Lecture 4: *Nuclear Structure 2*

- Independent vs collective models
- Collective Model
  - Rotation
  - Vibration
  - Coupled excitations
- Nilsson model

Nuclear Models

“There’s no small choice in rotten apples.” Shakespeare, *The Taming of the Shrew*

- No useful fundamental & universal model exists for nuclei
  - E.g. based on the nuclear interaction, how do we describe all nuclear properties?
  - Promising approaches include “ab initio” methods, such as Greens Function Monte Carlo, No-core shell model, Coupled cluster model, density functional theories
- Generally one of two classes of models is used instead
  - Independent particle models:
    - A nucleon exists within a mean-field (maybe has a few interactions)
    - E.g. Shell model, Fermi gas model
  - Collective models:
    - Groups of nucleons act together (maybe involves shell-model aspects)
    - E.g. Liquid drop model, Collective model

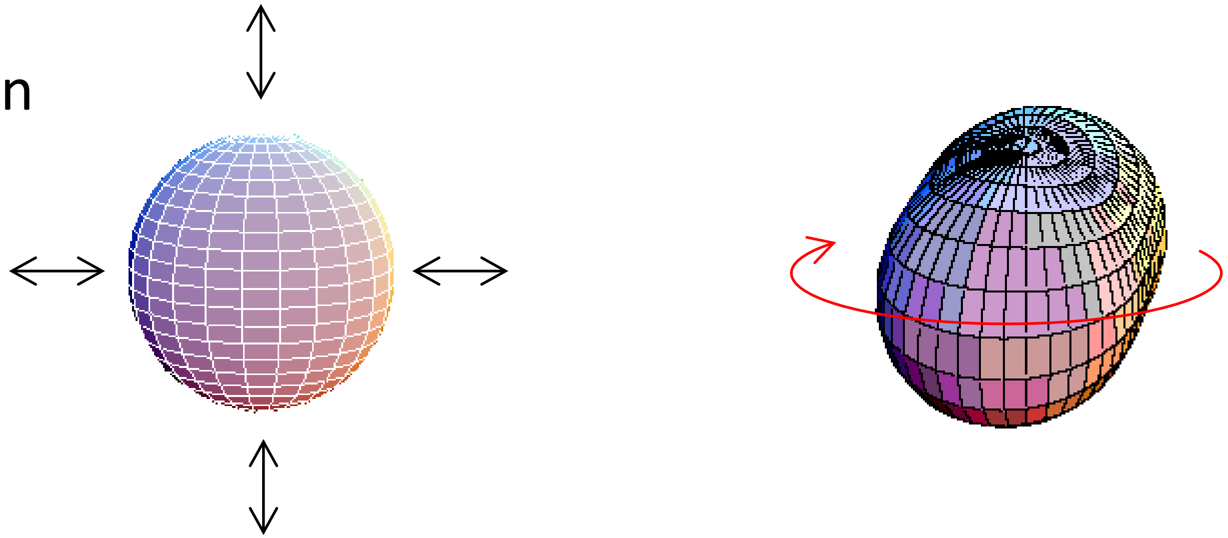
Collective Model

- There are compelling reasons to think that our nucleus isn't a rigid sphere
  - The liquid drop model gives a pretty successful description of some nuclear properties.

    …can't liquids slosh around?
  - Many nuclei have non-zero electric quadrupole moments (charge distributions)

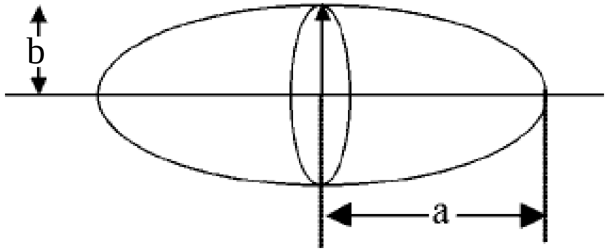
    …this means there's a non-spherical shape.

    …can't non-spherical things rotate?
- Then, we expect nuclei to be able to be excited rotationally & vibrationally
  - We should (and do) see the signature in the nuclear excited states
- The relative energetics of rotation vs vibration can be inferred from geometry
  - The rotational frequency should go as $\omega_r \propto \frac{1}{R^2}$ (because $I \equiv \frac{L}{\omega}$ and $I \propto MR^2$)
  - The vibrational frequency should go as $\omega_v \propto \frac{1}{(\Delta R)^2}$ (because it's like an oscillator)
  - So $\omega_r \ll \omega_v$



Rainwater's case for deformation

- A non-spherical shape allows for rotation …but why would a nucleus be non-spherical?
- Consider the energetics of a deformed liquid drop (J. Rainwater, Phys. Rev. (1950))
  - $BE_{SEMF}(Z,A) = a_{vol}A - a_{surf}A^{2/3} - a_{coul}\frac{Z(Z-1)}{A^{1/3}} - a_{asym}\frac{\left(Z-\frac{A}{2}\right)^2}{A} \pm a_{pair}i\sqrt{A}$
  - Upon deformation, only the Coulomb and Surface terms will change
    - Increased penalty for enlarged surface
    - Decreased penalty for Coulomb repulsion because charges move apart
    - The volume remains the same because the drop is incompressible



