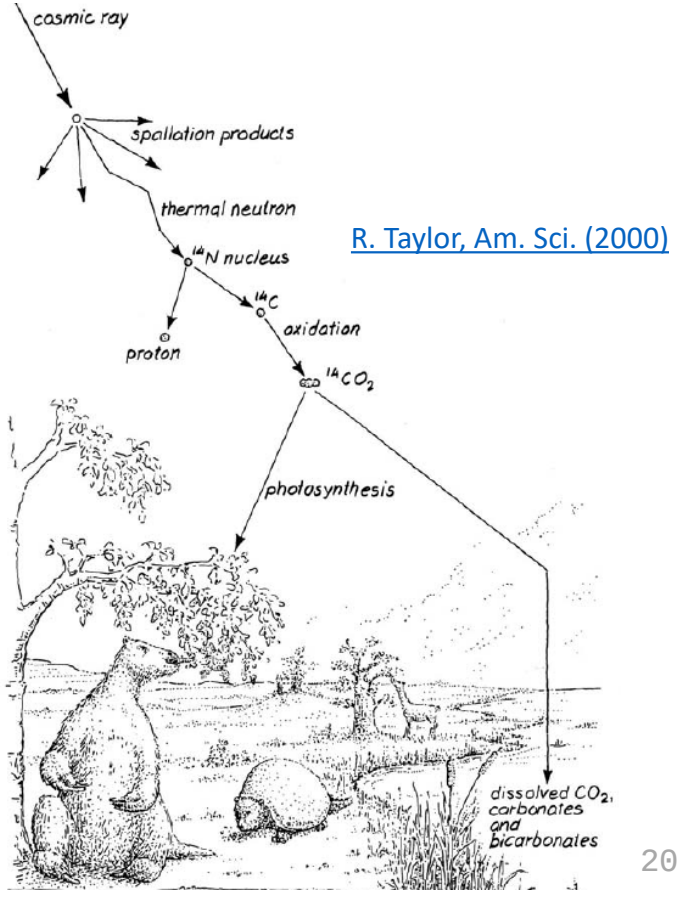


# Environmental Radioactivity

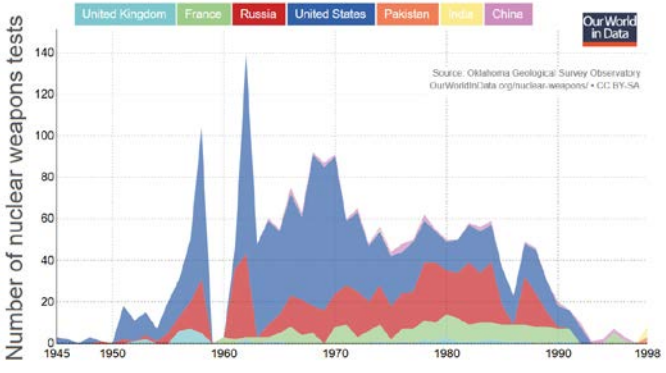
- Radioactivity is everywhere, coming from various sources:
  - ***Primordial***: Nuclei remaining since their formation
    - Ex. 1: U, Th, and K that provide most of the Earth's internal heat
    - Ex. 2: Deep-sea $^{244}$Pu from supernovae [A.Wallner et al. Nature Comm. (2015)]
  - ***Cosmogenic:*** Nuclei produced from cosmic ray interactions
    - Energetic particles from events out in space interact with nuclei in the atmosphere to make radioactive products
    - Important examples include $^{3}$H($t_{1/2}$~12yr), $^{7}$Be($t_{1/2}$~53d), and $^{14}$C($t_{1/2}$~5700yr)
  - ***Anthropogenic:*** Nuclei produced by human activities
    - Mostly from nuclear reactors and weapons tests

Arevalo Jr., McDonough, & Luong, Earth & Planetary Sci. Lett. (2009)

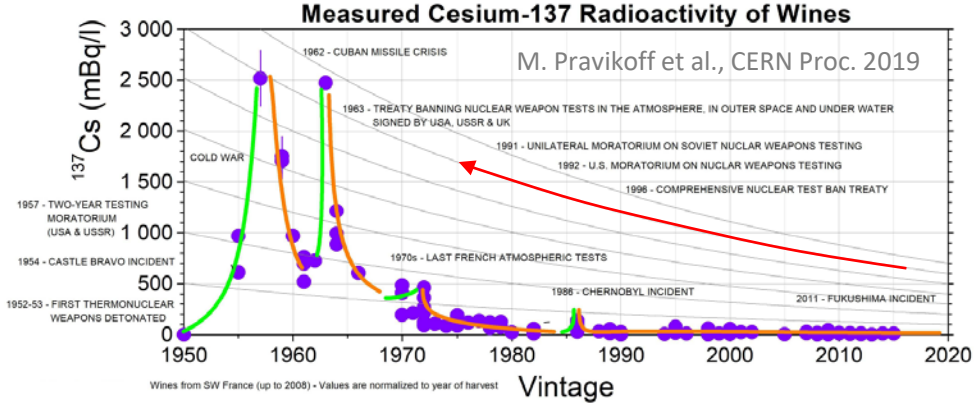



R. Taylor, Am. Sci. (2000)



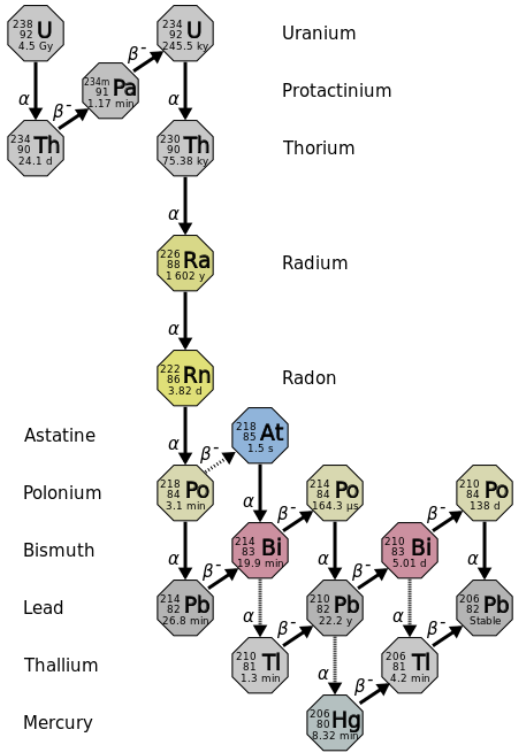

- …can be taken advantage of for dating:



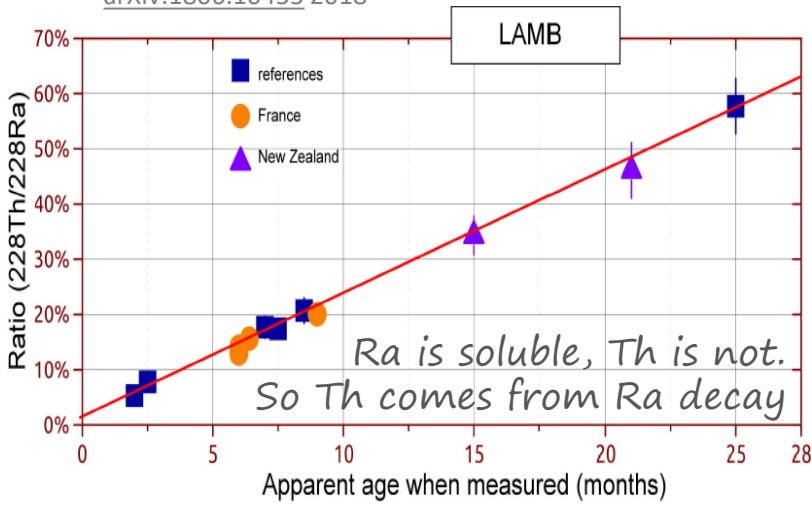

Neat time-lapse of weapons tests by location: https://youtu.be/LLCF7vPanrY

# Radionuclide Dating

- If an object contains or contained a radioactive nucleus, we can use this to tell how old it is
- From the equation for decay of a single species $N(t) = N(0)e^{-\lambda/t}$, we can solve for the time $t = \frac{\ln\left({}^{N(t)}/_{N(0)}\right)}{\lambda}$
- However, while determining $N(t)$ just requires counting ($A(t) = \lambda N(t)$), often we have no way to know $N(0)$
- The solution is to keep in mind that for the decay sequence, e.g. $1 \rightarrow 2$, the total number of nuclei in the decay sequence is constant: $N_1(t) + N_2(t) = N_{tot} = constant$
- As such, assuming $N_2(0) = 0$, then $t = \frac{1}{\lambda}\ln\left(1 + \frac{N_2(t)}{N_1(t)}\right)$, so we only need the ratio of daughter to parent nuclides in the sample
- The "parent" nuclide need not be the direct decay parent of the "daughter"; it just has to be earlier in the decay sequence
  - e.g. $^{206}Pb/^{238}U$ works well for old rocks
  - $^{228}Ra \rightarrow ^{228}Ac \rightarrow ^{228}Th$ can be used for meat

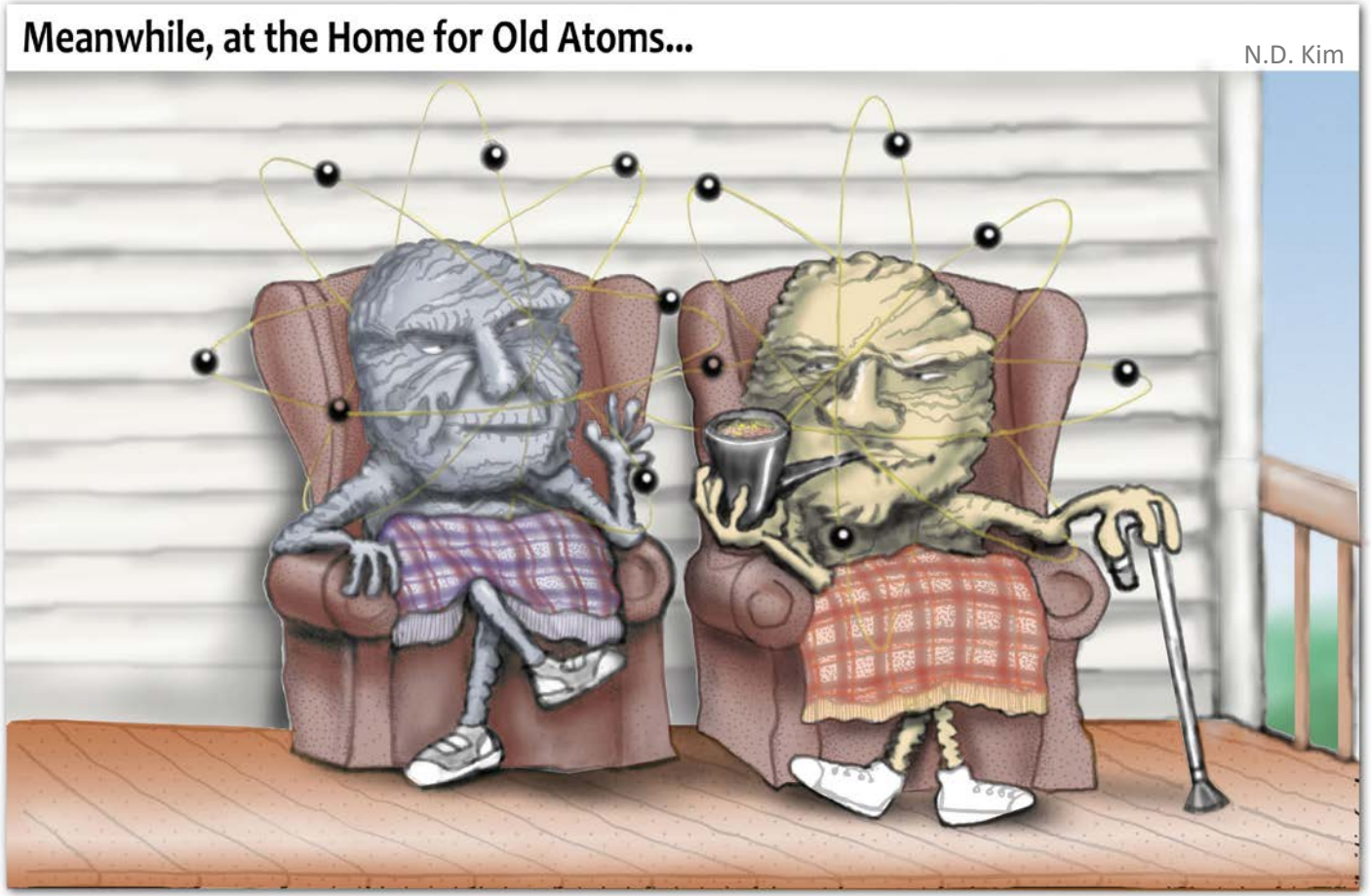



Pravikoff et al.
arXiv:1806.10455 2018



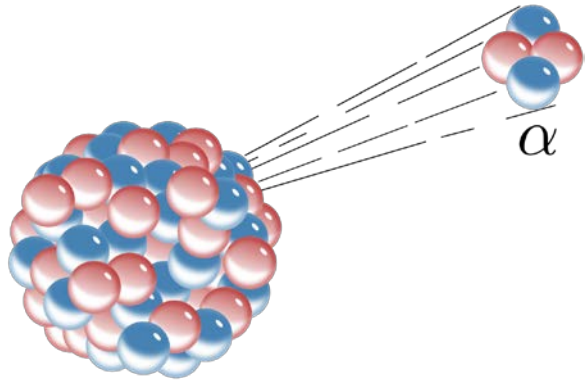
# Lecture 7: *α Decay*

- Energetics
- Geiger-Nuttall
- Tunneling through a barrier
- Decay hindrance
- Why α emission?



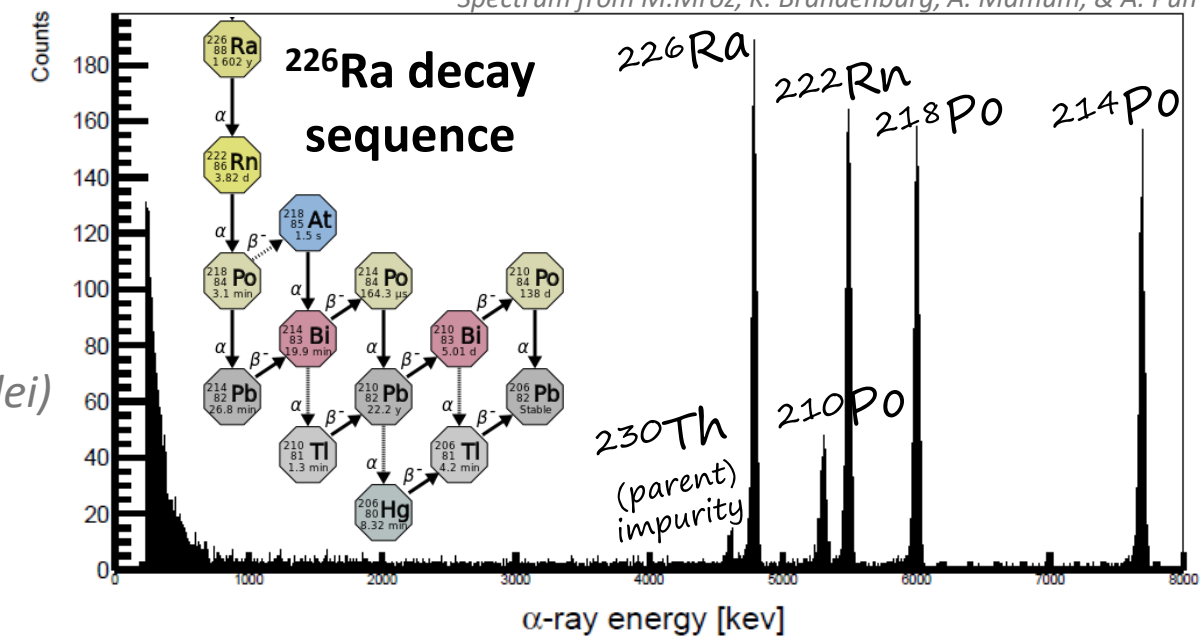

*"When I was young I was an atom of uranium-238. I felt so alive, so dangerous! Then one day I accidentally ejected an alpha particle. Now look at me – a spent old atom of lead-206. All my life since then has been nothing but decay, decay, decay..."*

# α energy from $Q_\alpha$




- When an α is emitted, it will share some energy with the heavy recoil, so $KE_\alpha$ isn't quite equal to $Q_\alpha$
- We just need to employ conservation of momentum and energy
  - $\vec{p}_{parent} = \vec{p}_{daughter} + \vec{p}_\alpha$
  - Conveniently $p_{parent} = 0$, so the daughter and α will move in opposite directions and $p_{daughter} = -p_\alpha$
  - $\frac{p_{parent}^2}{2A_{parent}} + Q_\alpha = \frac{p_{daughter}^2}{2A_{daughter}} + \frac{p_\alpha^2}{2A_\alpha} = \frac{p_\alpha^2}{2A_{daughter}} + \frac{p_\alpha^2}{2A_\alpha} = \frac{A_\alpha}{A_{daughter}} KE_\alpha + KE_\alpha$
  - $Q_\alpha = \frac{A_\alpha + A_{daughter}}{A_{daughter}} KE_\alpha = \frac{A_{parent}}{A_{daughter}} KE_\alpha$
  - $KE_\alpha = \frac{A_{daughter}}{A_{parent}} Q_\alpha$
  - So it's a pretty small effect, but one you should include anyway
    *(Not a small effect for β-delayed particle emission in lighter nuclei)*
- Conveniently, α sources typically have several $E_\alpha$ from the decay chain, and so they provide several energy calibration points

*Spectrum from M.Mroz, K. Brandenburg, A. Mamum, & A. Pun*

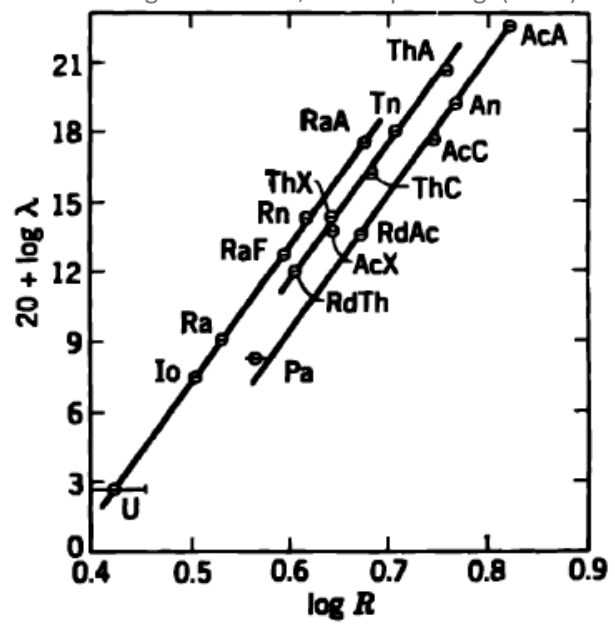


# Geiger-Nuttall relation

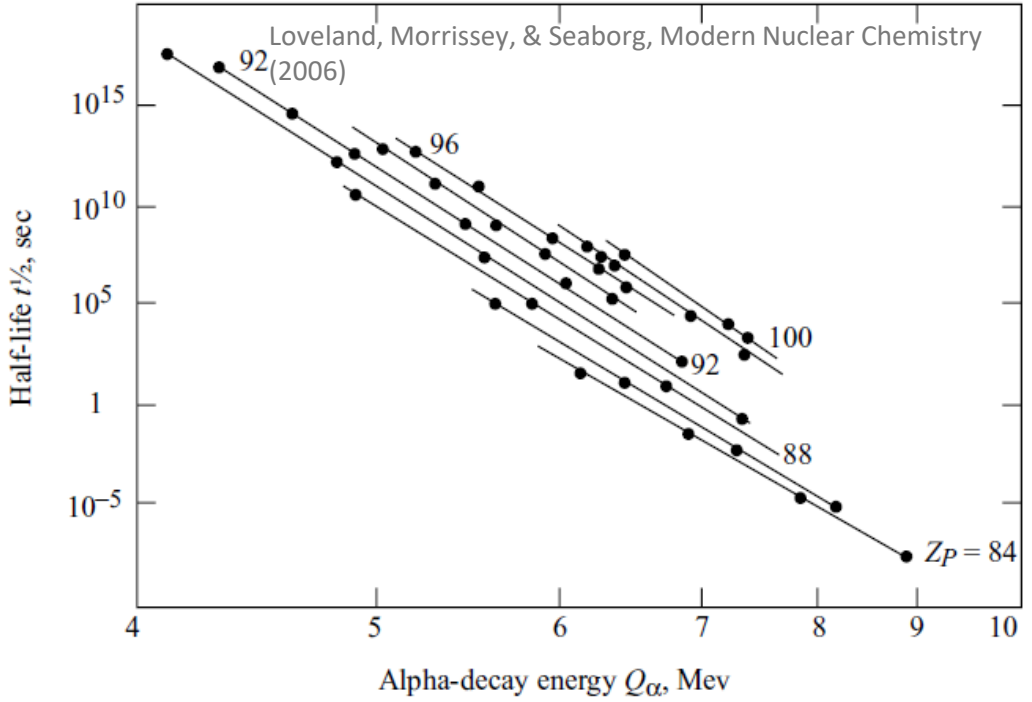
- In an early effort to characterize α-decay, Geiger & Nuttall
  (H. Geiger & J.M. Nuttall, Philisoph. Mag. (1911, 1912))
  compared the range of α particles in a material vs $t_{½}$ of the α-source and found a linear relationship in log-log space
- In modern terms, using $Q_\alpha$ instead of range, we get the Geiger-Nuttall relation: $\log_{10}(t_{½}) = a + bZQ_\alpha^{-½}$
  (Range is the integral of stopping power ($\frac{dE}{dx} \equiv S(E) \propto \frac{1}{E}$): $\mathrm{R} = \int \frac{1}{S(E)} \propto E^2$)
- Obviously the $\alpha$ energy somehow impacts $t_{½}$

  …incredibly strongly
- For $\sim \times 2$ increase in $Q_\alpha$, nearly 20 orders of magnitude decrease in $t_{½}$!!