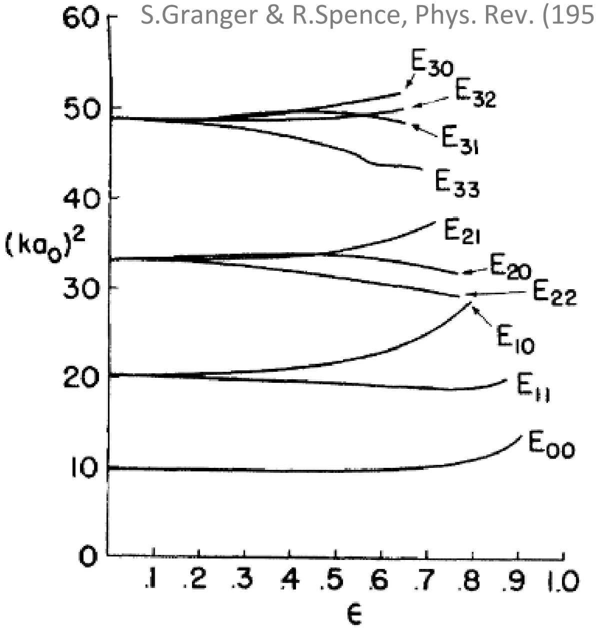
Loveland, Morrissey, Seaborg, Modern Nuclear Chemistry (2006)

- To change shape, but maintain the same volume, the spheroid's axes can be parameterized as
  - $a = R(1+\varepsilon)$; $b = \frac{R}{\sqrt{1+\varepsilon}}$; where $V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi ab^2$
- It turns out (B.R. Martin, Nuclear and Particle Physics), expanding the surface and Coulomb terms in a power series yields:
  - $E_s' = a_{surf}A^{2/3}\left(1+\frac{2}{5}\varepsilon^2+\cdots\right)$; $E_c' = a_{coul}\frac{Z(Z-1)}{A^{1/3}}\left(1-\frac{1}{5}\varepsilon^2+\cdots\right)$
- Therefore, the change in energy for deformation is:
  - $\Delta E = (E_s' + E_c') - (E_s + E_c) = \frac{\varepsilon^2}{5}\left(2a_{surf}A^{2/3} - a_{coul}\frac{Z(Z-1)}{A^{1/3}}\right)$

  ($\Delta E < 0$ is an energetically favorable change)
  - Written more simply, $\Delta E(Z,A) = -\alpha(Z,A)\varepsilon^2$

To get $\Delta E<0$, need $Z>116$, $A>270$!
So we do not expect deformation
from this effect alone.
Nonetheless, heavier nuclei are going
to be more susceptible to deformation.

Rainwater’s case for deformation



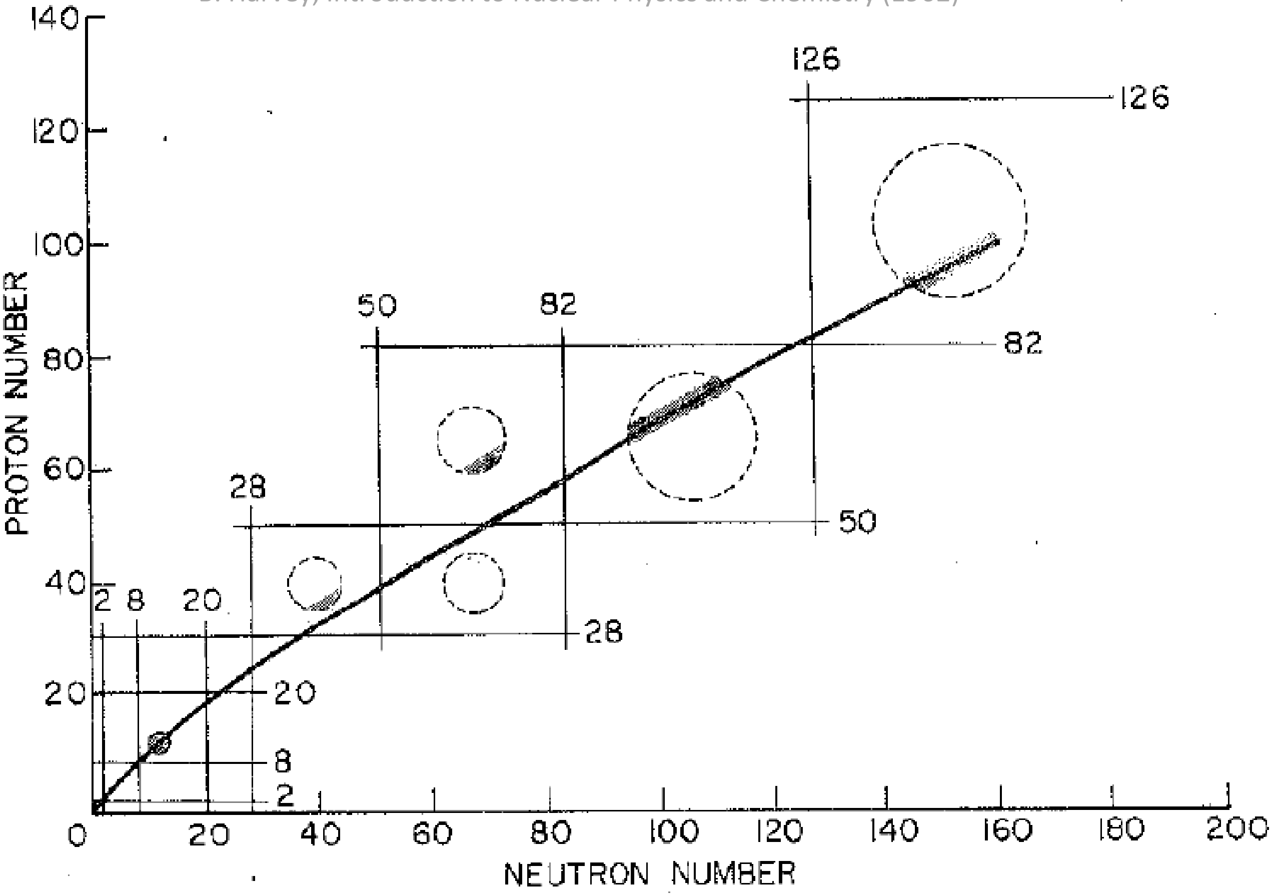
S.Granger & R.Spence, Phys. Rev. (1951)

FIG. 1. Energy levels for $n=1$. Levels are labeled with the quantum numbers l, m of the undistorted nucleus.

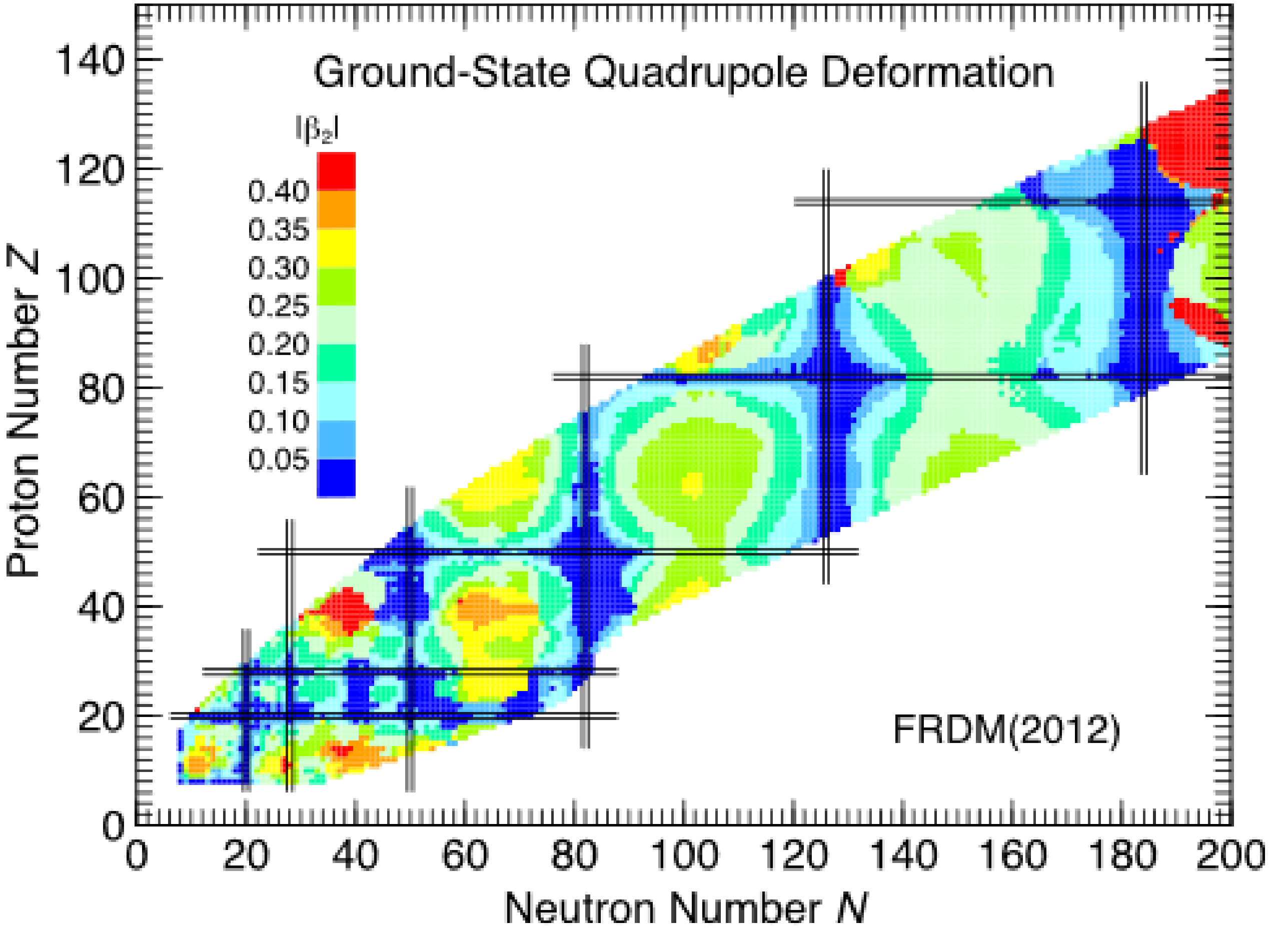
- So far we’ve only considered the deformation of the core
- However, we also need to consider any valence nucleons
  - A non-spherical shape breaks the degeneracy in m for a given l, where the level-splitting is linear in the deformation ε.
  - The strength of the splitting is found by solving the Schrödinger equation for single-particle levels in a spheroidal (rather than spherical) well and comparing the spheroidal eigenvalue to the spherical one.
- The total energy change for deformation then becomes: $\Delta E(Z,A) = -\alpha(Z,A)\varepsilon^2 - \beta\varepsilon$
- The core deformation favors a given m, reinforcing the overall deformation
  - i.e. the valence nucleon interacts with the core somewhat like the moon with the earth, inducing “tides”
- Taking the derivative with respect to ε, we find there is a favored deformation: $\varepsilon_{min} = \frac{-\beta}{2\alpha}$
- Since valence nucleons are necessary to amplify the effect, this predicts ground-state deformation occurring in between closed shells
- Note that the value for β is going to depend on the specific nuclear structure, which shell-model calculations are often used to estimate