Geiger & Nuttall, Philisoph. Mag. (1912)



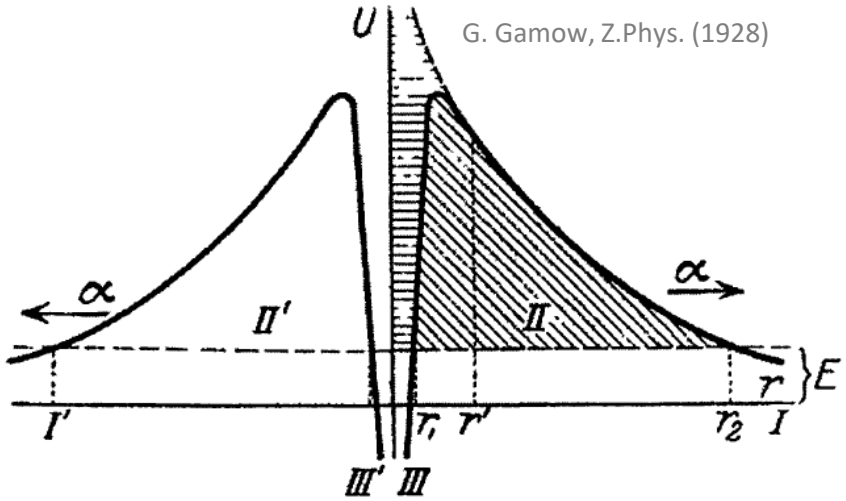

Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)



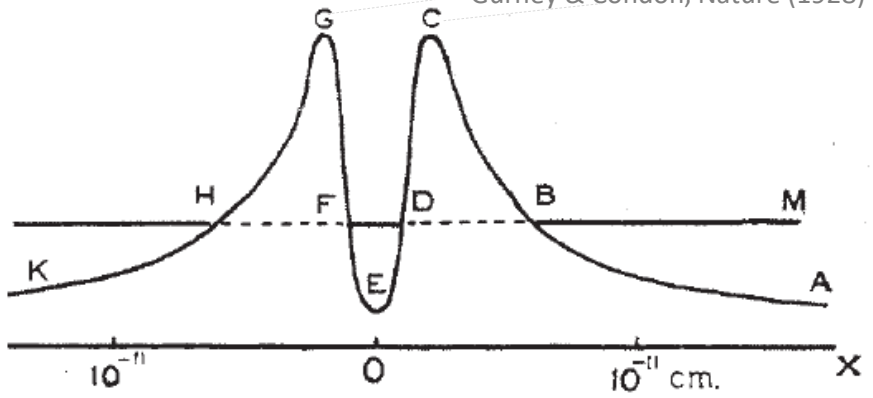

*What does this imply about useful α sources?*
*There's a relatively limited range of $E_\alpha$ available.*
- *Large $E_\alpha$ sources aren't active for long enough,*
- *while low $E_\alpha$ sources require huge amounts to have an appreciable activity (A=λN).*

# α decay from a quantum mechanical perspective

- Compared to the Coulomb barrier, $E_\alpha$ is pretty puny:
  - E.g. $^{226}$Ra: $V_c = \frac{Z_\alpha Z_{Ra226}}{R}\frac{e^2}{\hbar c}\hbar c = \frac{2 \cdot 88}{1.2\left(4^{1/3}+226^{1/3}\right)}\frac{197 MeV fm}{137}$
  $\approx 27.5 MeV$
  - Compare this to $E_\alpha \approx 5 MeV$
- So, classically an α emission couldn't happen
- Gamow (and simultaneously Gurney & Condon) realized this problem could be neatly described by quantum mechanical tunneling
- The basic picture is that an α particle is rattling around in the nucleus, doing laps with a velocity $v_\alpha = \sqrt{2E_\alpha/m_\alpha}$
- Each time the α hits the Coulomb barrier, formed by the nuclear core [which is the daughter of the α decay], it has some probability of tunneling through
- Therefore, the challenge is to calculate this transmission probability



G. Gamow, Z.Phys. (1928)



Gurney & Condon, Nature (1928)

*Interestingly, Gamow figured this out on a summer research trip he took out of frustration with his thesis research. This is the same trip where he conceived the liquid drop model and the theory of nuclear fusion.*
(*R. Stuewer, Plenary paper for 1997 Gamow Symposium*)

# Tunneling through a thick barrier

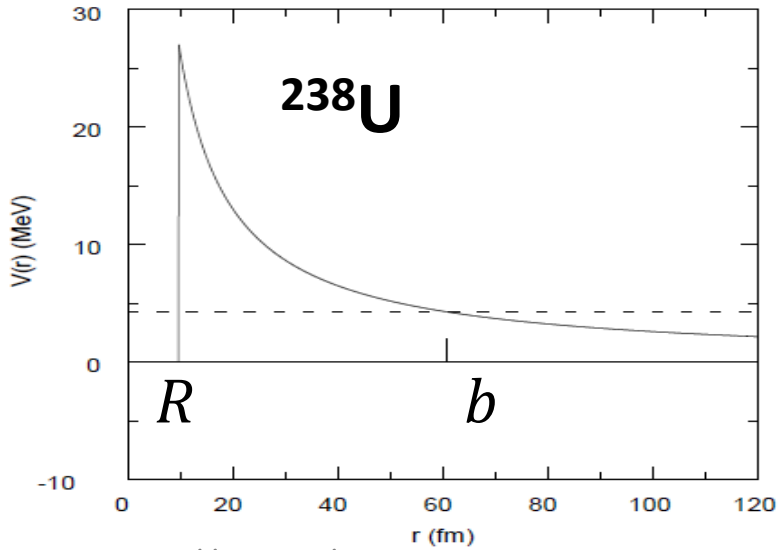
- $T = e^{-2G}$, where $2G = 2\frac{e^2}{\hbar c} Z_\alpha Z_{daughter} \sqrt{\frac{2\mu c^2}{Q_\alpha}} \left[ cos^{-1}\left(\sqrt{\frac{R}{b}}\right) - \sqrt{\frac{R}{b}\left(1-\frac{R}{b}\right)} \right]$ is pretty ugly
- Conveniently, for most cases $b \gg R$, so the stuff in [ ] $\rightarrow \frac{\pi}{2}\left(1 - \frac{4}{\pi}\sqrt{\frac{R}{b}}\right)$

  so the Gamow factor $2G \approx 2\pi \frac{e^2}{\hbar c} Z_\alpha Z_{daughter} \sqrt{\frac{\mu c^2}{2Q_\alpha}} \left( 1 - \frac{4}{\pi} \sqrt{\frac{r_0\left(A_\alpha^{1/3} + A_{daughter}^{1/3}\right)}{\frac{Z_\alpha Z_{daughter} e^2}{Q_\alpha}}} \right)$

this correction looks small and is often neglected in text books, but a factor of ~0.5 in an exponent can be a big deal

- Since the decay half-life will be inversely proportional to the tunneling probability, $t_{½} \propto e^{2G} \propto e^{Z/\sqrt{Q_\alpha}}$
- You may notice this is what Geiger & Nuttall told us all along $\log_{10}(t_{½}) = a + bZQ_\alpha^{-½}$

B.A. Brown, Lecture Notes in Nuclear Structure Physics (2005)




*Aside:*
Gamow realized this formalism would work just as well for a charged-particle tunneling in (i.e. for nuclear fusion). For nuclear fusion, 2G is often written instead as $2\pi\eta$, where $\eta$ is the Sommerfeld parameter.
In nuclear astrophysics, $T=P=\exp(-2\pi\eta)$ is convolved with the Maxwell-Boltzmann distribution to get the "Gamow window", the energy-window of astrophysical relevance.

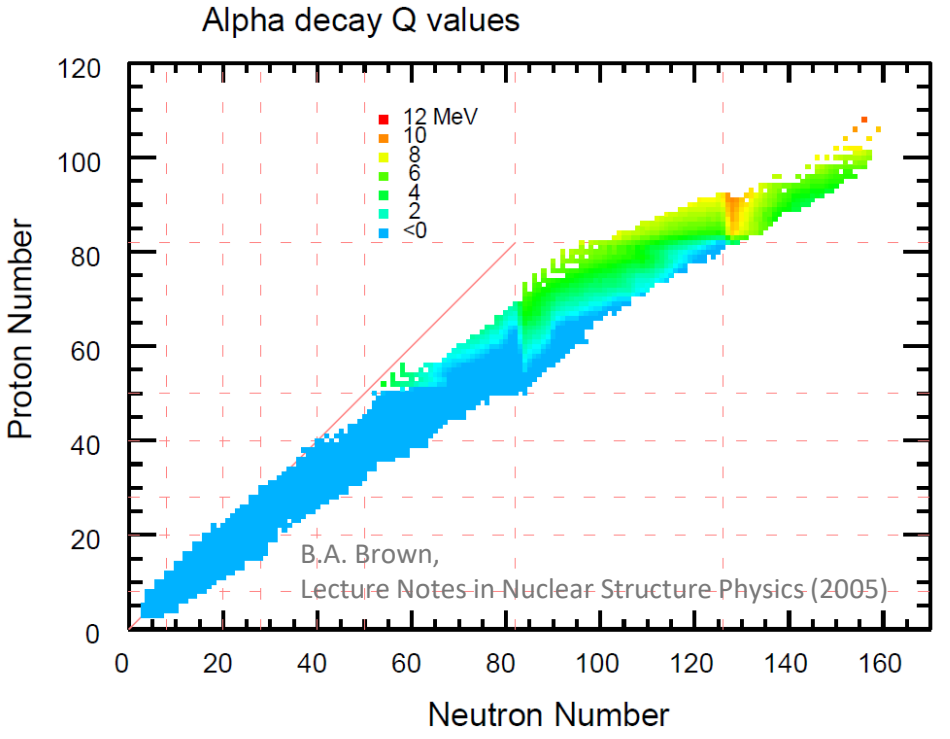
# Qualitative implications

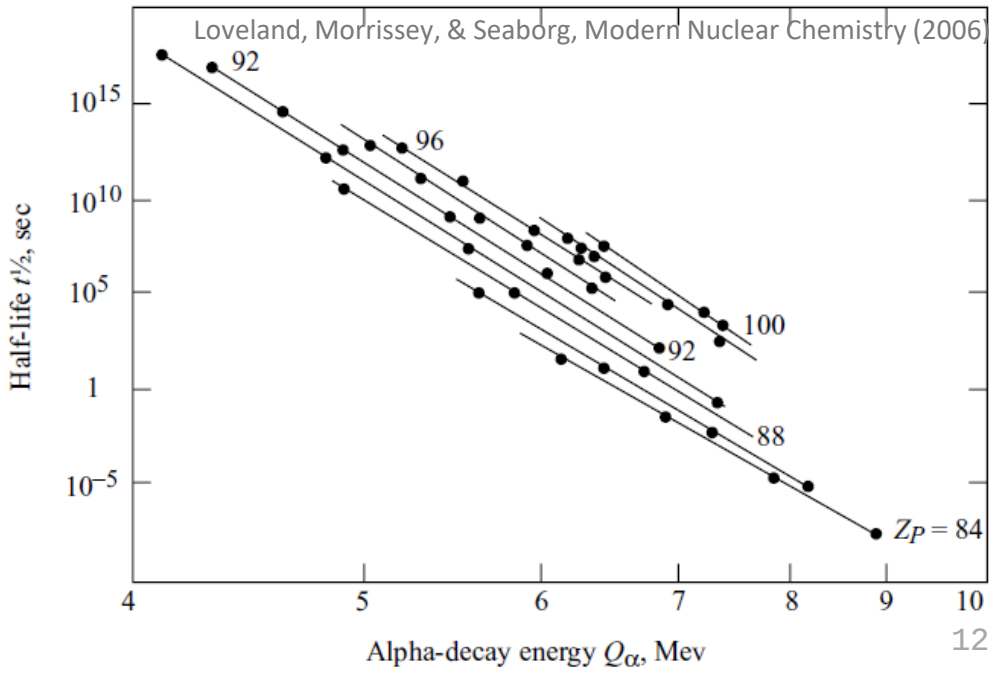
- $Q_\alpha$ generally increases with increasing $Z$ because there is increased penalty in the liquid drop model binding for increased $E_{Coul}$
- Increasing $Z$ means an increased Coulomb barrier height and therefore more of a barrier to tunnel through

  *…for the same $Q_\alpha$, increasing $Z$ increases $t_{1/2}$*
- Increasing $Q_\alpha$ means the $\alpha$ energy required to tunnel through the barrier is larger, therefore the $\alpha$ velocity is larger, therefore the $\alpha$ bombards the barrier more frequently

  *…for the same $Z$, increasing $Q_\alpha$ decreases $t_{1/2}$*

  *i.e. decays to excited states should have longer half-lives, provided parity needn't be violated*

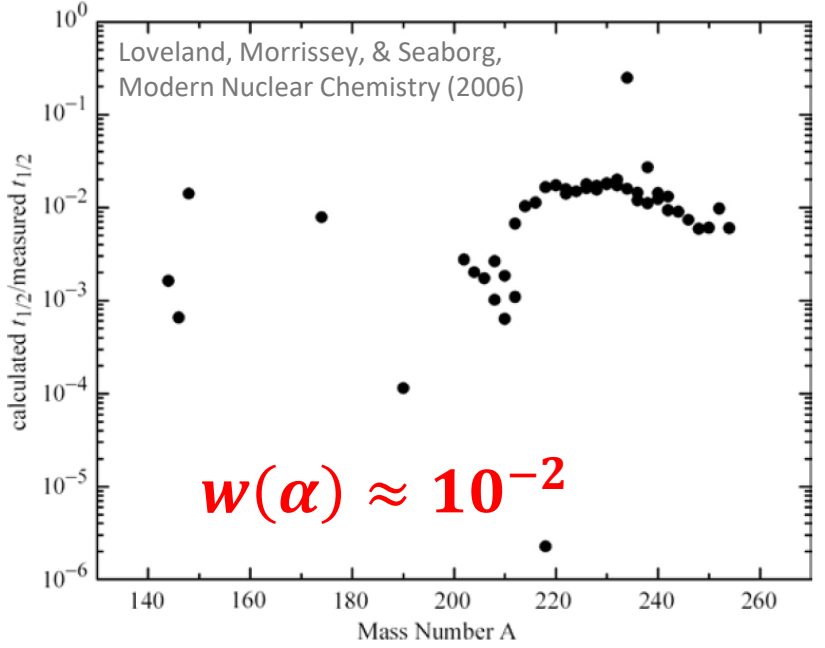


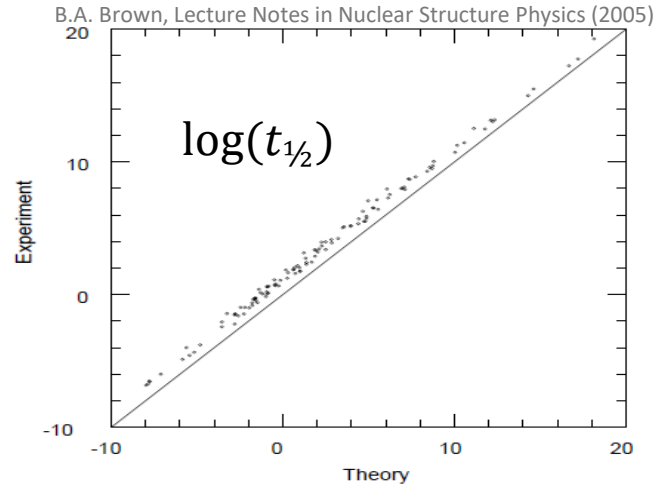



# Estimate for α-decay λ

- The decay constant is a product of the frequency an α will bombard the barrier $f$, the tunneling probability $T$, and the probability of forming an α within the nucleus $w(\alpha)$
  - $\lambda = fTw(\alpha)$
- We discussed *T* already: $T = \exp\left(-2\pi\frac{e^2}{\hbar c}Z_\alpha Z_{daughter}\sqrt{\frac{\mu c^2}{2Q_\alpha}}\left(1-\frac{4}{\pi}\sqrt{\frac{r_0\left(A_\alpha^{1/3}+A_{daughter}^{1/3}\right)}{\frac{Z_\alpha Z_{daughter}e^2}{Q_\alpha}}}\right)\right)$
- $f$ is just one over the time it takes for the α to travel across the nucleus $f = \frac{v_\alpha}{2R} = \sqrt{\frac{2(V_0+Q_\alpha)}{\mu}}\frac{1}{2r_0\left(A_\alpha^{1/3}+A_{daughter}^{1/3}\right)}$
- Setting $w(\alpha) = 1$ for the moment, consider $^{235}U$
  - $V_0 \approx 30MeV$ (from optical model fits), $Q_\alpha \approx 4.68MeV$
  - $f = \frac{\sqrt{2(30MeV+4.68MeV)/(3.93amu*931.5MeV/c^2/amu)}}{2(9.3fm)} \approx 2.3\times10^{21}s^{-1}$
  - $T = \exp\left(-2\pi\frac{1}{137}(2)(90)\sqrt{\frac{(3.93)\left(931.5MeV/c^2\right)c^2}{2(4.68MeV)}}\right)\left(1-\frac{4}{\pi}\sqrt{\frac{1.2fm\left(4^{\frac{1}{3}}+231^{\frac{1}{3}}\right)}{\frac{(2)(90)197MeVfm}{2(4.68MeV)137}}}\right) \approx 1.08\times10^{-34}$
  - $\lambda \approx 2.4\times10^{-13}s^{-1}$ …i.e. $t_{½} \approx 9\times10^4 yr$

…actual: $7\times10^8 yr$




*results can be very sensitive to $r_0$

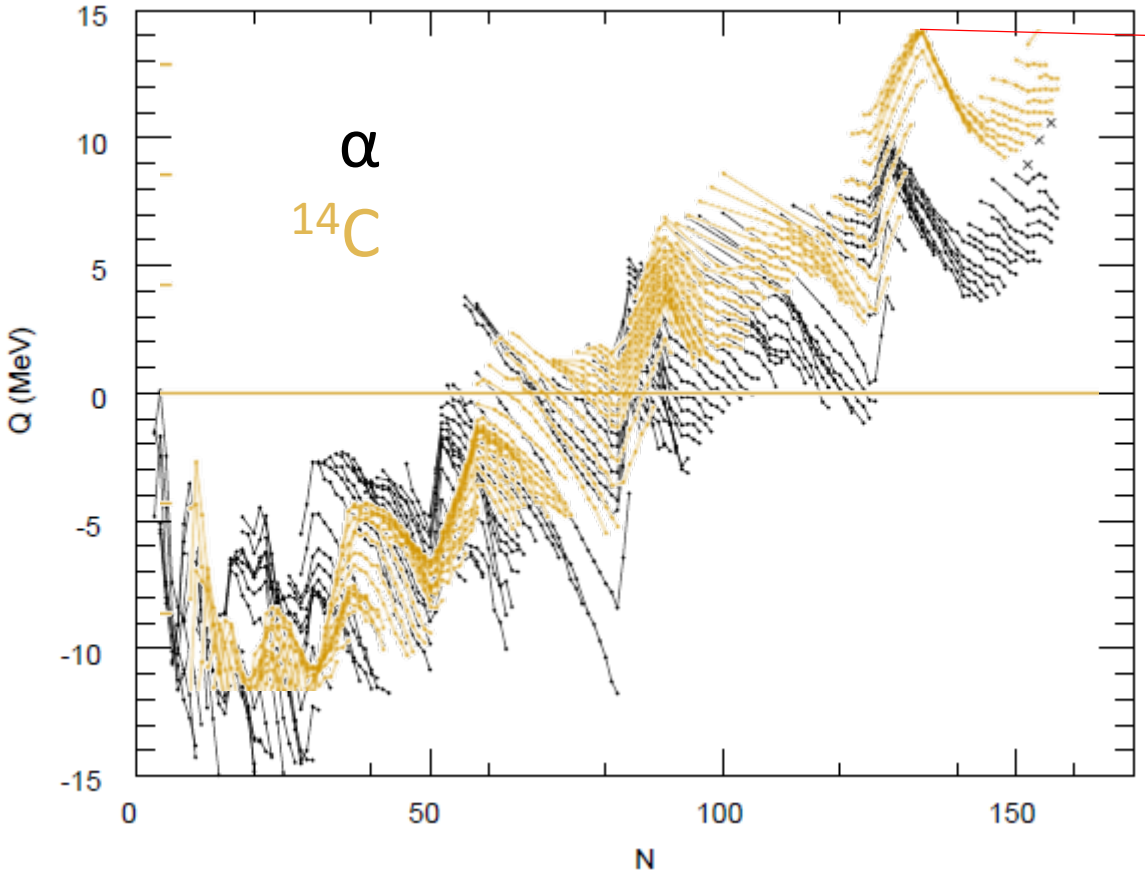
# Why is it α particles that are being emitted?