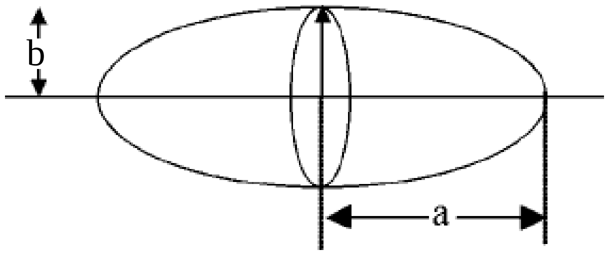
Predicted regions of deformation

B. Harvey, Introduction to Nuclear Physics and Chemistry (1962)



P. Möller et al. ADNDT (2016)



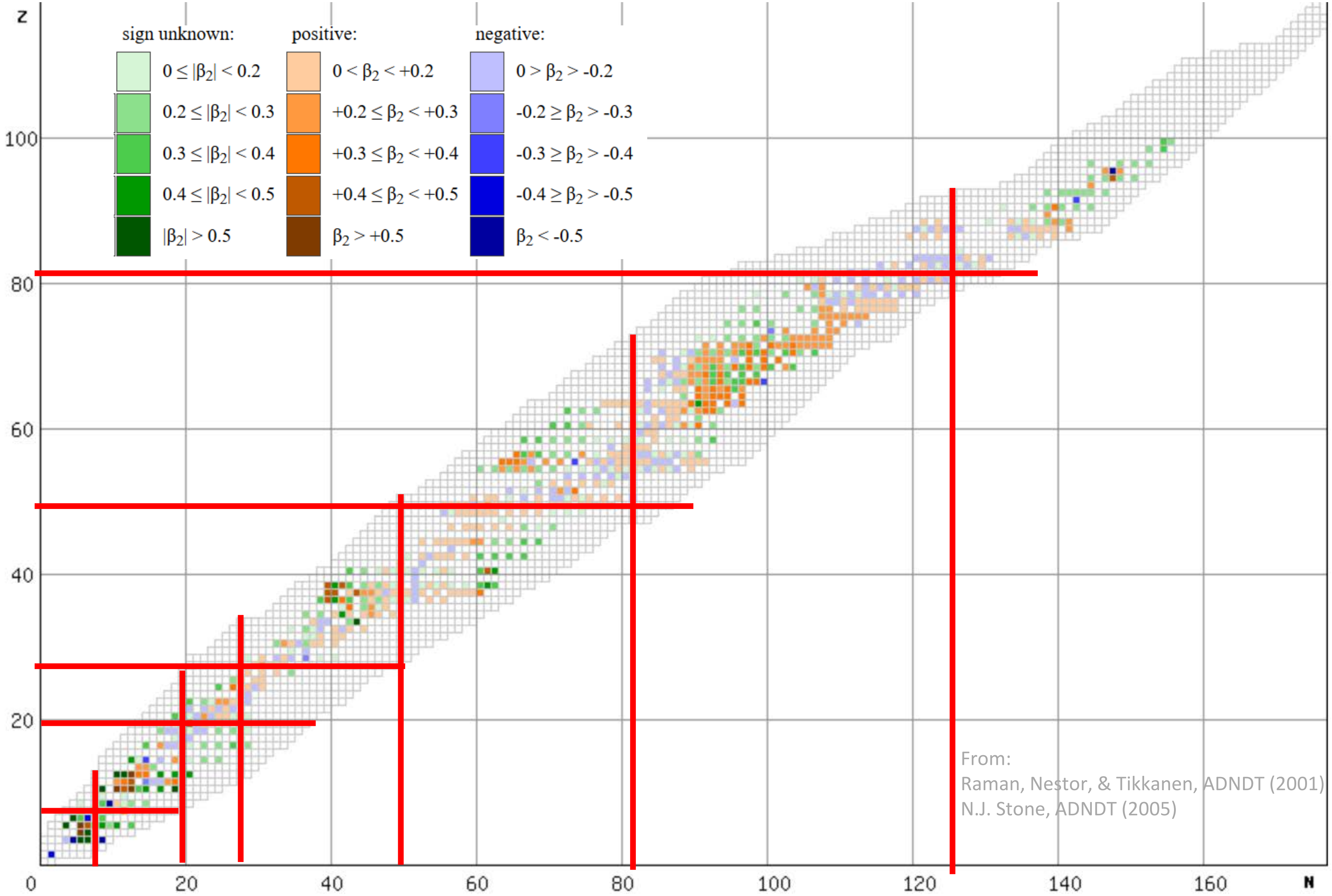


$$\beta_2 \equiv \alpha_{2,0} = \frac{4}{3}\sqrt{\frac{\pi}{5}}\frac{(a-b)}{R_{sph}}$$

Loveland, Morrissey, Seaborg, Modern Nuclear Chemistry (2006)

Measured regions of deformation

Center for Photonuclear Experiments Data



From:
Raman, Nestor, & Tikkanen, ADNDT (2001)
N.J. Stone, ADNDT (2005)

Rotation: *Rigid rotor*

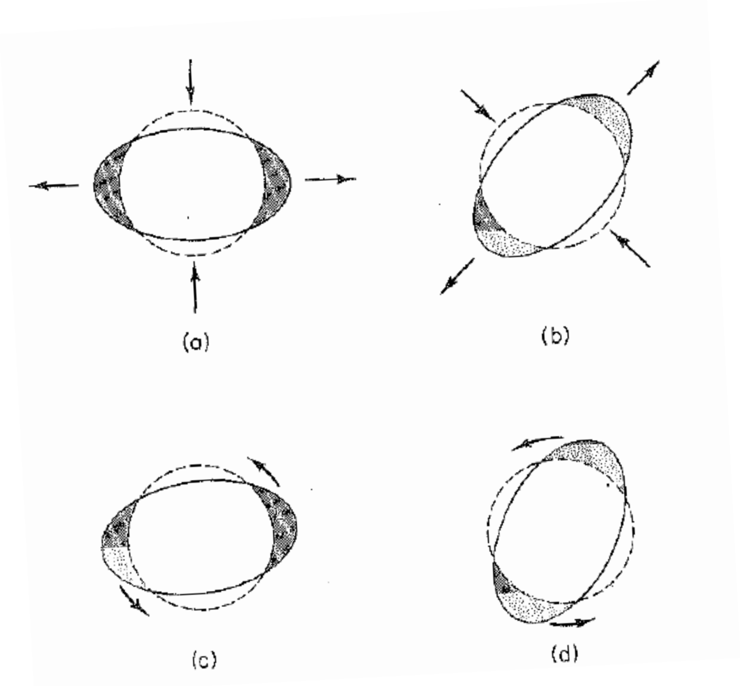
- The energy associated with a rotating object is: $E_{rot} = \frac{1}{2} I \omega^2$
- We’re working with quantum stuff, so we need angular momentum instead where $J = I\omega$
  - So, $E_{rot} = \frac{1}{2}\frac{J^2}{I}$
  - …and J is quantized, so $E_{rot} = \frac{\hbar^2 j(j+1)}{2I}$
- Thus our rotating nucleus will have excited states spaced as $j(j+1)$ corresponding to rotation