- So far we've been smugly pleased with ourselves about our ability to describe α decay …but why α decay? Why not proton decay, or $^{3}$He decay, or $^{12}$C decay?
- The short answer is Q-values, Coulomb barriers, and clustering probabilities
  - *Q-value*: The cluster decay must be energetically favorable
  - *Coulomb barrier*: Higher-Z particles will have a larger barrier to tunnel through
  - *Clustering probability:* It's less likely for more nucleons to congregate within a nucleus

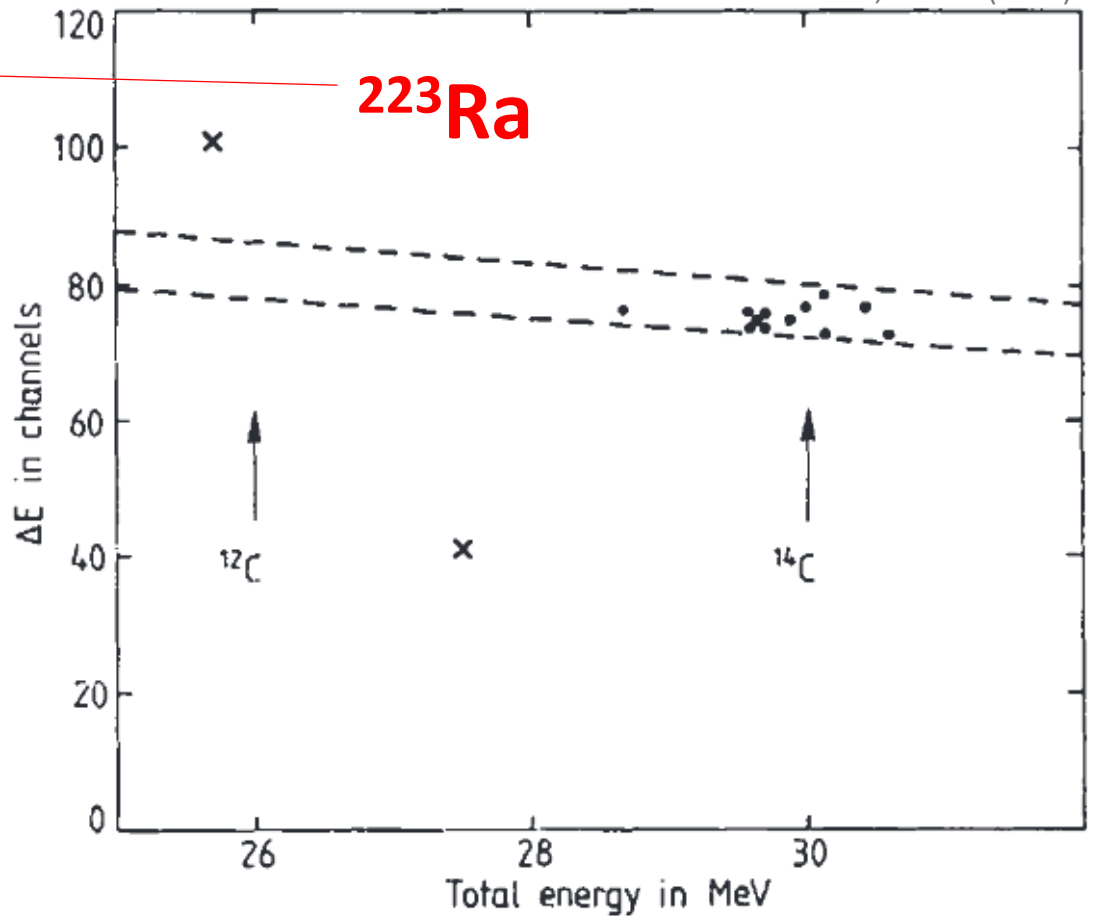
# Why is it α particles that are being emitted? Q-value

- Decay is a spontaneous process that only occurs because there's a lower energy state that is available; i.e. a positive Q-value is required for a decay to occur
- The more positive the better (since this means more energy to tunnel)

B.A. Brown, Lecture Notes in Nuclear Structure Physics (2005)

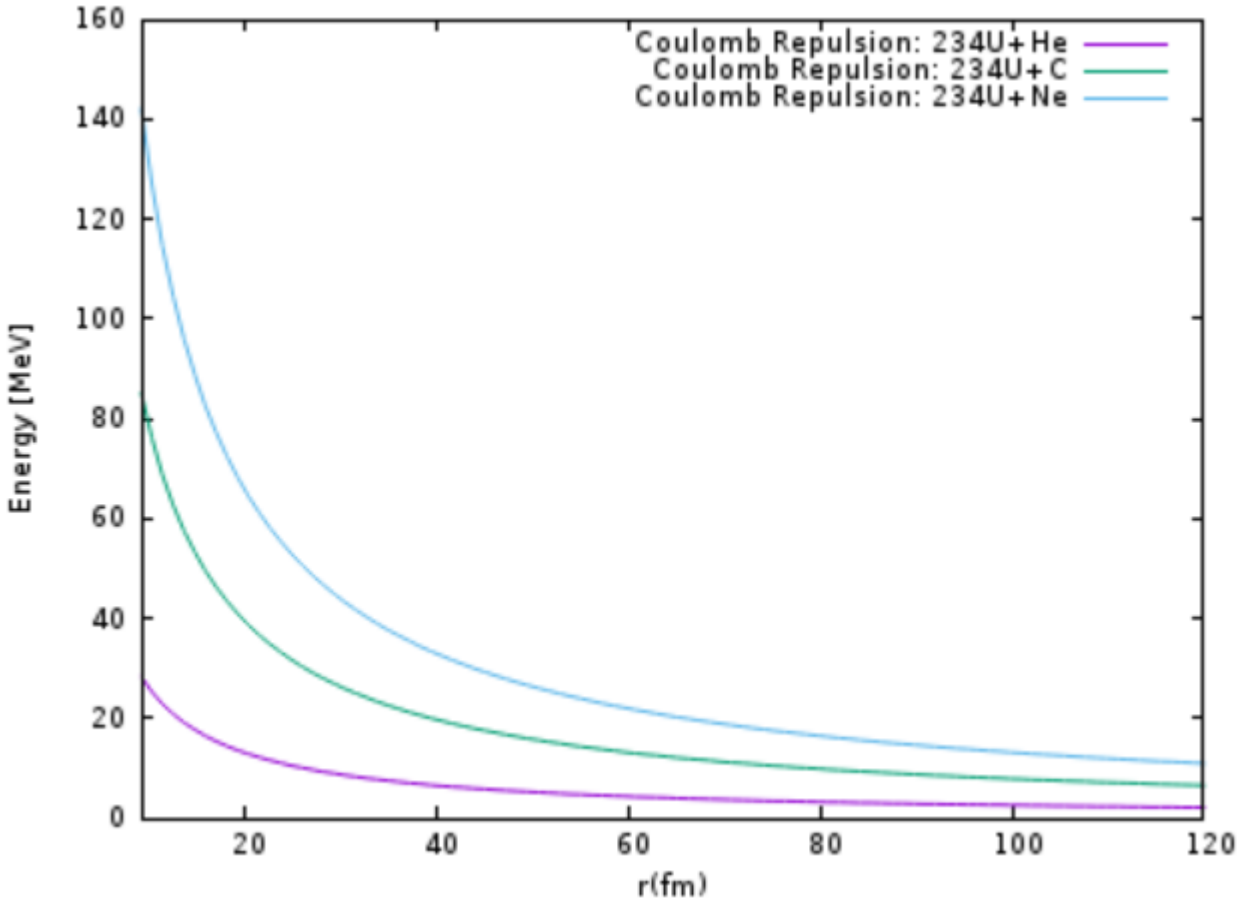



Rose & Jones, Nature (1984)



# Why is it α particles that are being emitted? Coulomb barrier

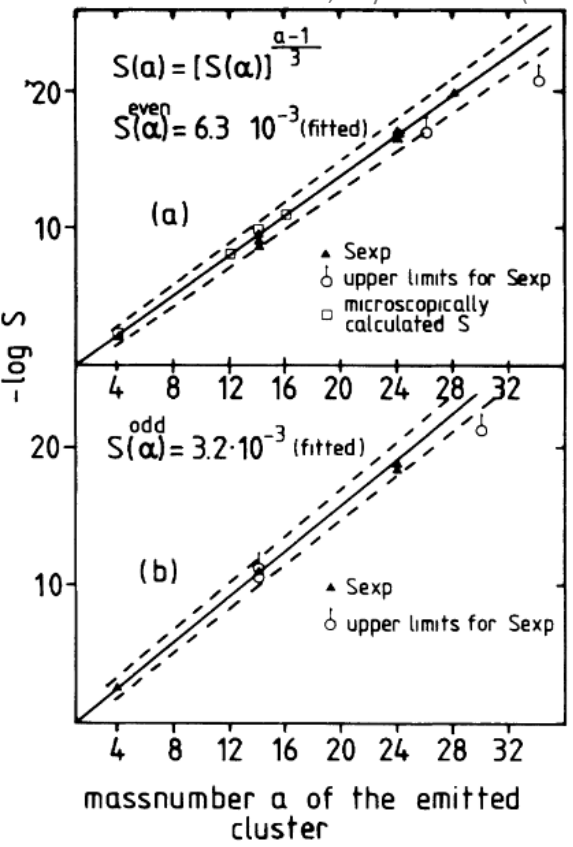
- The Coulomb barrier height scales with the charge of the particle being emitted
- It takes a much larger Q-value to make larger Z decay have any chance at tunneling through



# Why is it α particles that are being emitted? Clustering probability

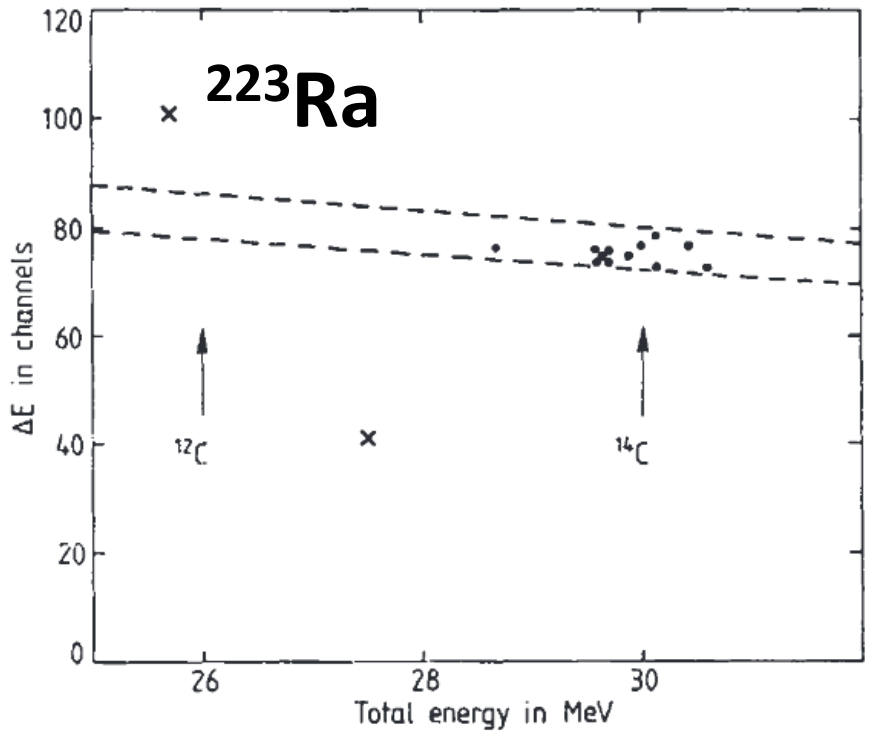
- The likelihood of forming a cluster of nucleons within a nucleus is the preformation factor
- Fancy calculations (which agree with some measurements) show that the cluster preformation probability relative to clustering for an α (for $A_c$<28) goes as (Blendowske & Walliser, Phys. Rev. Lett. (1988)**:**
  - $w(A_c) = w(\alpha)^{(A_{cluster}-1)/3}$, where $w_{even}(\alpha) = 6.3 \times 10^{-3}$ and $w_{odd}(\alpha) = 3.2 \times 10^{-3}$

Blendowske & Walliser, Phys. Rev. Lett. (1988):



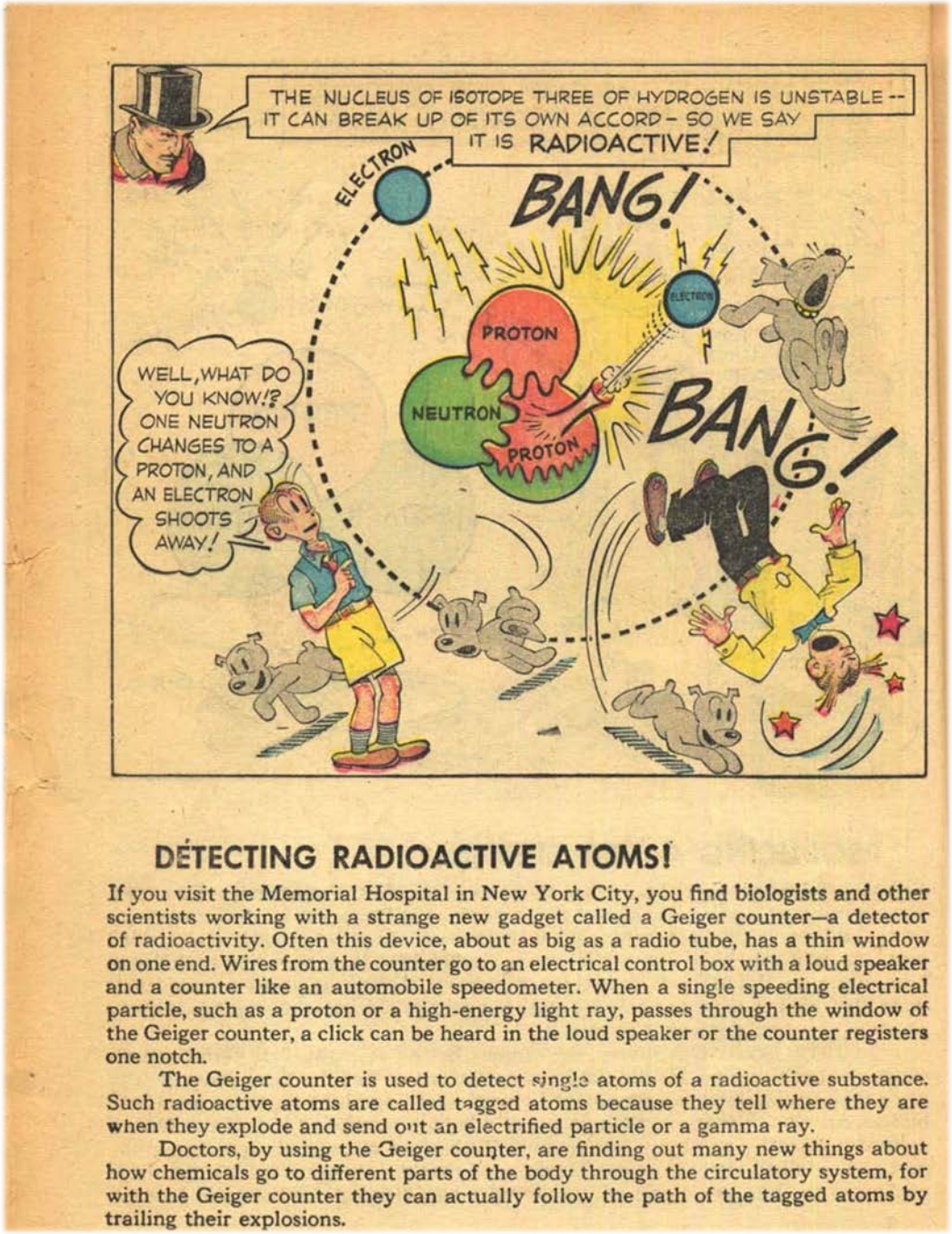

*...though neutron-richness matters, since heavy nuclides will favor more neutron-rich decay products, e.g. the lack of $^{12}C$ emission from $^{223}Ra$*

Rose & Jones, Nature (1984)



# Lecture 8: *β Decay*

- Basic process & energetics
- Fermi theory
- *ft*-values
- Electron-capture
- Parity violation
- Special cases




**DETECTING RADIOACTIVE ATOMS!**

If you visit the Memorial Hospital in New York City, you find biologists and other scientists working with a strange new gadget called a Geiger counter—a detector of radioactivity. Often this device, about as big as a radio tube, has a thin window on one end. Wires from the counter go to an electrical control box with a loud speaker and a counter like an automobile speedometer. When a single speeding electrical particle, such as a proton or a high-energy light ray, passes through the window of the Geiger counter, a click can be heard in the loud speaker or the counter registers one notch.

The Geiger counter is used to detect single atoms of a radioactive substance. Such radioactive atoms are called tagged atoms because they tell where they are when they explode and send out an electrified particle or a gamma ray.

Doctors, by using the Geiger counter, are finding out many new things about how chemicals go to different parts of the body through the circulatory system, for with the Geiger counter they can actually follow the path of the tagged atoms by trailing their explosions.

# What is β decay?

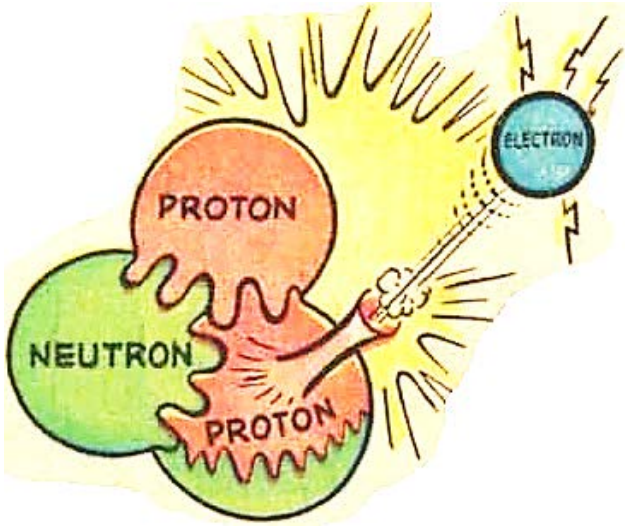
a) Nucleus perspective:

$\beta^-$: ${}^A_Z X \rightarrow {}^A_{Z+1} X' + e^- + \bar{\nu}_e$

$\beta^+$: ${}^A_Z X \rightarrow {}^A_{Z-1} X' + e^+ + \nu_e$

$EC$: $e^- + {}^A_Z X \rightarrow {}^A_{Z-1} X' + \nu_e$

$e^-$ can be from atomic shells (terrestrial cases) or from surrounding electron gas (e.g. neutron star outer crust)



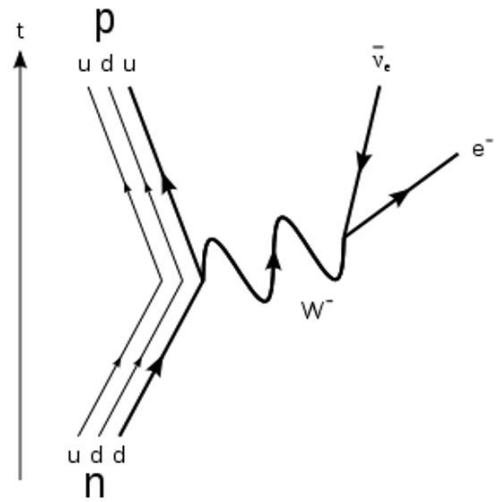

b) Nucleon perspective:

$\beta^-: n \rightarrow p + e^- + \bar{\nu}_e$

$\beta^+: p \rightarrow n + e^+ + \nu_e$

$EC: e^- + p \rightarrow n + \nu_e$

Free neutron decay takes ~15min (though there is some controversy), while $t_{1/2}(p) > 10^{34}$yr.




I'm going to let you finish, but the process really has to do with quarks and bosons...



c) Quark perspective:

$\beta^-: d \rightarrow u + e^- + \bar{\nu}_e$

$\beta^+: u \rightarrow d + e^+ + \nu_e$

$EC: e^- + u \rightarrow d + \nu_e$

# β decay energetics

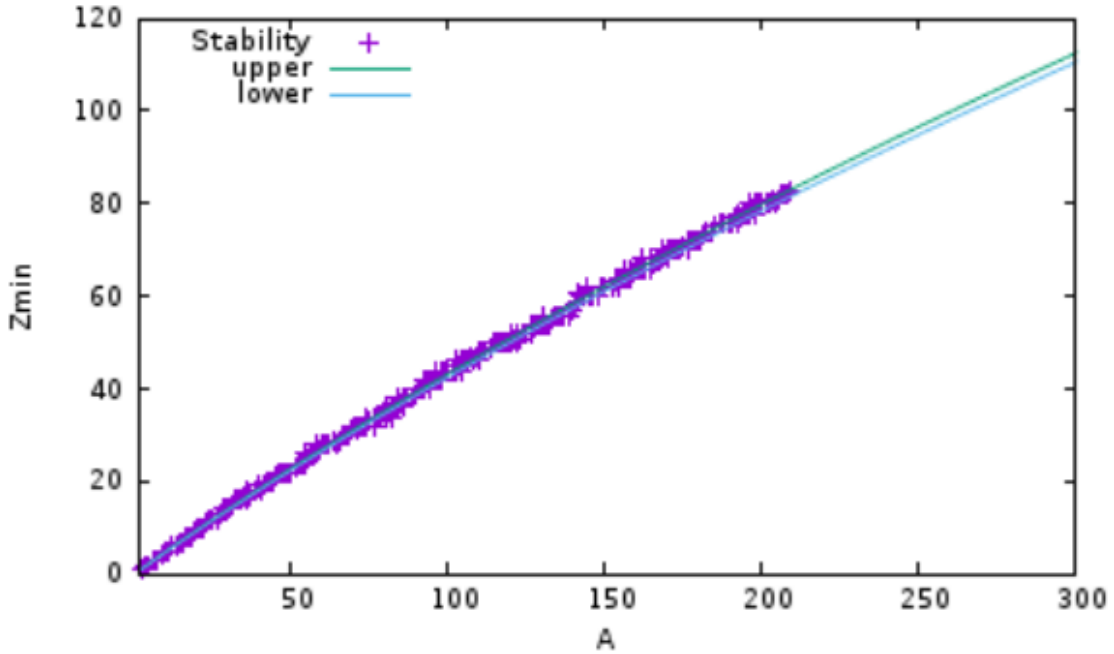
- β-decay can proceed if energetics allow it ($Q_\beta > 0$):
  - $Q_{\beta^-} = ME(Z,A) - ME(Z+1,A)$
  - $Q_{\beta^+} = ME(Z,A) - ME(Z-1,A) - 2m_e$
  - $Q_{EC} = ME(Z,A) - ME(Z-1,A)$

*Atomic mass excesses have the electron masses included, so it's only for positron emission that we have to take into account that we will not be gaining an electron with the atom, but instead effectively losing one.*

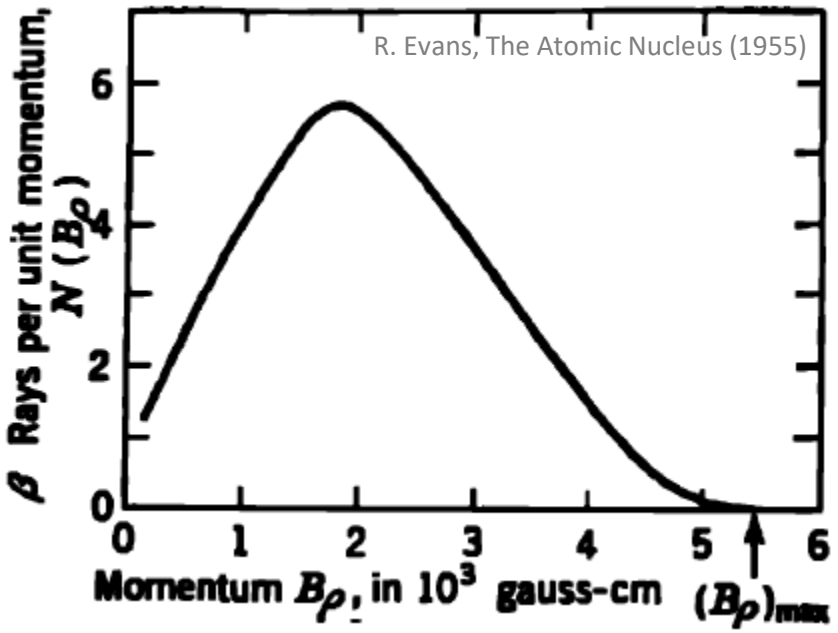
- For several cases near stability, more than one type is possible, e.g.




- An estimate for β instability for the nuclear landscape can be determined by finding the minimum $Z$ for a given $A$ using the semi-empirical mass formula
  *(See homework #3)*
  - Unsurprisingly, the agreement with the valley of β stability is excellent:



# β decay spectrum, spin conservation, and the neutrino




- Early experiments investigating the "β ray" showed that it was not emitted with a singular energy, like the "α ray", but rather in a continuum of energies
  - Though the maximum energy is equal to the decay Q-value
- Furthermore, the reaction $n \rightarrow p + e^-$ doesn't conserve spin!
  - $J_n = J_p = J_e = \frac{1}{2}$ … so $0 \leq J_p + J_e \leq 1 \quad \neq \frac{1}{2}$
- To remedy this issue, Pauli proposed the involvement of a 3rd hypothetical particle, the neutrino ν *Like a proper old-timey physicist, he made this proposal not in a paper, but in a letter to physicist Lisa Meitner*
- Given the above considerations, it was postulated that ν is a spin-½ particle ("fermion") that it is massless* and electrically neutral
  *(of course this isn't quite true, but true enough for our purposes)*
- In one of his last works before switching to primarily performing experimental work, Fermi postulated (E. Fermi, Z. Phys. 1934) that nucleons could act as sources & sinks of electrons and neutrinos, in analogy to charged particles acting as sources and sinks of photons in quantum electrodynamics (the only successful theory of interactions between quantum particles at that point)

*For what it's worth, Nature rejected Fermi's paper for being "too remote from physical reality"*

# Fermi theory of β decay

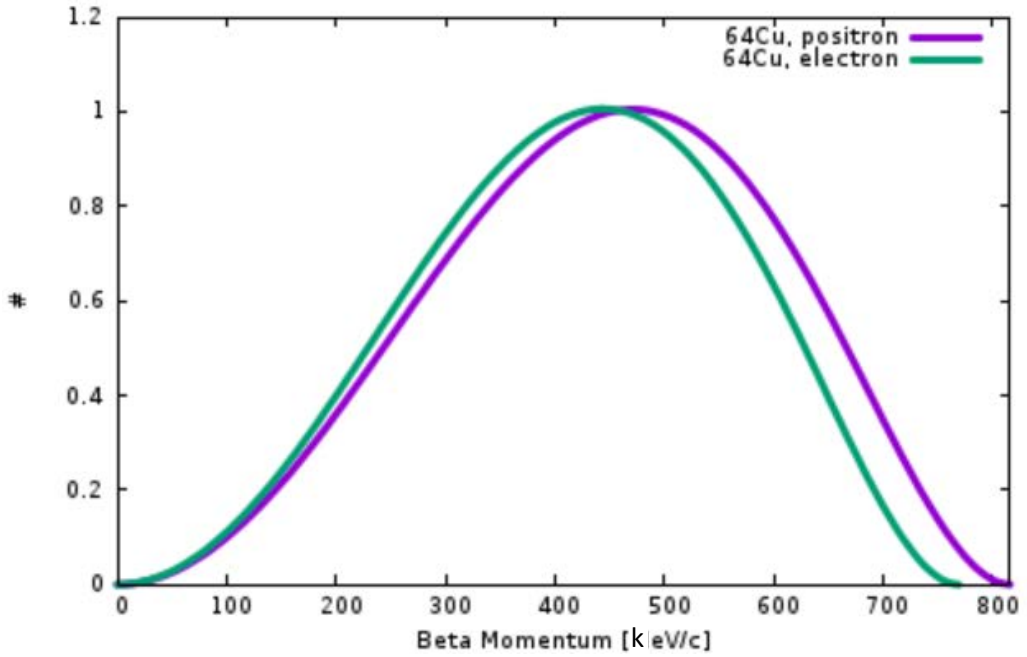
- Fermi posited that small perturbative interaction precipitated the β decay
- The initial state is described by the wave function of the parent nucleus in state $j$, $\Psi_i = \psi_{p,j}$, whereas the final state is the product of the daughter nucleus in state $k$, $\psi_{d,k}$, the electron $\varphi_e$, and the neutrino* $\varphi_\nu$, $\Psi_f = \psi_{d,k}\varphi_e\varphi_\nu$

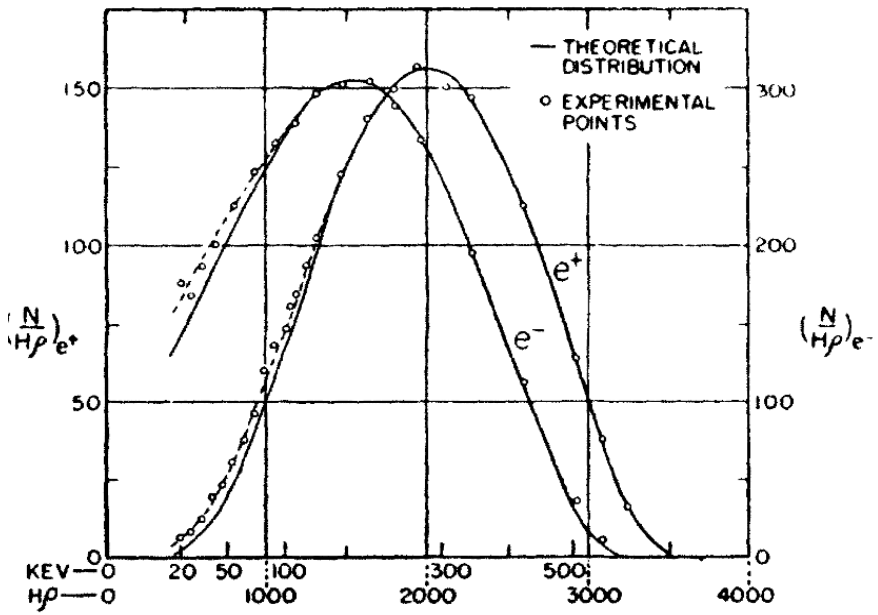
*(of course it's an antineutrino, but that's cumbersome to say and it doesn't matter here)*

- The transition rate for such a case is derived by solving the Schrödinger equation to 1st order in time-dependent perturbation theory, where the result is known as the Second Golden Rule:
  (or often as Fermi's Golden Rule …though he didn't originally derive it)
  $\lambda = \frac{2\pi}{\hbar}\left|\langle\Psi_{final}|H'|\Psi_{initial}\rangle\right|^2\rho(E) = \frac{2\pi}{\hbar}\left|\int\Psi^*_{final}H'\Psi_{initial}d\tau\right|^2\rho(E_f)$
- The qualitative picture is that the transition rate is constant and depends only on two parts:
  - the density of final states to which the decay can proceed
    (i.e. # of ways $e^-$ and ν can share the decay energy $E_f$)
  - the "matrix element" describing the interaction
    (i.e. description of initial state $i$ to one of many possible final states $f$)
- $\rho(E_f)$ is a phase-space factor described by kinematics
- The matrix element describes the wave-function overlap between initial and final states, so a theoretical description requires calculating these wave functions

# β decay phase space factor & the β energy spectrum

- Considering the β decay rate for an electron momentum within $p_e + dp_e$,
  $\lambda(p_e)dp_e = \frac{2\pi}{\hbar}\left|\langle\Psi_{final}|H'|\Psi_{initial}\rangle\right|^2\rho(E)$
- The matrix element is just some number, so the functional form is from $\rho(E)$
- Therefore, we expect $\lambda(p_e)dp_e \propto (Q - KE_e)^2 p_e^2$




Wu & Albert, Phys. Rev (1949)

FIG. 2. Momentum spectra of $Cu^{64}$ negatrons and positrons.

*Not too bad, but what effect are we forgetting that will cause positrons and electrons to behave differently?*

*Coulomb repulsion!*

# Coulomb distortion to the β spectrum

- Fermi realized that the protons in the daughter nucleus would repel $e^+$ and attract $e^-$, modifying the resultant spectrum by a factor $F\left(Z_{daughter}, p_e\right)$
  (in an unfortunate convention, this coulomb distortion factor is often called the Fermi function …or a better term is the Fermi screening factor)
- $F\left(Z_{daughter}, p_e\right)$ is pretty nasty to calculate, so numerical tables are generally used instead
  (See e.g. J.Reitz, Phys. Rev. (1949))
- A non-relativistic approximation that works for nuclides with $Z_{daughter} \ll \sqrt{2} \cdot 137$ is
  (See R. Evans, The Atomic Nucleus (1955) for the relativistic version that works well for all Z)

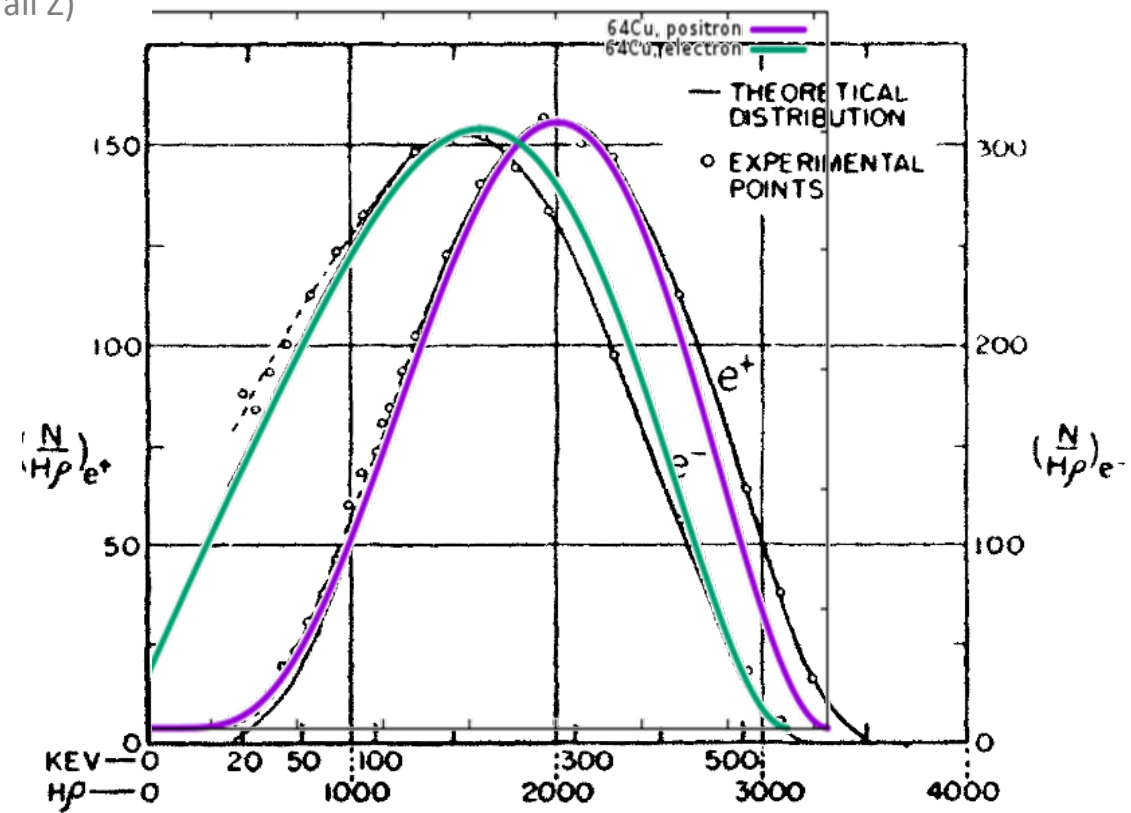
$$F\left(Z_{daughter}, p_e\right) \approx \frac{2\pi y}{1-e^{-2\pi y}},$$

where $y \equiv \frac{\pm Z_{daughter}\alpha}{p_e c} E_e$, $\alpha \approx 1/137$,