- For a solid constant-density ellipsoid, $I_{rigid} = \frac{2}{5} M R^2 (1 + 0.31\beta + 0.44\beta^2 + \cdots)$

  where $\beta = \frac{4}{3}\sqrt{\frac{\pi}{5}} \frac{(a-b)}{R_{sph}}$ (A.Bohr & B.Mottelson, Dan. Matematisk-fysiske Meddelelser (1955))

Rotation: *Irrotational Motion*

- Rather than the whole nucleus rotating, a tide-like effect could produce something like rotation
- Here nucleons just move in and out in a synchronized fashion, kind of like people doing “the wave” in a stadium



B. Harvey, Introduction to Nuclear Physics and Chemistry (1962)

- Since nucleons aren’t orbiting, but are just bobbing in and out, this type of motion is called “irrotational”
- Thankfully, Lord Rayleigh worked-out the moment of inertia for continuous, classical fluid with a sharp surface (also in D.J. Rowe, Nuclear Collective Motion (1970))
- $I_{irro} = \frac{9}{8}\pi M R^2 \beta^2$

Moment of inertia comparison

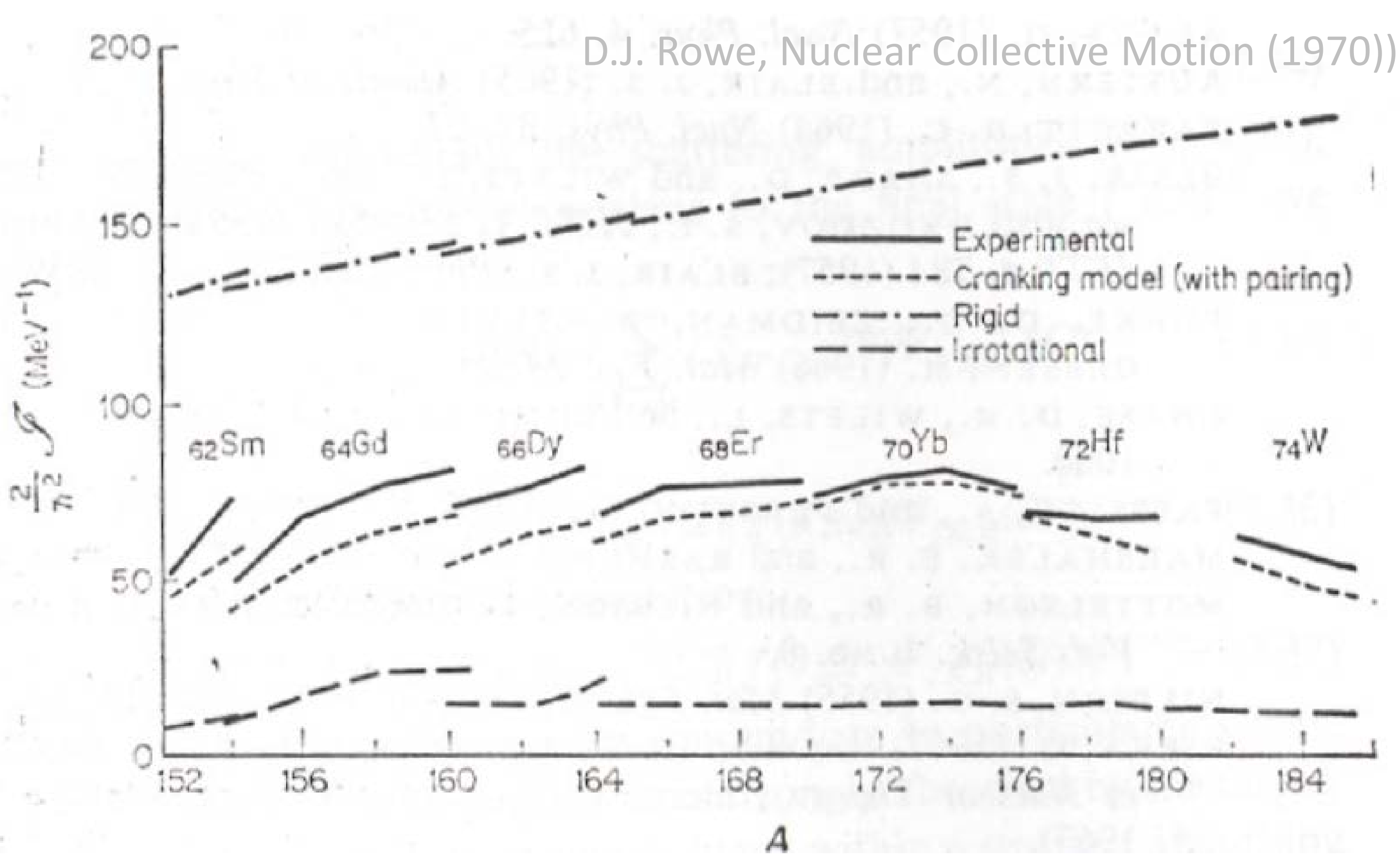
- As an example, we can calculate the moment of inertia for ^{238}Pu.
  - The NNDC chart says for this nucleus $\beta = 0.285$
  - $I_{rigid} = \frac{2}{5}MR^2(1 + 0.31\beta + 0.44\beta^2 + \cdots) \approx \frac{2}{5}A\left(r_0A^{1/3}\right)^2(1 + 0.31\beta + 0.44\beta^2)$
    $= \frac{2}{5}(1.2fm)^2\left(A^{5/3}amu\right)(1 + 0.31\beta + 0.44\beta^2) = 5874\ amu\ fm^2$
  - $I_{irro} = \frac{9}{8}\pi MR^2\beta^2 \approx \frac{9}{8}\pi A\left(r_0A^{1/3}\right)^2\beta^2$
    $= \frac{9}{8}\pi(1.2fm)^2\left(A^{5/3}amu\right)\beta^2 = 3778\ amu\ fm^2$