$E_e = m_e c^2 + KE_e$,

and $+$ is for $\beta^-$ decay and $-$ is for $\beta^+$ decay



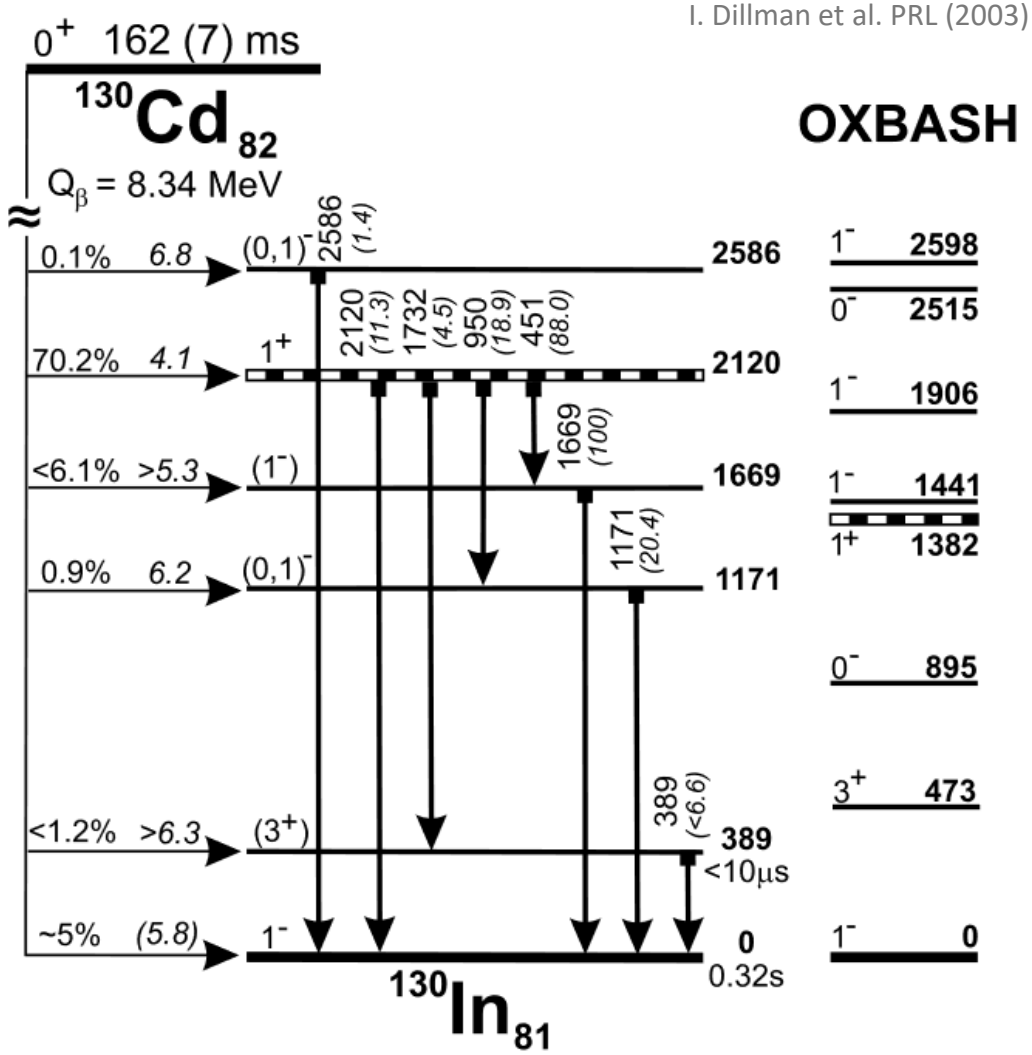
# Using $ft$ for spectroscopy

- Since we established $ft$ is linked to a particular $\Delta J, \Delta\pi$, a measurement of $ft$ can constrain $J^\pi$ of an unknown state if the β decay proceeds from a state with known $J^\pi$
- Nuclear masses combined with γ ray energy measurements provide the transition Q-value, i.e. $f$
- The % of the time the decay proceeds through a given state (the branching ratio) gives the partial half-life for that decay ($t_{½}{}' = \frac{t_{½}}{Branching}$), i.e. $t$
- Given knowledge of the parent $J^\pi$ (always 0+ for even-even nuclei), one can infer $J^\pi$ for the daughter state

Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

| Transition Type | log $ft$ | $L_\beta$ | $\Delta\pi$ | Fermi $\Delta I$ | Gamow–Teller $\Delta I$ |
|---|---|---|---|---|---|
| Superallowed | 2.9–3.7 | 0 | No | 0 | 0 |
| Allowed | 4.4–6.0 | 0 | No | 0 | 0, 1 |
| First forbidden | 6–10 | 1 | Yes | 0, 1 | 0, 1, 2 |
| Second forbidden | 10–13 | 2 | No | 1, 2 | 1, 2, 3 |
| Third forbidden | >15 | 3 | Yes | 2, 3 | 2, 3, 4 |

I. Dillman et al. PRL (2003)



# λ for Electron capture

$$\lambda = \frac{2\pi}{\hbar}\left|\int \Psi^*_{final} H' \Psi_{initial} d\tau\right|^2 \rho(E_f)$$

- Rather than a nucleon undergoing transmutation by its lonesome, instead $e^-$-capture can occur
- This is either due to a capture of a low-lying (usually the "K-shell") electron or due to the electron Fermi energy in an electron-degenerate environment being high enough to overcome the electron-capture Q-value
- The decay constant for electron-capture decay is a bit different than for β decay, because the final state only consists of a nucleon and a neutrino … i.e. $KE_e = 0$ and $\Psi_f = \psi_{d,k}\varphi_\nu$
- The decay constant is then: $\lambda = \frac{G_F^2}{2\pi^3\hbar^3c^3}\left|M_{fi}\right|^2 T_\nu^2 |\varphi_K(0)|^2$,
  where $\varphi_K(0)$ is the wave-function for the inner-most atomic electron (the one in the "K-shell")
- You may recall from your Quantum class, $\varphi_K(0) = \frac{1}{\sqrt{\pi}}\left(\frac{Zm_e e^2}{4\pi\varepsilon_0\hbar^2}\right)^{3/2}$
- As such, the ratio of electron-capture to $\beta^+$ decay for a nucleus goes as $\frac{\lambda_K}{\lambda_{\beta^+}} \propto Z^3$

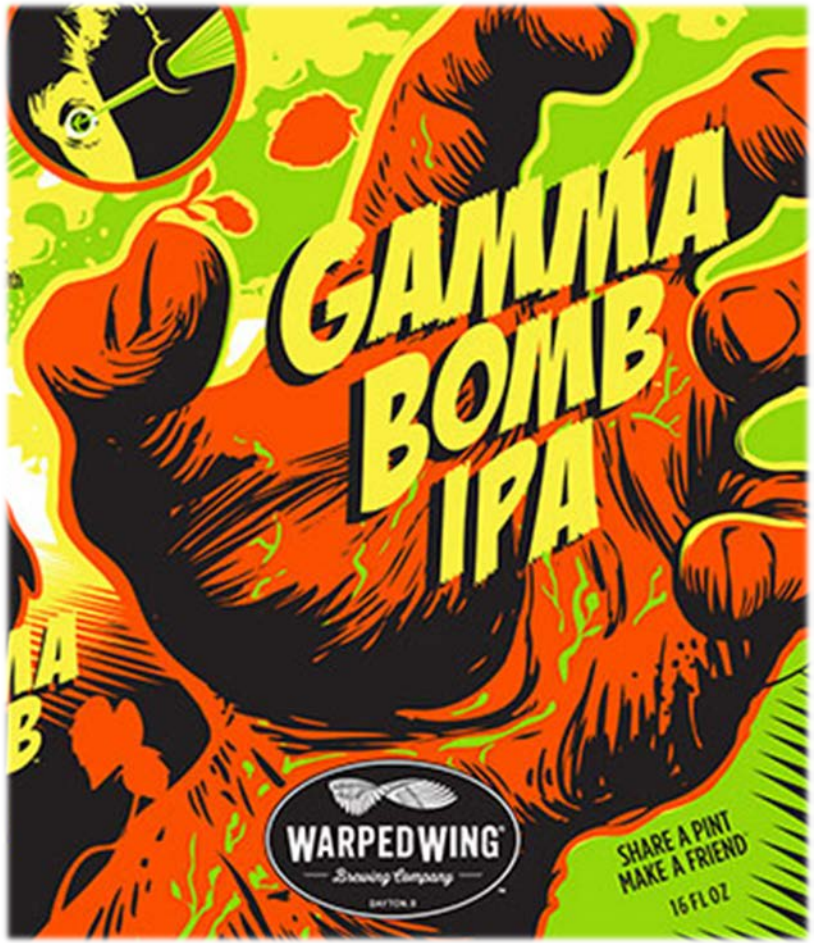
  *(of course, $Q_\beta > 2m_e$ is a requirement for $\beta^+$ decay to be possible in the first place)*

Since EC decay only emits a neutrino, which will be almost impossible for us to detect, how do you figure EC decay is usually detected?

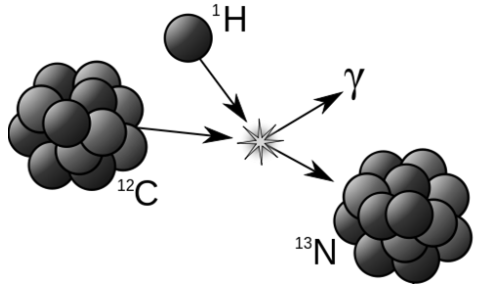
X-ray and Auger electron emission due to atomic electrons filling the vacated orbital

# Lecture 9: $\gamma$ *Decay*

- Basics
- Energetics
- Transition rates
- Angular correlations
- $\gamma$ strength functions



# γ decay energetics




- γ decay can be used to probe excited state energies, but is $E_\gamma = E_{xs}$?
- For a decay from a higher-lying excited state to a lower-energy one, $M_{higher}c^2 = M_{lower}c^2 + E_\gamma + KE_{recoil}$
- $M_{higher}c^2 - M_{lower}c^2 \equiv E_{xs} = E_\gamma + KE_{recoil}$
- $KE_{recoil} = \frac{p_{recoil}^2}{2M_{recoil}}$
- Conservation of momentum dictates $p_\gamma = p_{recoil}$, so $KE_{recoil} = \frac{p_\gamma^2}{2M_{recoil}}$




- Recall that for a massless particle, $E = pc$, so $KE_{recoil} = \frac{E_\gamma^2}{2M_{recoil}c^2}$
- Consider $E_\gamma = 2\text{MeV}, \text{A} = 50$: $KE_{recoil} = \frac{4\text{MeV}^2}{2 \cdot 50 \cdot 931.5\text{MeV}} \approx 43\text{eV}$
- For, a $5\text{MeV}$ $\gamma$ from $^3$He *(e.g. populated by d+p)*, $KE_{recoil} \approx 4.5\text{keV}$



# γ decay types

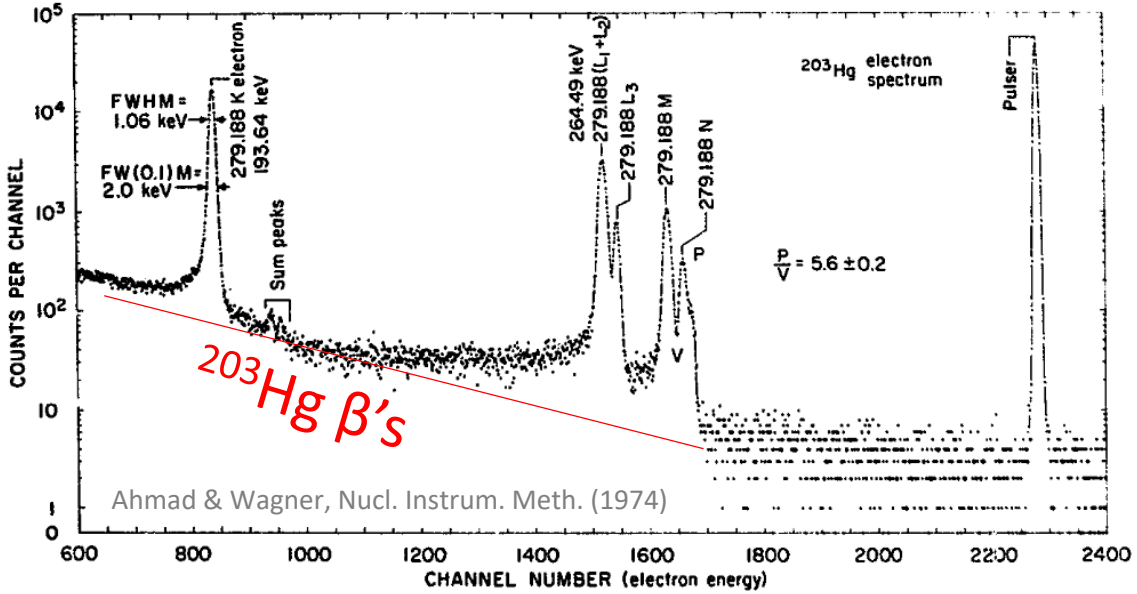
- Parity and angular momentum are conserved during γ decay
- Photons carry some integer angular momentum with a minimum $l = 1$, where $l$ is referred to by the multipole $2^l$
  - $l = 1: dipole, l = 2\ quadrupole, \cdots$
- A photon's parity depends not only on $l$, but also on the decay type
- A photon decay corresponds to shift in the nucleus's charge and matter distribution
  - Shift in the charge distribution = change in electric field = *Electric*
  - Shift in the current distribution [i.e. orbitals of protons]= change in magnetic field = *Magnetic*
- The selection rules corresponding to a particular decay type are:

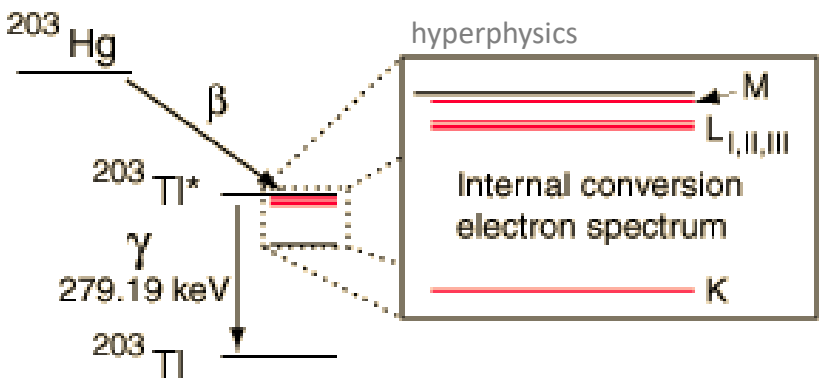
| Radiation Type | Name | $l = \Delta I$ | $\Delta\pi$ |
|---|---|---|---|
| E1 | Electric dipole | 1 | Yes |
| M1 | Magnetic dipole | 1 | No |
| E2 | Electric quadrupole | 2 | No |
| M2 | Magnetic quadrupole | 2 | Yes |
| E3 | Electric octupole | 3 | Yes |
| M3 | Magnetic octupole | 3 | No |
| E4 | Electric hexadecapole | 4 | No |
| M4 | Magnetic hexadecapole | 4 | Yes |

# How does $0^+ \rightarrow 0^+$ happen? *Internal conversion*




- Since $l_{min} = 1$ for a photon, de-excitation by photon emission isn't possible
- Instead the process of internal conversion can happen, whereby a nucleus interacts electromagnetically with an orbital electron and de-excites by ejecting that orbital electron
- This process operates in competition with γ decay for any transition, not just $0^+ \rightarrow 0^+$
- The energy of the emitted electron is: $E_{IC} = E_{xs} - E_{BE,e^-}$, where $E_{xs}$ is the decay transition energy, and $E_{BE,e^-}$ is the electron binding energy




*A similar, but different phenomenon is Internal Pair Conversion, where a photon emitted by the nucleus with $E_\gamma > 2m_ec^2$ interacts with the coulomb field of the nucleus to create an $e^+$-$e^-$ pair. See e.g. A. Wuosmaa et al. PRC(R) 1998.*

# Weisskopf (a.k.a. single-particle) estimates for λ

- $\lambda(l_\gamma, J_i, \pi \to J_f, \pi) = \frac{8\pi(l_\gamma+1)}{l_\gamma[(2l_\gamma+1)!!]^2}\frac{\left(E_\gamma/\hbar c\right)^{2l_\gamma+1}}{\hbar} B(l_\gamma, J_i, \pi \to J_f, \pi),$
- Reduced transition probabilities assuming the initial to final state transition is due to a single nucleon re-orienting itself within a nucleus of uniform density with $R = r_0 A^{1/3}$ are:
  - $B_{s.p.}(E, l_\gamma) = \frac{1}{4\pi}\left[\frac{3}{l_\gamma+3}\right]^2 r_0^{2l_\gamma} A^{2l_\gamma/3} e^2 (fm)^{2l_\gamma}$
  - $B_{s.p.}(M, l_\gamma) = \frac{10}{\pi}\left[\frac{3}{l_\gamma+3}\right]^2 r_0^{(2l_\gamma-2)/2} \mu_n^2 (fm)^{2l_\gamma-2}$, where the nuclear magneton $\mu_n = \frac{e\hbar}{2m_p c}$

The units for B change with $l_\gamma$!

- Note: There is a steep dependency of λ on $l$, so only one multipole of a decay type will matter
- The equations above are still a huge pain to work with and more noble souls have worked-out the decay constant for various situations.
- Using $Q$ in MeV, $\lambda_\gamma$ in $s^{-1}$ for a nucleus with mass number $A$ is given by:

L. van Dommelen, Quantum Mechanics for Engineers (2012)

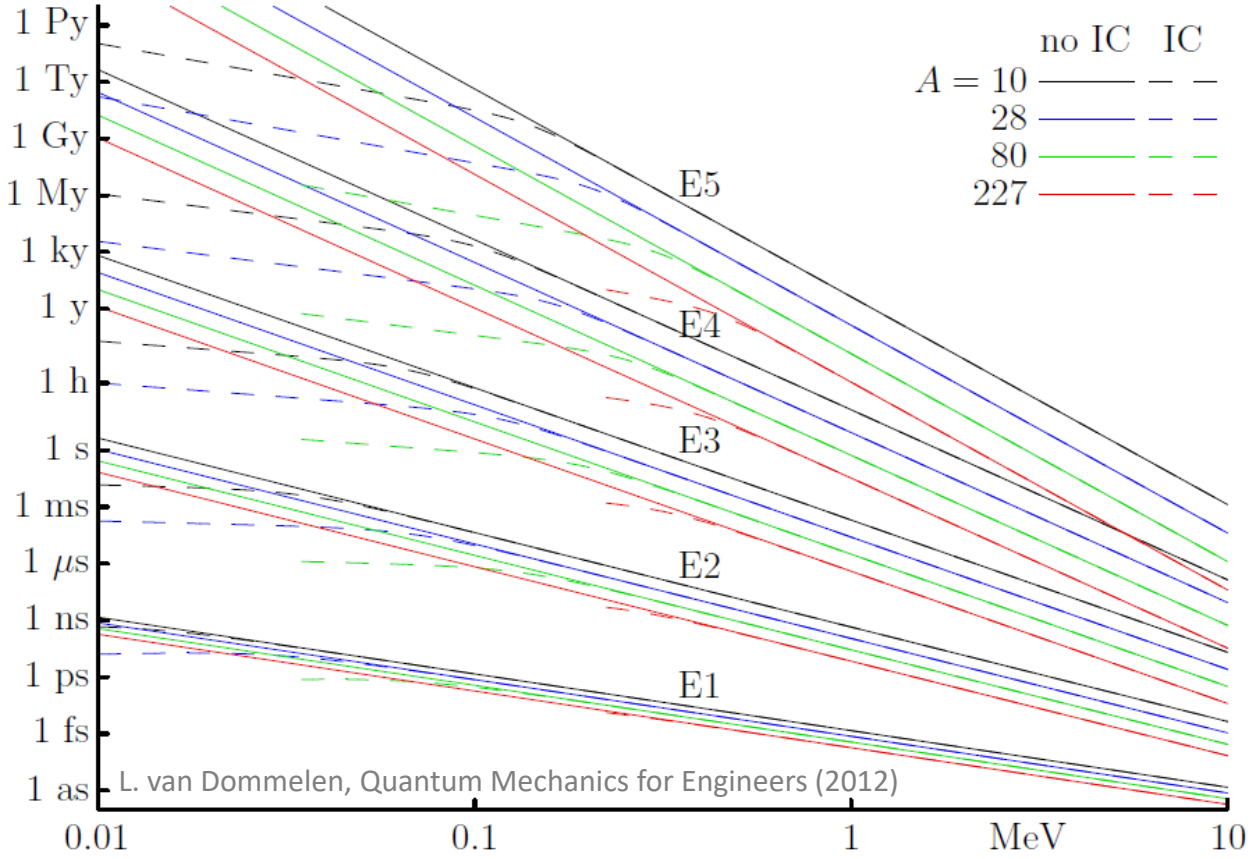
$$\lambda^{\mathrm{E}\ell} = C_{\mathrm{E}\ell} A^{2\ell/3} Q^{2\ell+1} \qquad \lambda^{\mathrm{M}\ell} = C_{\mathrm{M}\ell} A^{(2\ell-2)/3} Q^{2\ell+1}$$

| $\ell$ : | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $C_{\mathrm{E}\ell}$ : | $1.0\,10^{14}$ | $7.3\,10^{7}$ | 34 | $1.1\,10^{-5}$ | $2.4\,10^{-12}$ |
| $C_{\mathrm{M}\ell}$ : | $3.1\,10^{13}$ | $2.2\,10^{7}$ | 10 | $3.3\,10^{-6}$ | $7.4\,10^{-13}$ |

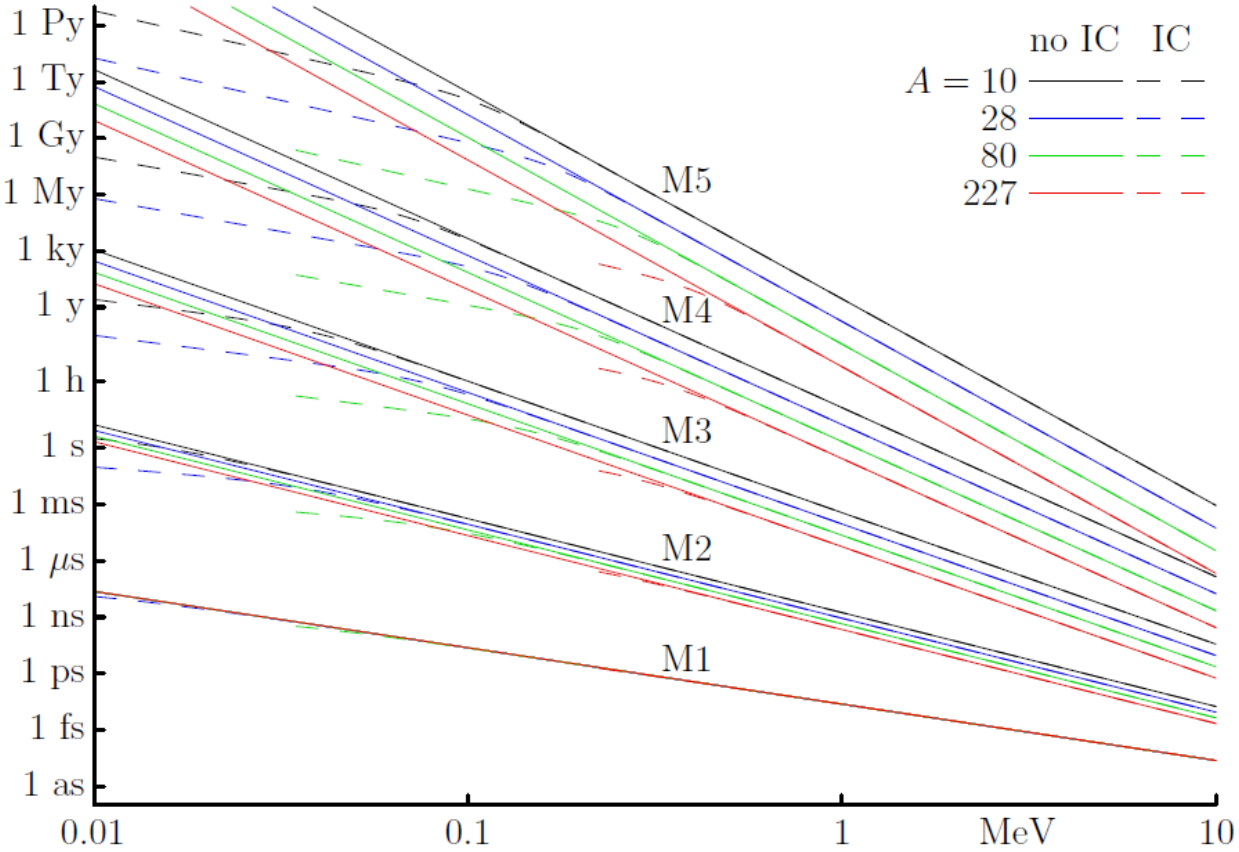
- Weisskopf estimates are generally within an order of magnitude of the real answer, so γ decay constants are often quoted as the ratio to this estimate in “Weisskopf Units” [w.u.]

# Weisskopf (a.k.a. single-particle) estimates for $t_{1/2}$

$t_{1/2}(E_\gamma)$ *for* ***Electric*** *Transitions (from Weisskopf)*




$t_{1/2}(E_\gamma)$ *for* ***Magnetic*** *Transitions (from Moszkowski)*




*Note that E transitions of a given multipole and $E_\gamma$ are ~100X faster than M transitions with the same $E_\gamma$,ℓ*

*Now we see how it is that low-energy high-spin states exist as isomeric states.*

# Internal conversion coefficient, α

- Don't forget about our old friend, internal conversion, which competes with γ decay
- Competition between the two is described by the internal conversion coefficient $\alpha = \frac{\lambda_{IC}}{\lambda_\gamma}$, so $\lambda = \lambda_{IC} + \lambda_\gamma = \lambda_\gamma(1+\alpha)$
- $\alpha$ depends on the density of electrons near the nucleus, and so some friendly atomic physicists have done the dirty work of calculating the following approximate formulas:
  - $\alpha(El) = \frac{Z^3}{n^3}\left(\frac{l}{l+1}\right)\alpha_{f.s.}^4\left(\frac{2m_ec^2}{Q}\right)^{l+5/2}$; $\alpha(Ml) = \frac{Z^3}{n^3}\alpha_{f.s.}^4\left(\frac{2m_ec^2}{Q}\right)^{l+3/2}$, where $\alpha_{f.s.} \approx \frac{1}{137}$, $Q$ is the transition Q-value, and $n$ is the principal quantum number of the orbital electron being ejected

  These rely on the Born approximation, so Z<<137 ought to apply
- The atomic orbitals $K, L, M, N, O, \cdots$ correspond to $n = 1, 2, 3, 4, 5, \cdots$
- Clearly this process is favored for high-$Z$ nuclei, …but also for $Q < 1.022MeV$ $l = 0$ transitions
- For $0^+ \rightarrow 0^+$ transitions, $\lambda_{E0} = 3.8 \cdot Z^3 A^{4/3} Q^{1/2}$, with Q in MeV and λ in $s^{-1}$

# $\lambda_{E0}$ predictions compared to data

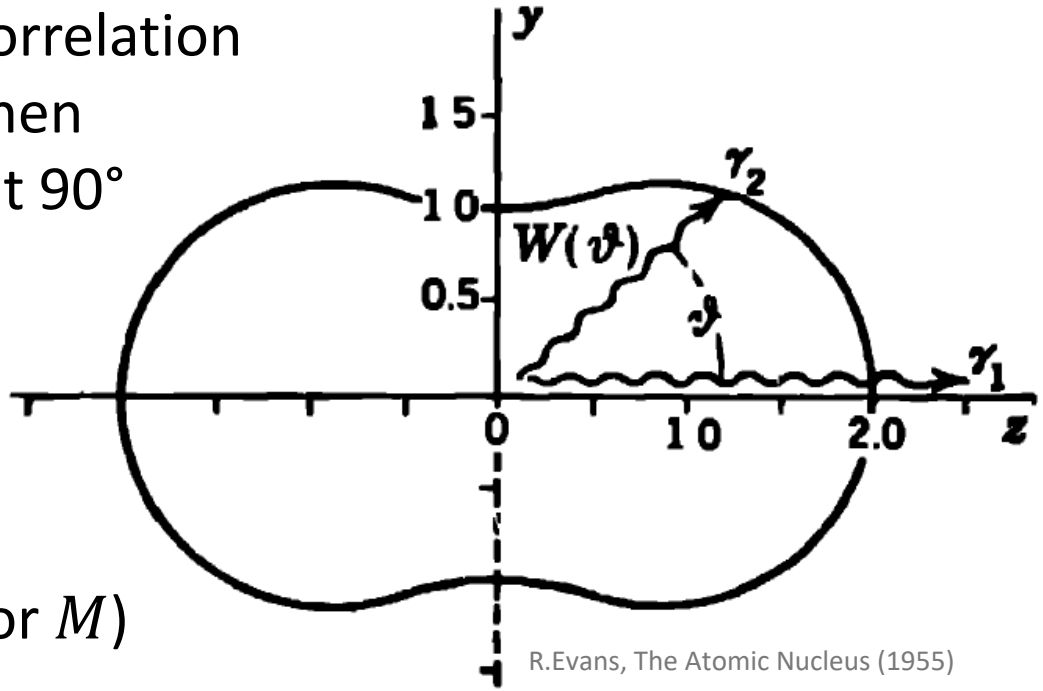
| Nucleus | $Q$ | $\tau_{1/2}$ $\mu s$ | $\tau_{1/2,\mathrm{red}}$ $\mu s$ | I.C. Theory $\mu s$ |
|---|---|---|---|---|
| $^{184}_{80}$Hg | 0.375 | 0.00062 | 0.80 | 0.72 |
| $^{72}_{36}$Kr | 0.671 | 0.0263 | 0.88 | 0.54 |
| $^{72}_{32}$Ge | 0.691 | 0.4442 | 10. | 0.53 |
| $^{98}_{42}$Mo | 0.735 | 0.0218 | 1.8 | 0.51 |
| $^{192}_{82}$Pb | 0.769 | 0.00075 | 1.1 | 0.50 |
| $^{98}_{40}$Zr | 0.854 | 0.064 | 4.4 | 0.47 |
| $^{194}_{82}$Pb | 0.931 | 0.0011 | 1.6 | 0.45 |
| $^{96}_{40}$Zr | 1.582 | 0.0380 | 2.6 | 0.35 |
| $^{90}_{40}$Zr | 1.761 | 0.0613 | 3.8 | 0.33 |
| $^{68}_{28}$Ni | 1.770 | 0.2760 | 4.0 | 0.33 |
| $^{40}_{20}$Ca | 3.353 | 0.00216 | 0.0057 | 0.24 |
| $^{32}_{16}$S | 3.778 | 0.00254 | 0.0025 | 0.23 |
| $^{16}_{8}$O | 6.048 | 0.00007 | 0.000003 | 0.18 |

For these decays $e^+$–$e^-$ pair creation is a significant decay mode

L. van Dommelen, Quantum Mechanics for Engineers (2012)

# γ angular correlations

- Radioactive decay, including γ decay, is isotropic when nuclei are oriented at random, which is generally the case in a laboratory setting
- However, the γ angular distribution relative to a fixed axis on a given nucleus, e.g. the angular momentum projection of the state the nucleus originates from, is anisotropic
- Measuring one γ-ray essentially identifies the nuclear axis, so measuring the next γ-ray, which has an angular distribution relative to the angular momentum projection of the state populated by the first γ-ray, will result in an angular correlation
- We choose the original γ-ray to be at $\theta_{CM} = 0,$ and then the relative intensity to the intensity of the 2nd γ-ray at 90° $W(\theta) = \frac{Intensity\ at\ \theta}{Intensity\ at\ \theta_{cm}=90°}$ is described by a sum of several Legendre polynomials, since after all the populated $m$ and photon multipolarity $l$ correspond to a given Legendre polynomial
- Note $W(\theta)$ does not depend on the γ-decay type ($E$ or $M$)

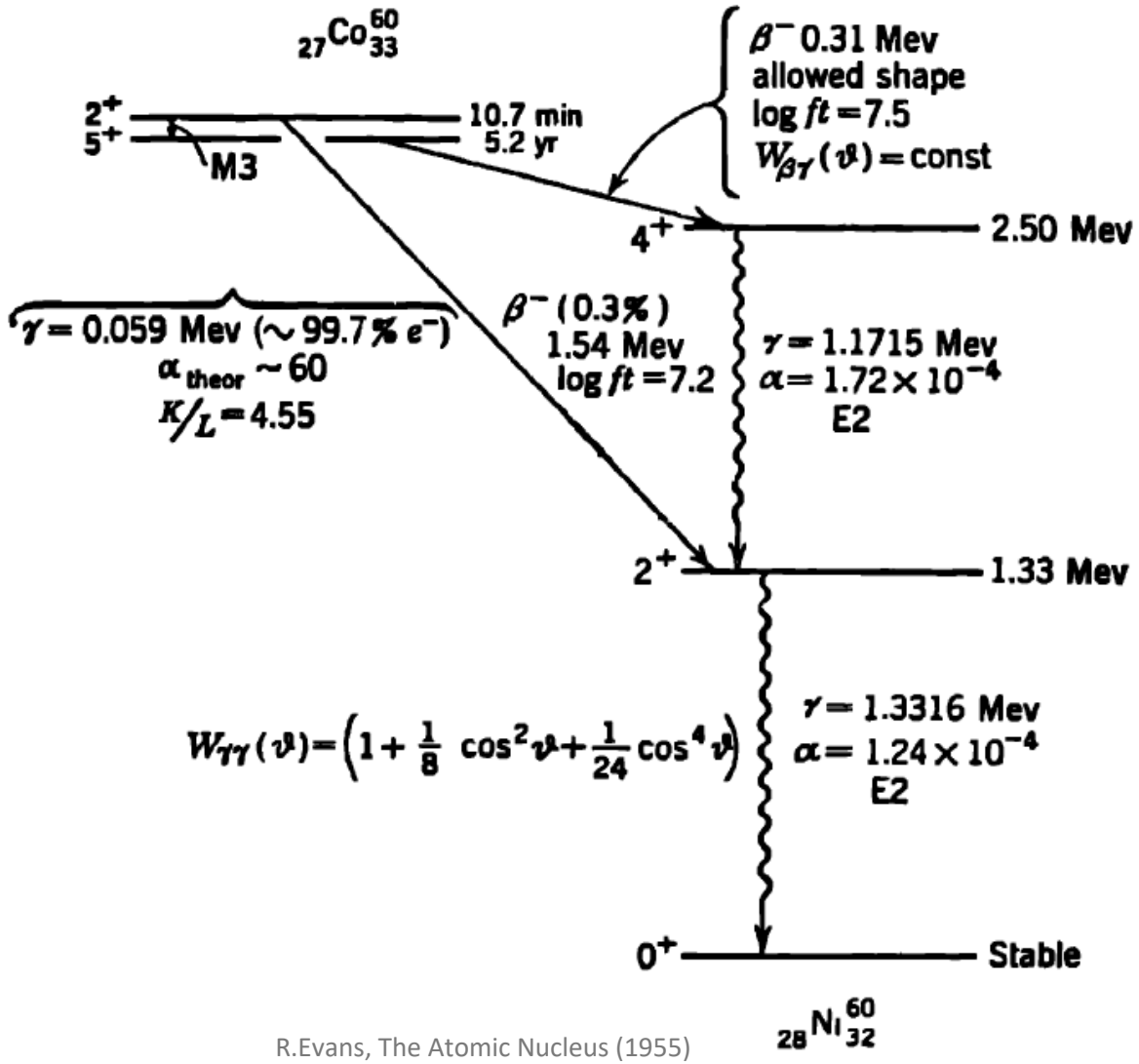



R.Evans, The Atomic Nucleus (1955)

In all this and the following talk of W(θ), θ refers to the center of mass angle …however, as we showed earlier, the nuclear recoil is mostly negligible, so $\theta_{CM} \approx \theta_{lab}$.

# Spectroscopy with γ decay

- Particularly when combined with $ft$ from β-decay, $\lambda_\gamma$ and/or $W(\theta)$ provide powerful tools to determine or at least constrain $J^\pi$ for nuclei

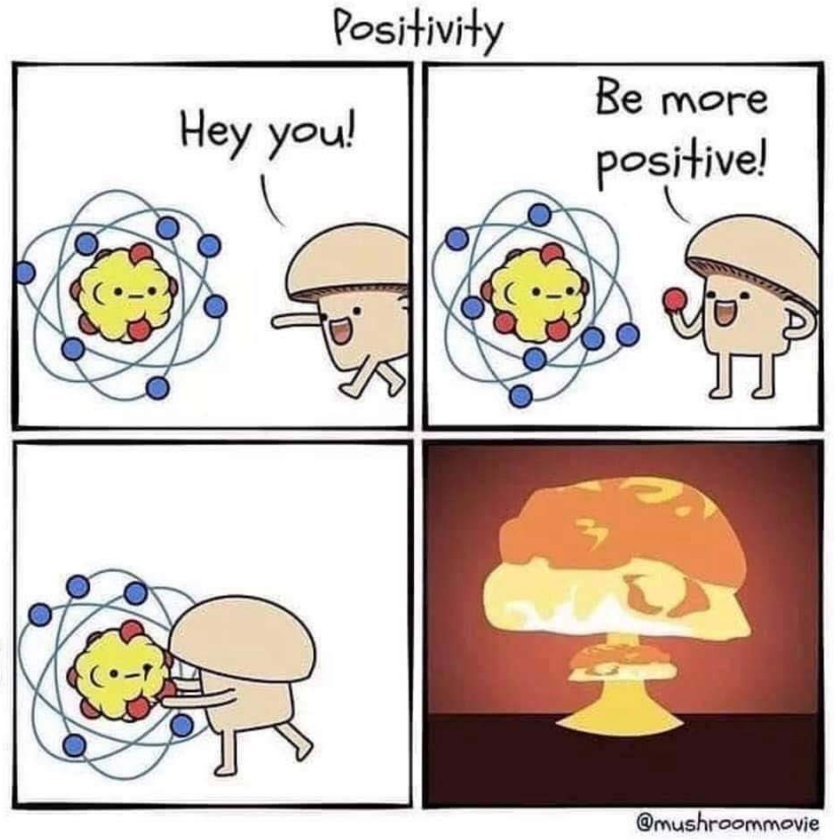



R.Evans, The Atomic Nucleus (1955)

# Lecture 10: Fission

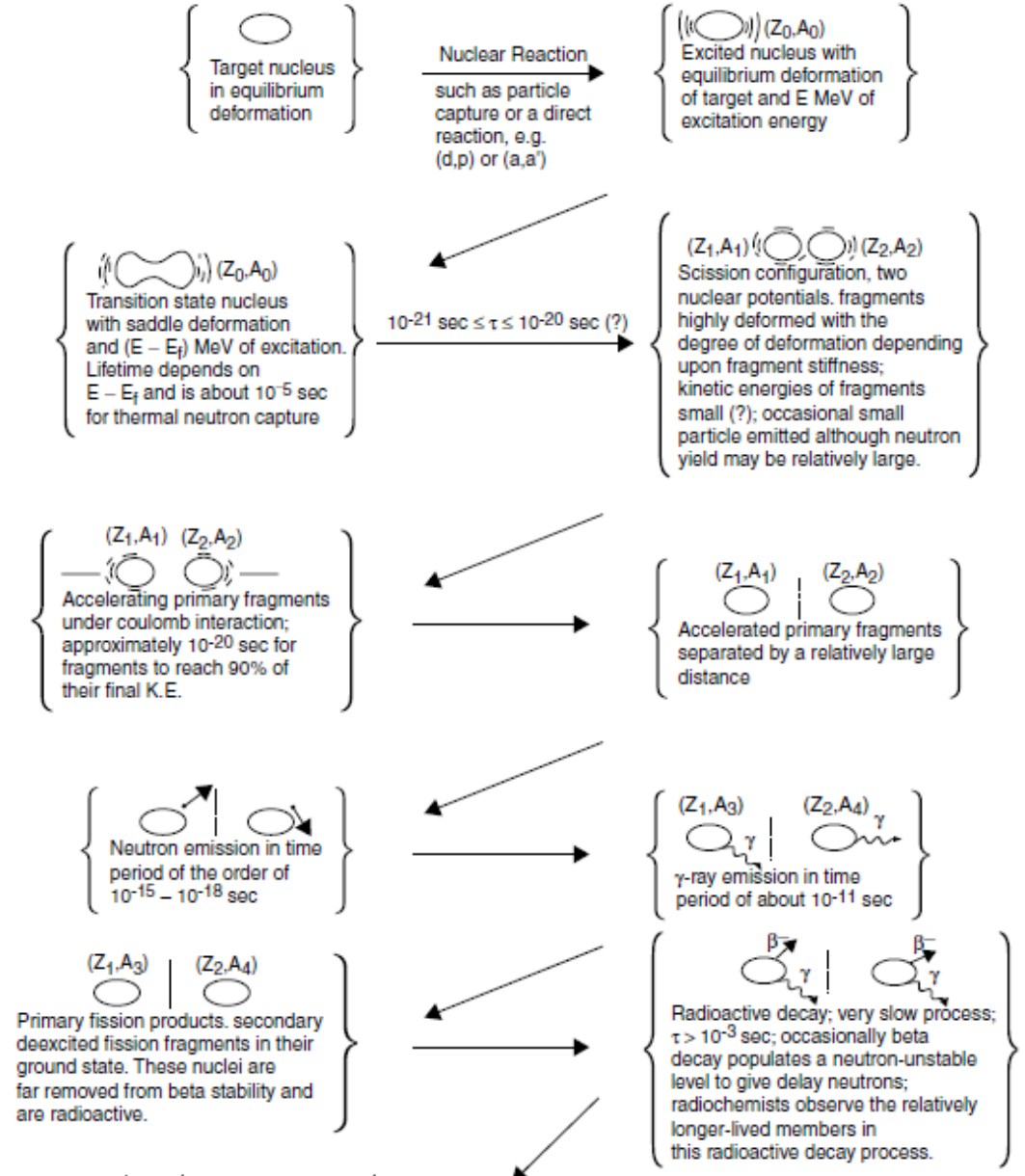
- Conceptual process
- Fissionability
- Decay rate
- Decay branching
- Mass distribution
- Kinetic energy
- Neutrons