- We can obtain an empirical rotation constant for ^{238}Pu
  - The energy associated with excitation from the 1st 2^+ excited state to the 1st 4^+ state is:
    $\Delta E = E_{rot}^{4+} - E_{rot}^{2+} = \frac{\hbar^2}{2I}\left(4(4+1)\right) - \frac{\hbar^2}{2I}\left(2(2+1)\right) = 7\frac{\hbar^2}{I}$
  - From NNDC, $E(2_1^+) = 44keV$ & $E(4_1^+) = 146keV$, so $I_{expt} = \frac{7}{102}\hbar^2 keV^{-1}$
  - Take advantage of fact that $\hbar c \approx 197MeV\ fm$ and $1amu \approx 931.5MeV/c^2$,
    $I_{expt} = \frac{14}{0.102MeV}\hbar^2\frac{1amu}{931.5MeV/c^2} = 0.074\frac{\hbar^2c^2}{MeV^2}amu = 0.074\frac{(197MeV\ fm)^2}{MeV^2}amu$
    $= 2859\ amu\ fm^2$

…closer to irrotational

Empirical moment of inertia

- $E_{rot} = \frac{\hbar^2 j(j+1)}{2I}$, so measuring ΔE between levels should give us I
- It turns out, generally: $I_{irro} < I_{expt} < I_{rigid}$



D.J. Rowe, Nuclear Collective Motion (1970))

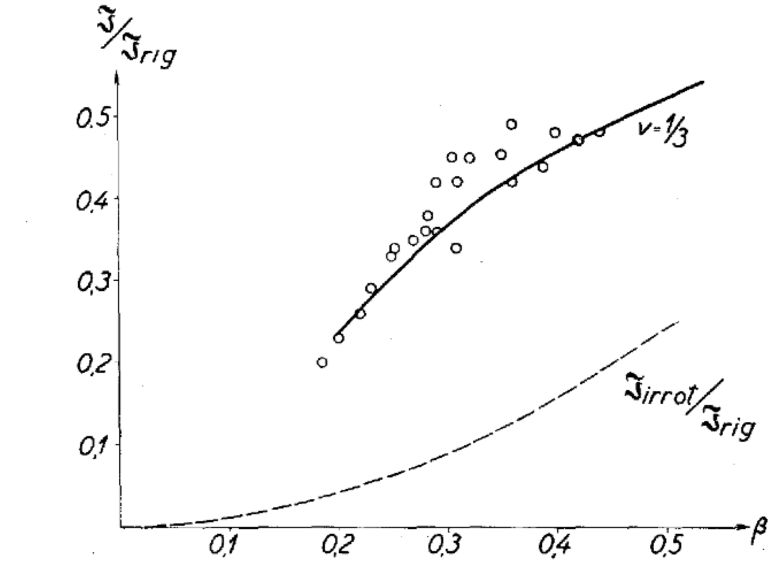
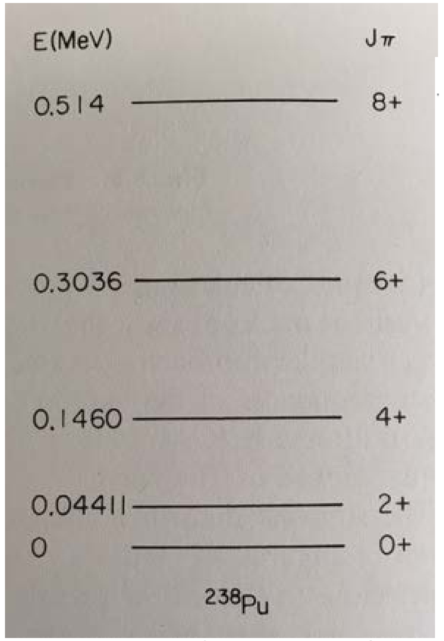


Fig. 2. Dependence of Nuclear Moments of Inertia on the Nuclear Deformation. The empirical moments of inertia for even-even nuclei in the region $150 \leqslant A \leqslant 188$ are plotted as a function of the nuclear deformation. The moments of inertia, obtained from the data in Table I, are given in units of the rigid moment (18), while the deformation parameters β are obtained from the Q_0-values in Table I by means of (19). The nuclear radius has been taken to be $R_0 = 1.2\ A^{1/3}\ 10^{-13}$ cm. The full-drawn curve represents a theoretical estimate, based on the two-nucleon model with an interaction parameter $v = 1/3$ (cf. Fig. 1). For comparison, the moment of inertia corresponding to irrotational flow is shown by the dotted curve.