# Steps of fission

1. A nucleus becomes deformed either due to an external perturbation that brings in energy or an internal cluster rattling around within the potential well
2. The energy is absorbed as a collective excitation that manifests itself as a drastic shape change, elongating the nucleus into a peanut shape
3. The separation of the two lobes of the peanut becomes great enough that the two repel each other, splitting apart at the scission point
4. The coulomb repulsion accelerates the two fragments apart
5. The two fragments are each highly excited and de-excite initially via neutron emission, followed by γ emission
   *(meaning prompt neutrons will be emitted along the direction of the fragments)*
6. The neutron-rich fragments will then β decay back to stability, possibly emitting delayed neutrons via β-delayed neutron emission



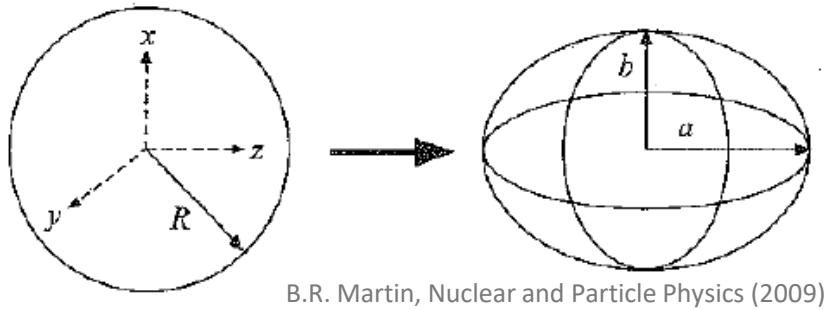

*Why do you figure there is neutron emission at first and finally only γ emission?*

*A massive particle is better suited to remove the large angular momentum present in high-lying excited states. At lower excitation energies, a particle has to tunnel out of the nucleus, while the γ doesn't.*

Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

# Energetics of shape change: *Liquid drop picture*

- For the deformed shape, which is an ellipsoid in this picture, the nuclear radius can be parameterized as an expansion in terms of Legendre polynomials
  *[which for axial symmetry will only keep the $l = 2$ term]*
  - $R(\theta) = R_0[1 + \alpha_2 P_2(\cos\theta)]$
  - $\alpha_2$ is the quadrupole distortion parameter, which is related to the quadrupole deformation by $\alpha_2 = \sqrt{\frac{5}{4}\pi}\beta_2$, and the ellipsoid axes by: $a = R_0(1 + \alpha_2)$ , $b = R_0 \frac{1}{\sqrt{1+\alpha_2}}$



B.R. Martin, Nuclear and Particle Physics (2009)

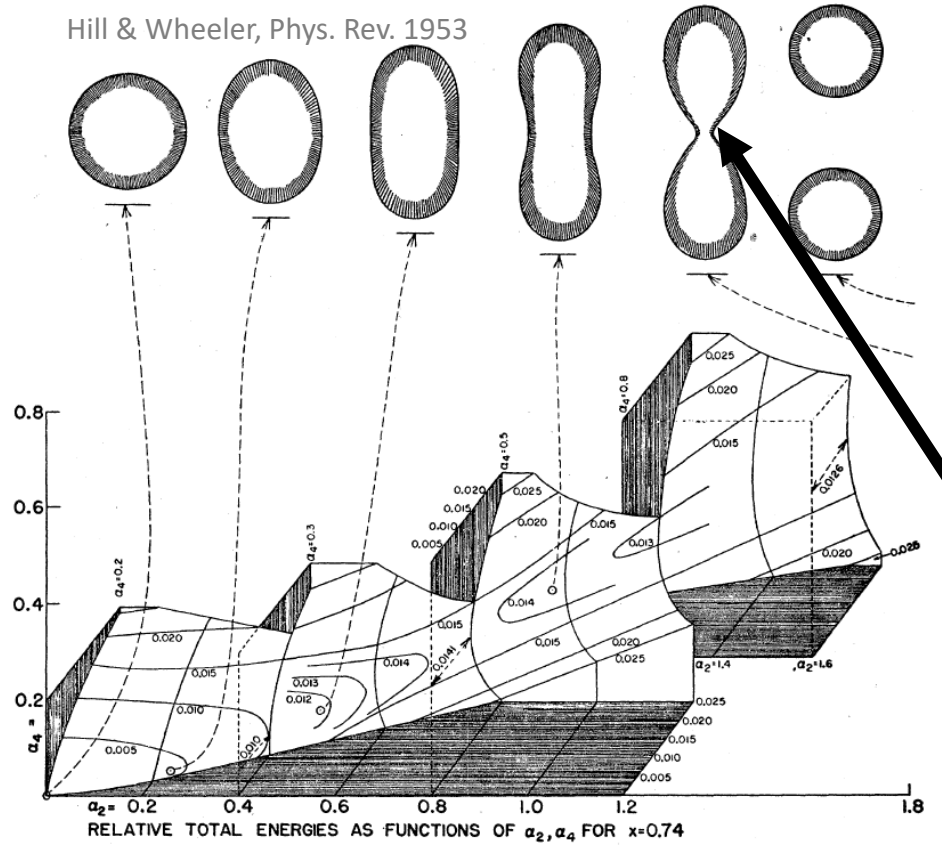
- It turns out, expanding the Coulomb and surface energy terms as a power series in $\alpha_2$ yields
  - $E_c' \approx a_c \frac{Z^2}{A^{1/3}}\left(1 - \frac{1}{5}\alpha_2^2\right)$ and $E_s' \approx a_s A^{2/3}\left(1 + \frac{2}{5}\alpha_2^2\right)$
- Meaning the energy cost for deformation is $\Delta E = \frac{\alpha_2^2}{5}\left(2a_s A^{2/3} - a_c \frac{Z^2}{A^{1/3}}\right)$ *which we determined back in lecture 4*
- So, when the non-deformed Coulomb energy is twice the non-deformed surface energy or greater, there is zero energetic cost (or even an energetic gain) to deform (and ultimately fission!)
- The fissionability parameter $x = \frac{E_c}{2E_s} = \frac{a_c}{2a_s}\frac{Z^2}{A} \equiv \frac{Z^2/A}{\left(Z^2/A\right)_{critical}}$ is a measure of fission favorability

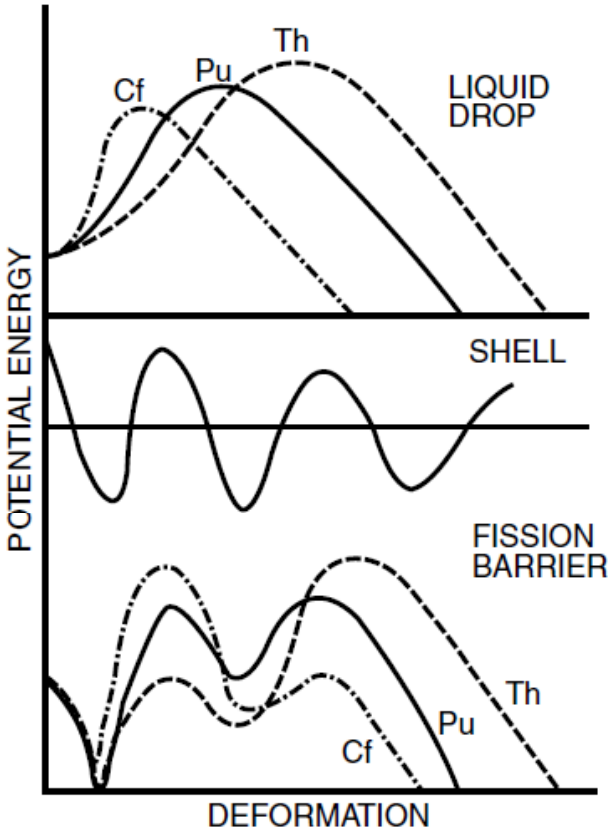
# Fission barrier

- To quantify the probability of fission occurring, we can consider the potential energy associated with a particular shape of the nucleus
- This potential energy is dominated by liquid drop energetics,

  …but with a major correction for enhanced binding near closed shells

  …the magnitude of which is altered by the nuclear deformation
- So, the peaks and valleys in potential energy space are associated by the competition between liquid-drop and shell-model modified by deformation
- The likelihood of transitioning from one shape to another depends on the energetic cost associated with overcoming *(or tunneling through)* the barrier between them



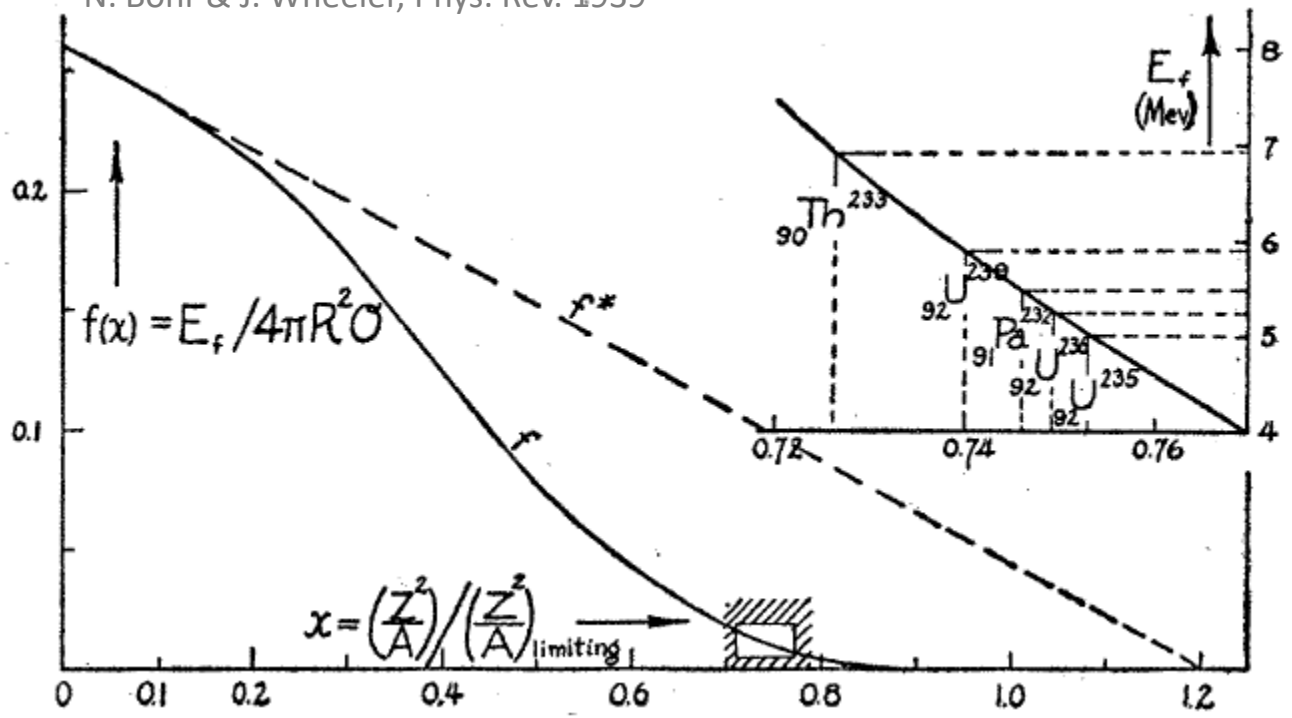

Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

*scission occurs at the saddle point in the potential*

# Fission barrier height

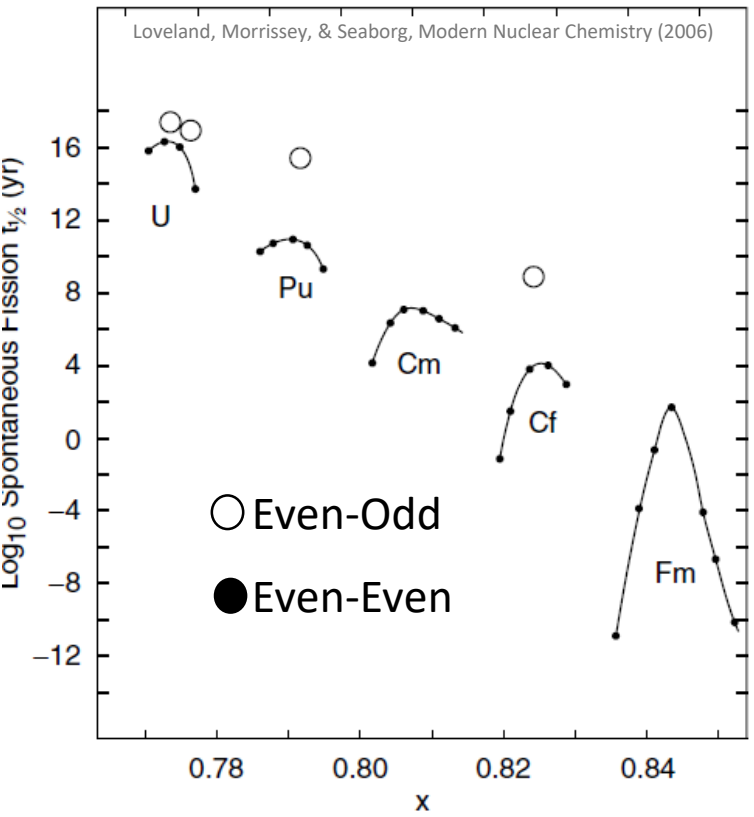
- In a liquid drop estimate,
  the fission barrier height is the difference between the ground-state binding energy of a system and the energy of two spherical nuclei made by splitting that system
- Near the longest-lived isotopes of Uranium and Thorium, $E_f \sim 5 - 7 MeV$
- Note this simple method yields one useful prediction already:
  - $^{235}$U has a much shorter fission barrier height than $^{238}$U and so the $A = 235$ isotope is more useful when a lot of fission is what you're looking for

N. Bohr & J. Wheeler, Phys. Rev. 1939

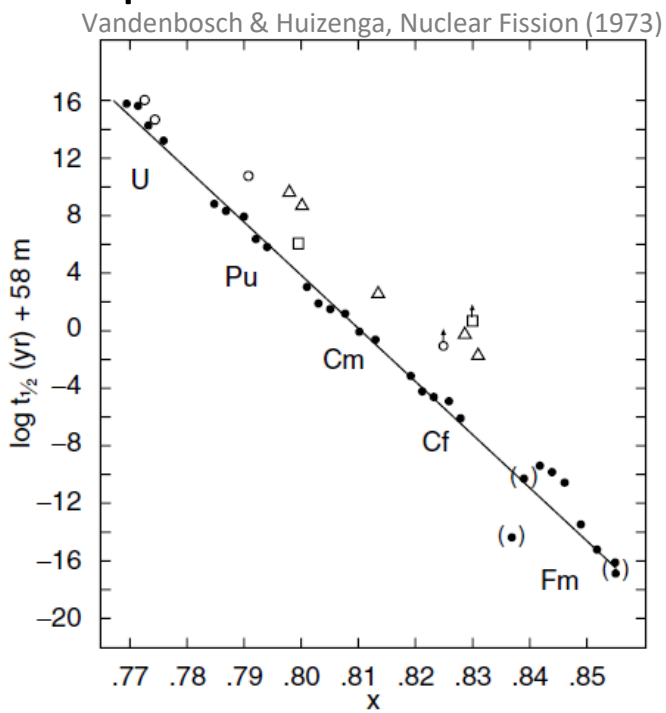


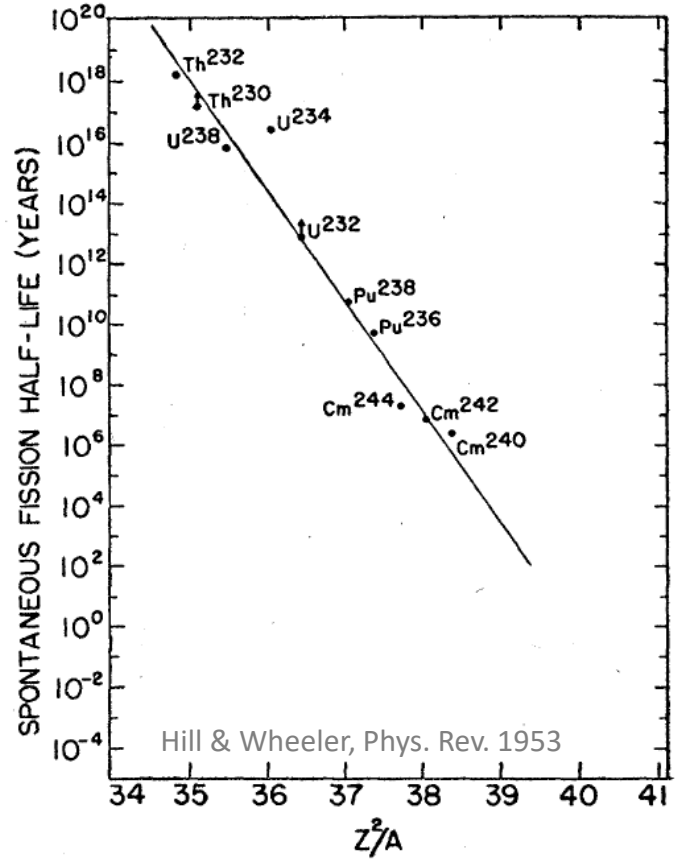
# Spontaneous fission rate

- The height of the fission barrier is related to the fissionability parameter $x$ *(recall x~1, i.e. $Z^2/A$~48, means fission immediately)*
- So without resorting to fancy calculations of $E_f$ and $\hbar\omega$, the half-life for spontaneous fission can be estimated empirically
- But this method is extremely rough




….using shell-corrected masses gives some improvement:



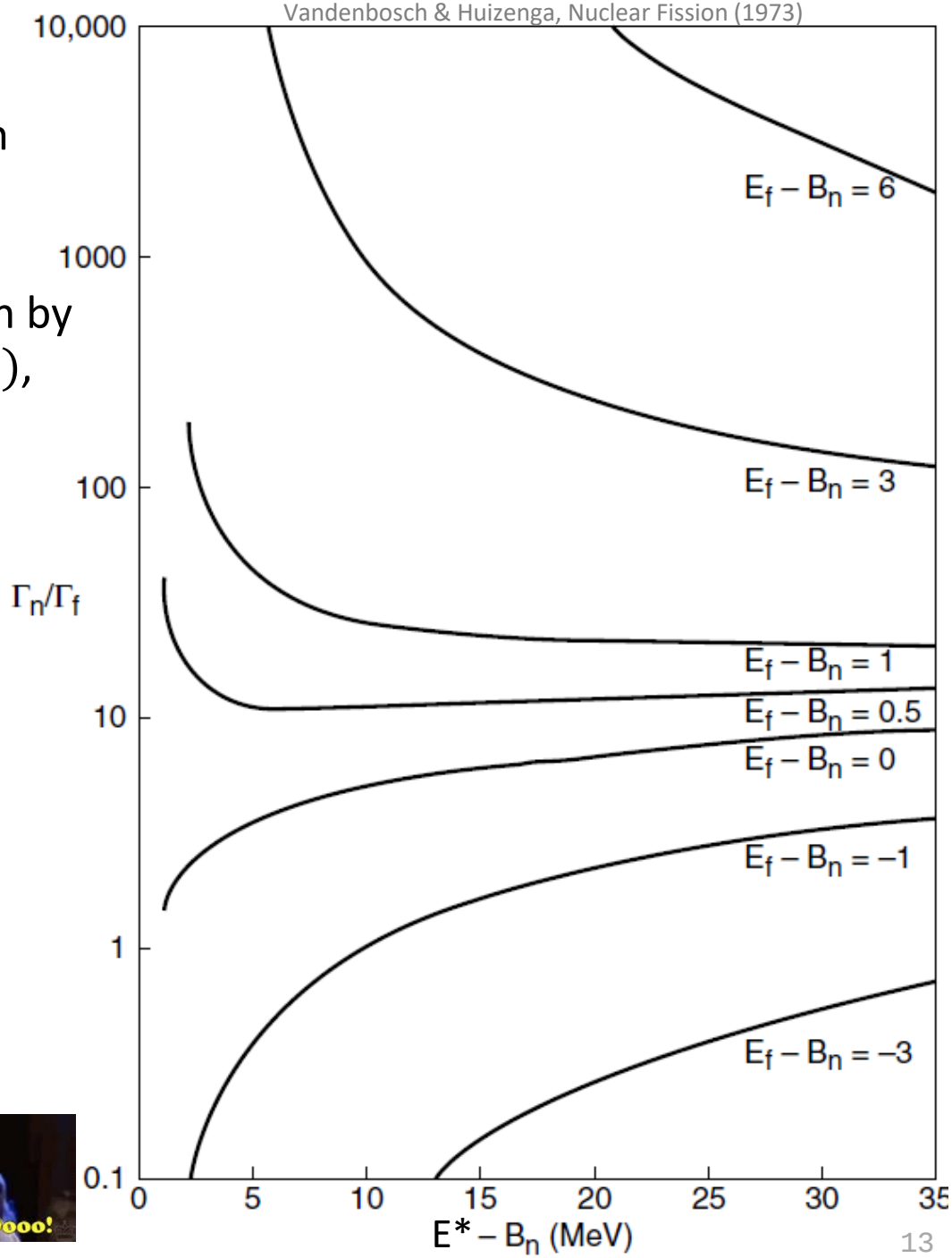
# Fission decay branching ratio

- Since neutrons have no Coulomb barrier to tunnel through and can generally carry away more angular momentum than γ rays, the neutron decay branch is the primary competition for fission
- The probabilities for neutron emission and fission are roughly given by a Boltzmann distribution, $P_f \propto \exp(-B_f/T)$ and $P_n \propto \exp(-B_n/T)$, so the ratio of neutron emission to fission is $\propto \exp(B_f - B_n/T)$
- Higher excitation energies $E^* = E_{CM} + Q$ corresponds to higher $T$. At higher $T$, more of the phase space is sampled evening-out the competition between n-emission and fission. E.g. if n-emission is more likely, increasing E* makes fission more competitive
- Exact calculations involved detailed statistical model calculations, so the figure on the right can be used for a rough estimate
- E.g. $^{238}$U bombarded by a 42MeV α
  - $E^* = 42\frac{238}{242} + (ME(U238) + ME(\alpha) - ME(Pu242))c^2 = 36.3MeV$
  - $B_n(Pu242) = (ME(Pu241) + ME(n) - ME(Pu242))c^2 = 6.3MeV$
  - $E_f(Pu242) = 3.3MeV$ *[From Myers & Swiatecki PRC 1999 fit]*
  - $E^* - B_n = 30MeV;\ \ E_f - B_n = -3 \quad \rightarrow \frac{\Gamma_n}{\Gamma_f} \sim 0.5$
  - … so fission occurs 2/3 of the time for this case

(actual answer [J. Wing et al. Phys Rev 1959] is ~80% of the time)

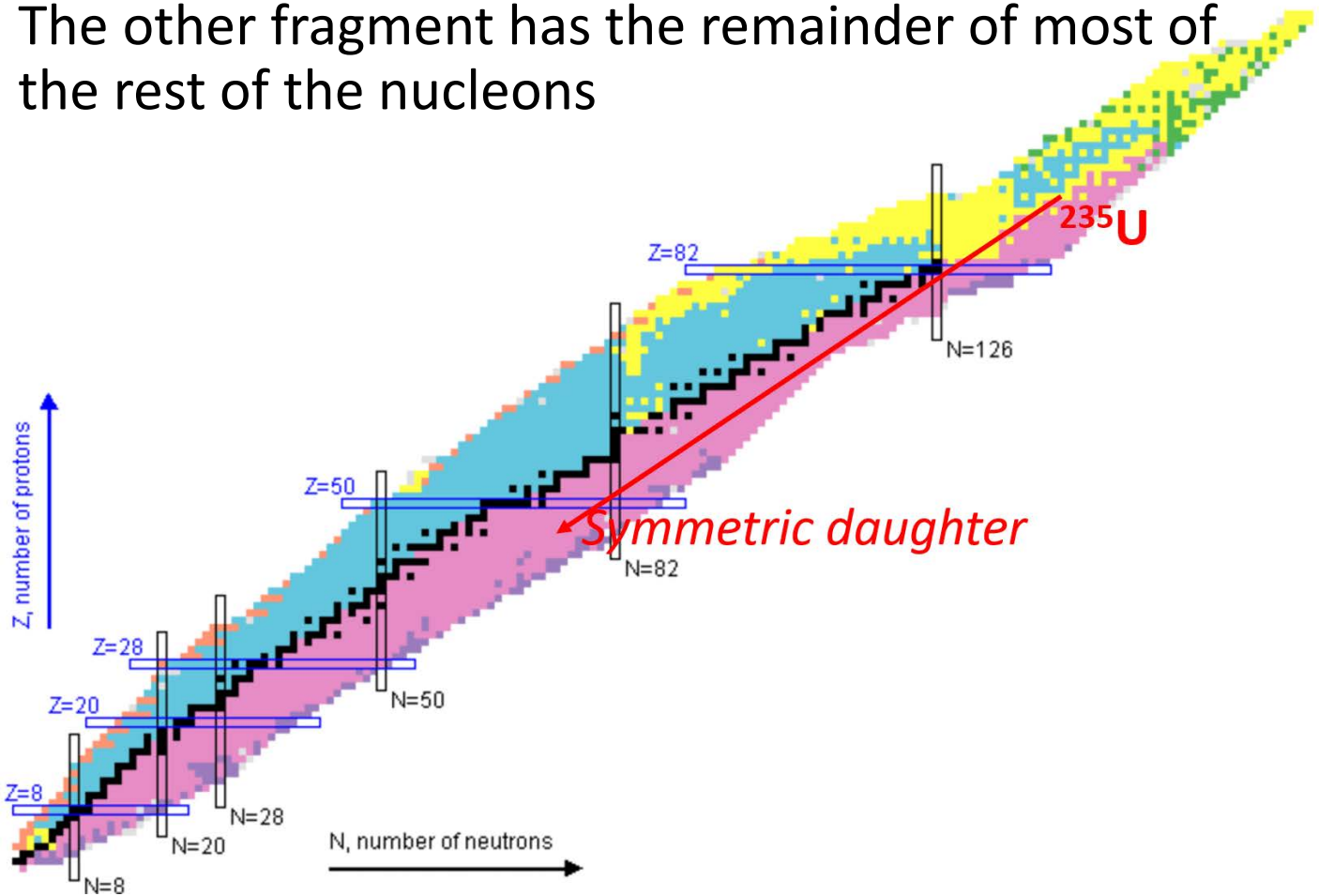



Vandenbosch & Huizenga, Nuclear Fission (1973)

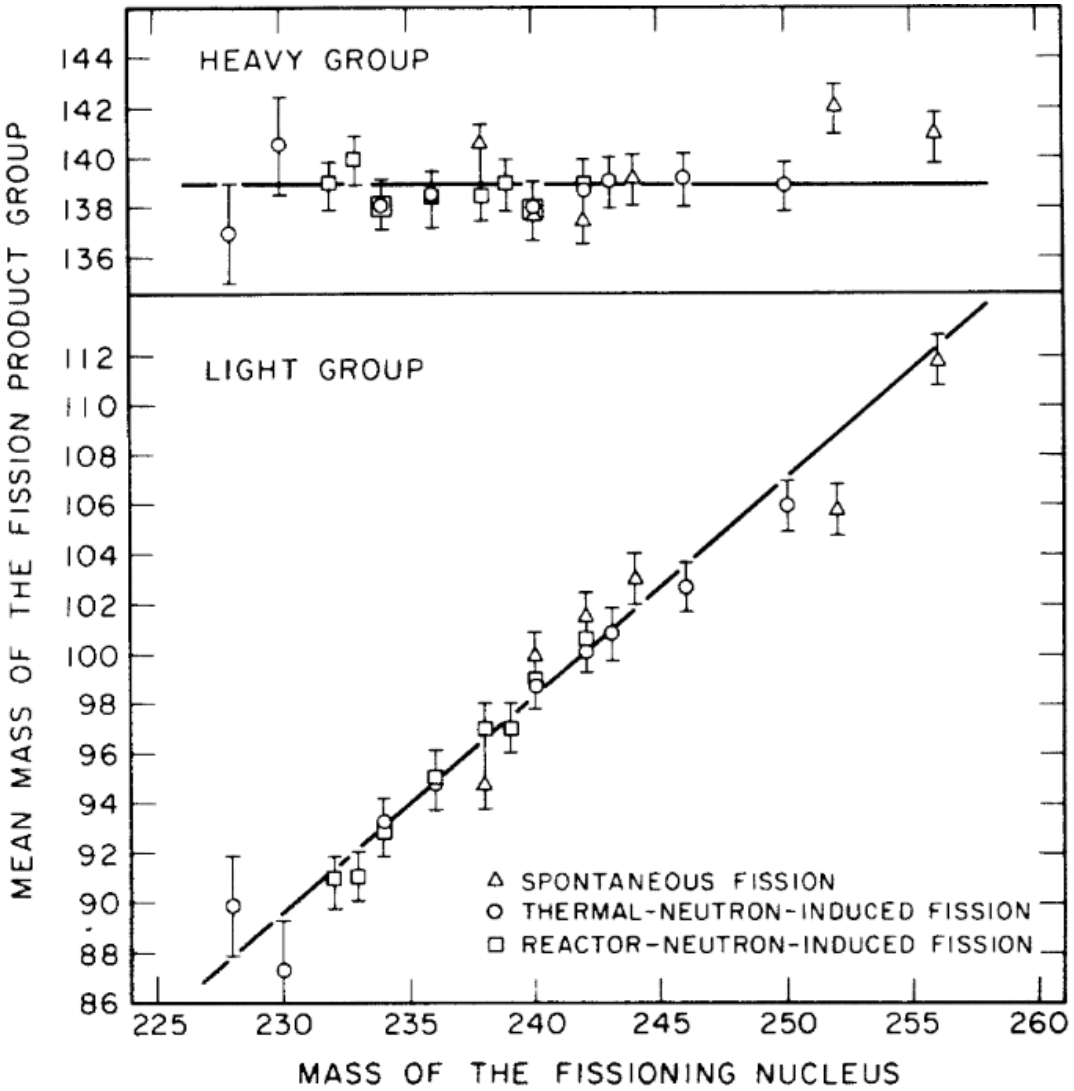


# Fission fragment mass distribution

- The culprit for the asymmetric mass distribution are the $Z = 50$, $N = 82$ shell closures, which favors nuclei in this range for one of the fission fragments.
- The other fragment has the remainder of most of the rest of the nucleons




K. Flynn et al. PRC Vol 5 1972

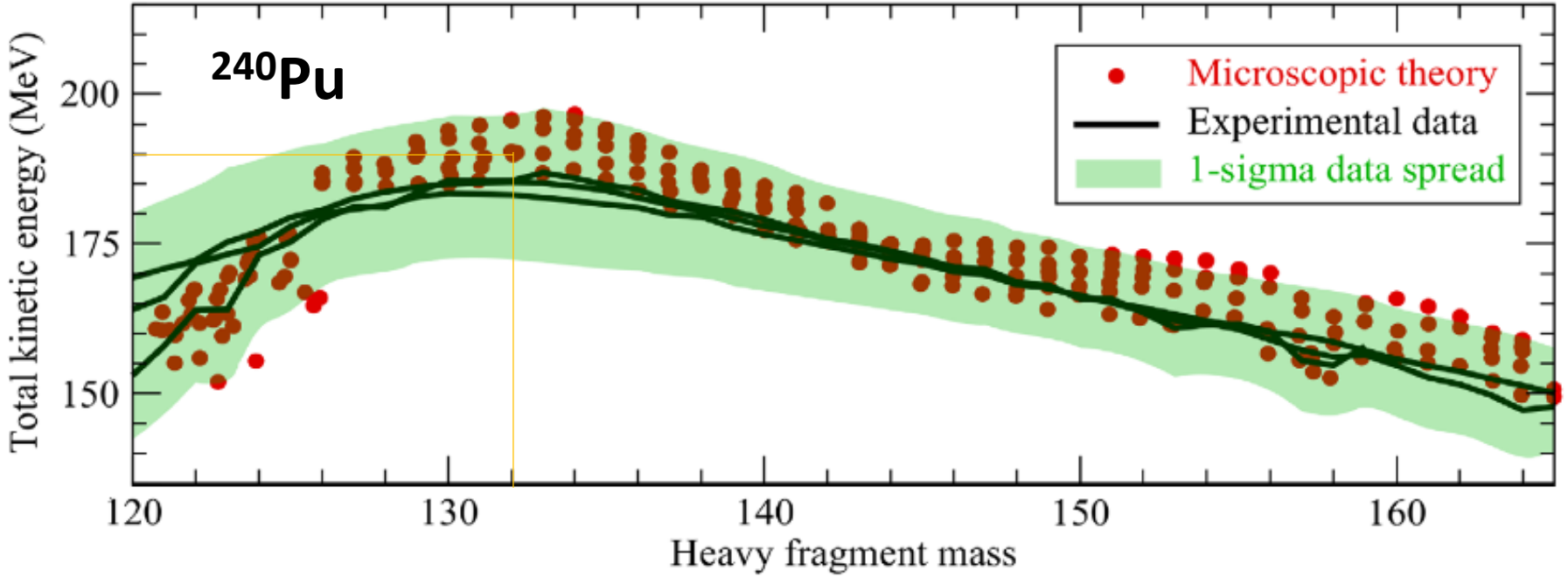


# Fission fragments: *kinetic energy*

- The total kinetic energy of fission products will roughly be the coulomb repulsion energy of the two main fission fragments,
  - $TKE = \frac{Z_1 Z_2 \alpha \hbar c}{1.8\left(A_1^{1/3}+A_2^{1/3}\right)}$ *…where $r_0$=1.8fm is used instead of 1.2 because of the strong deformation at scission*
- E.g. $^{240}$Pu
  - $A/Z \approx 2.55$ . If $Z_{high} = 50, N_{high} = 82$, then $Z_{low} = 94 - 50 = 44, N_{low} = 240 - 132 - 44 = 64$
  - $TKE = \frac{(50)(44)\alpha \hbar c}{1.8\left(132^{1/3}+108^{1/3}\right)} \approx 178 MeV$




Schunk & Robledo, Rep. Prog. Phys. (2016)

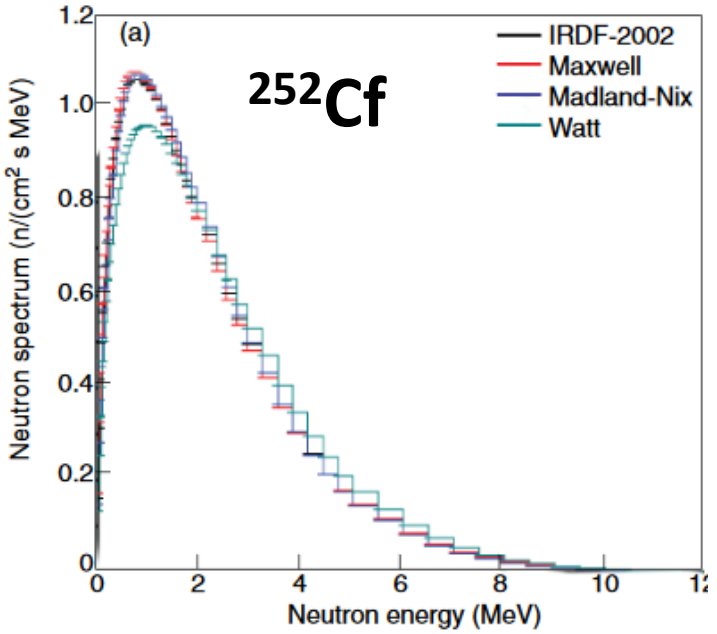


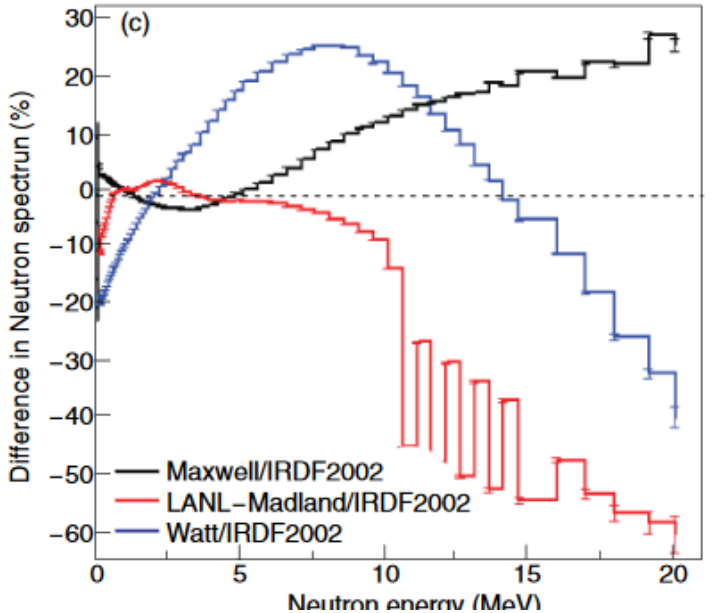
# Fission neutrons kinetic energy

- Neutrons are emitted from the nearly fully accelerated fission fragments and so are emitted along the direction of the fragments
- The neutron kinetic energy (typically $\sim 2MeV$) is just described by a Maxwell-Boltzmann distribution, because the nucleus is "evaporating" neutrons
  $P(KE_n) = KE_n \exp(-E_n/k_B T)$
- Transforming into the lab frame, the result is the Watt fission spectrum,

  $P(KE_n) = \exp(-KE_n/k_B T)\sinh\left(\sqrt{4E_n E_f/(k_B T)^2}\right)$, where $E_n$ and $E_f$ are the laboratory neutron and fission fragment energies in MeV/u and $T = \sqrt{\frac{8E^*}{A}}$ is the nuclear temperature
- Parameterized forms of the Watt fission spectrum, with the associated parameter values for a given fission fragment are available. E.g. in the appendix of the MCNP user manual

L. Snoj et al. Nucl. Eng. & Design (2012)



# Fission fragment angular distributions

- Induced fission is slow enough that the nucleus reaches statistical equilibrium
- This results in an angular distribution of fission fragments that is symmetric about 90° in the center of mass, where 90° marks the plane perpendicular to the direction of motion of the fissioning system.
- This is because forward and backward angles maximize the amount of angular momentum the fragments carry away. Prior to scission, the fissioning nucleus will be highly excited relative to the absolute ground state, and higher excitations correspond on average to larger angular momenta.
- The proper formula for the angular distribution is complicated and related to the deformation of the fissioning nucleus
  *[See Loveland, Morrissey, & Seaborg]*
- From momentum conservation, we know that the fragments themselves are emitted 180° apart (in the CM), which is handy for gating on fission events in reaction-induced fission studies

G. Gordon et al. Phys. Rev. 1960