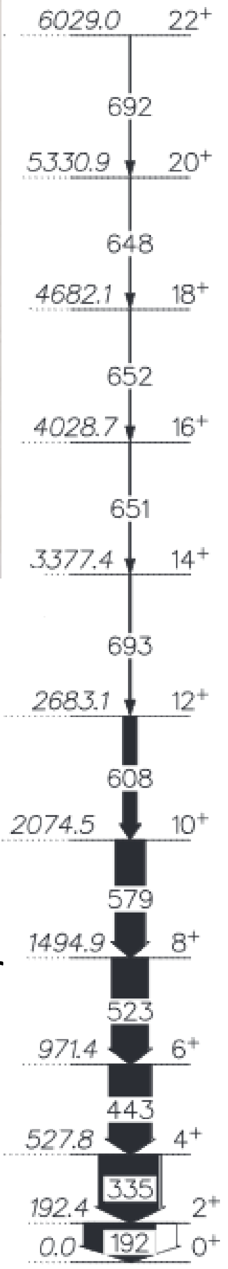
A.Bohr & B.Mottelson, Dan. Matematisk-fysiske Meddelelser (1955))

Rotational bands: sequences of excited states

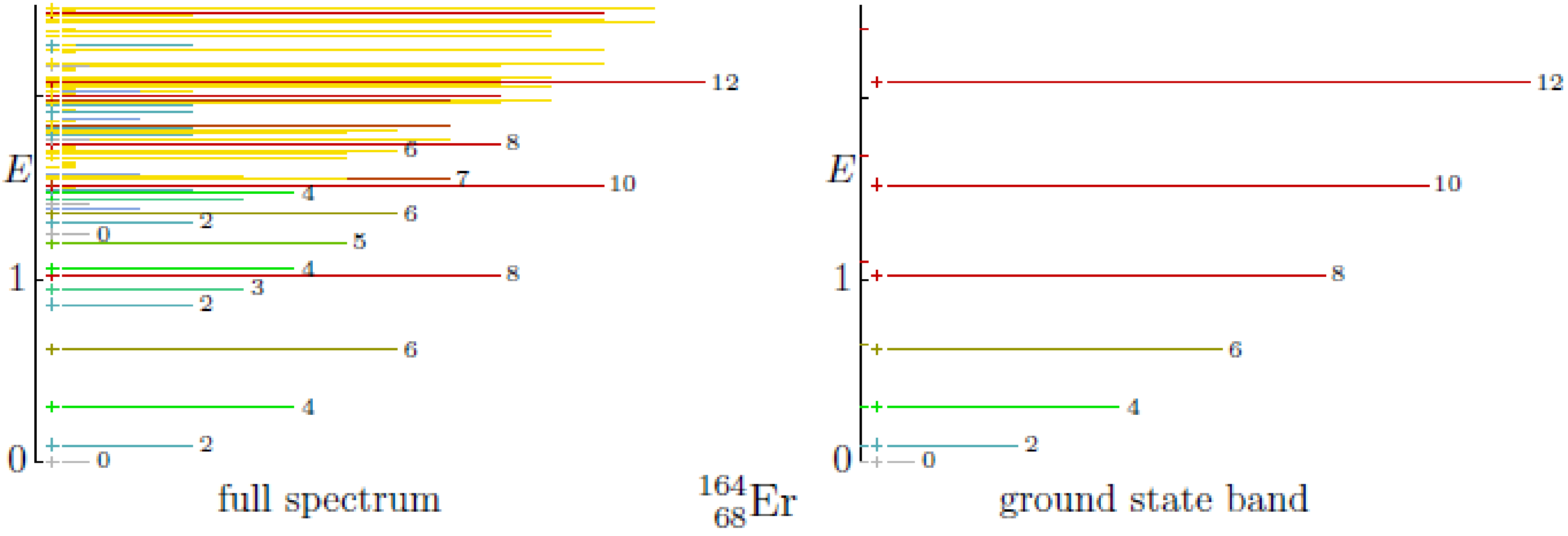
- $E_{rot} = \frac{\hbar^2 j(j+1)}{2I}$, so for a given I, $\Delta E \propto j(j+1)$
- Note that parity needs to be maintained because rotation is symmetric upon reflection and so 0^+ ground-states can only have j=0,2,4,… (because $\pi = (-1)^J$)
- Without observing the decay scheme, picking-out associated rotational states could be pretty difficult
- Experimentally, coincidence measurements allow schemes to be mapped



B. Harvey, Introduction to Nuclear Physics and Chemistry (1962)



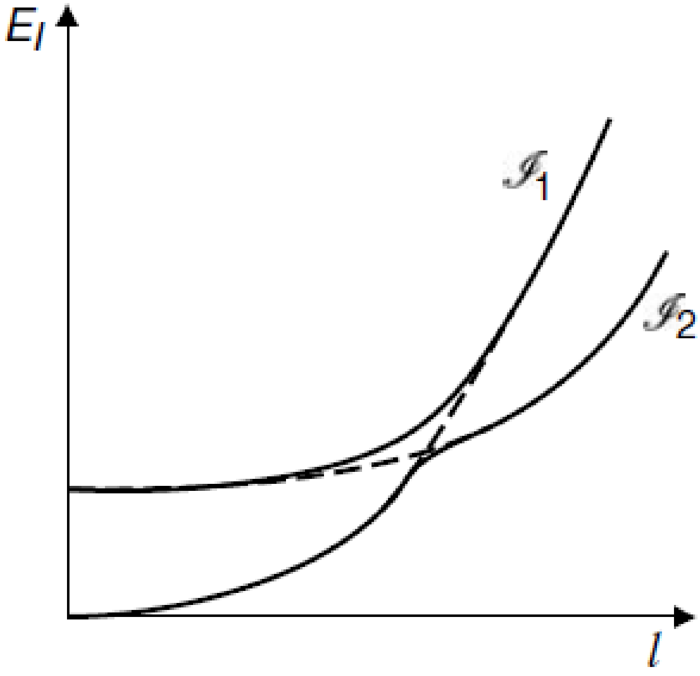
T.Dinoko et al. EPJ Web Conf. (2013)



L. van Dommelen, Quantum Mechanics for Engineers (2012)

Rotational bands

- Rotation can exist on top of other excitations
- As such, a nucleus can have several different rotational bands and the moment of inertia $\mathcal{I}$ is often different for different bands
- The different $\mathcal{I}$ lead to different energy spacings for the different bands



Loveland, Morrissey, Seaborg, Modern Nuclear Chemistry (2006)

T.Dinoko et al. EPJ Web Conf. (2013)

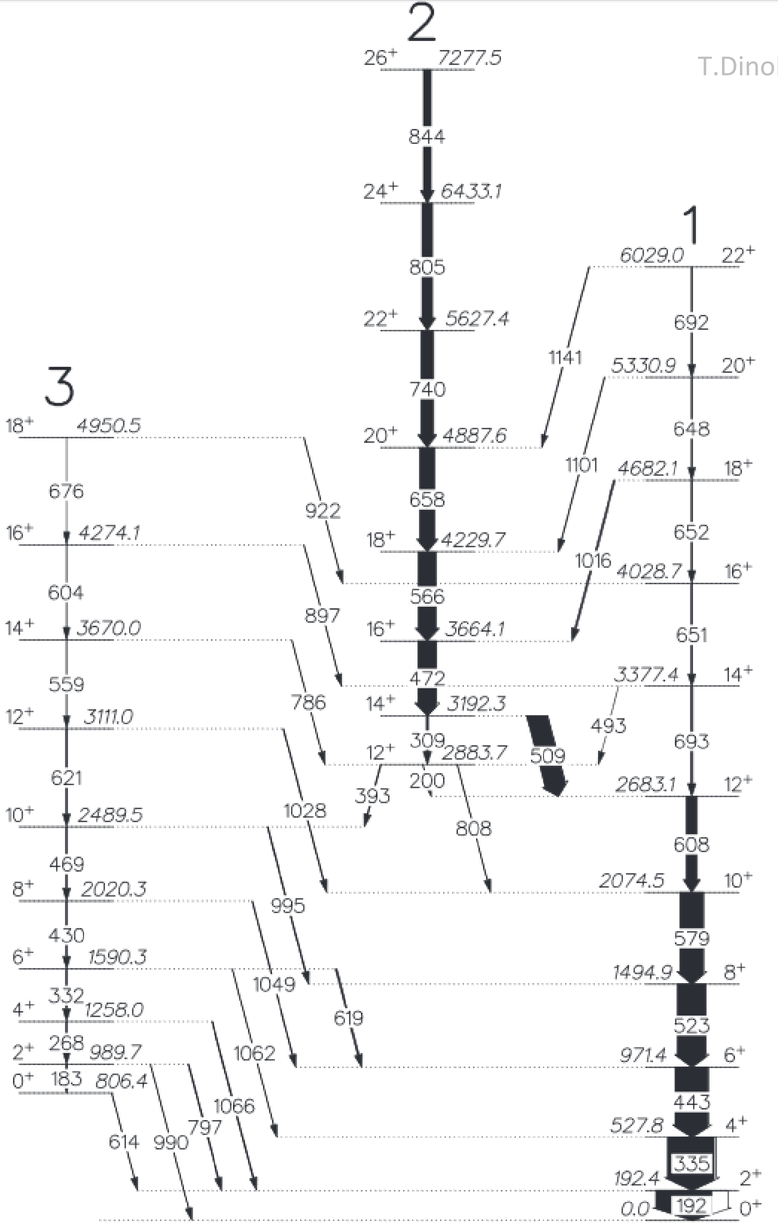
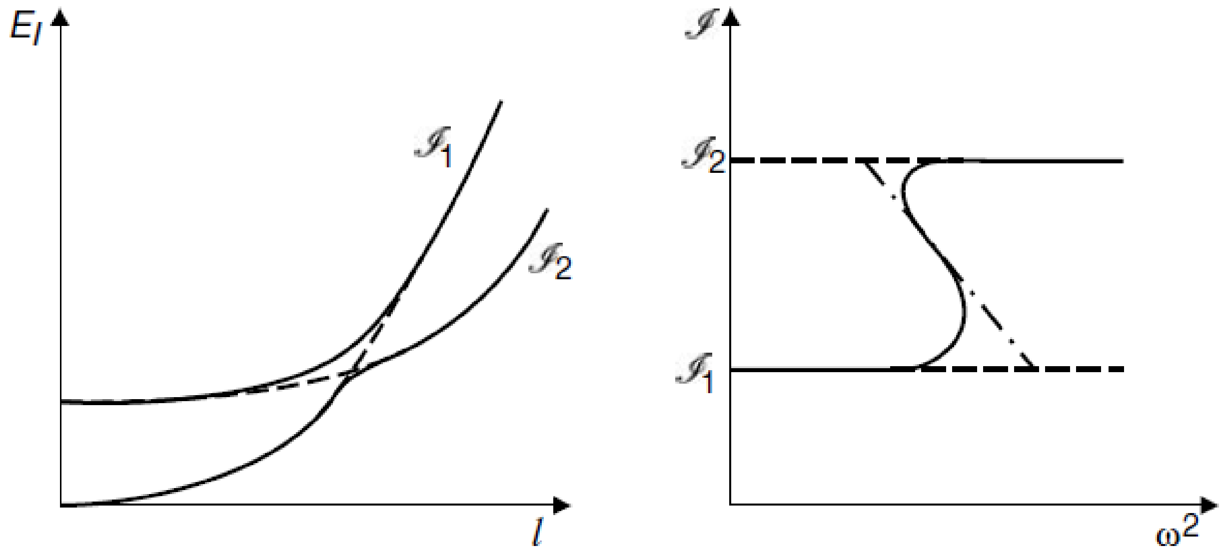


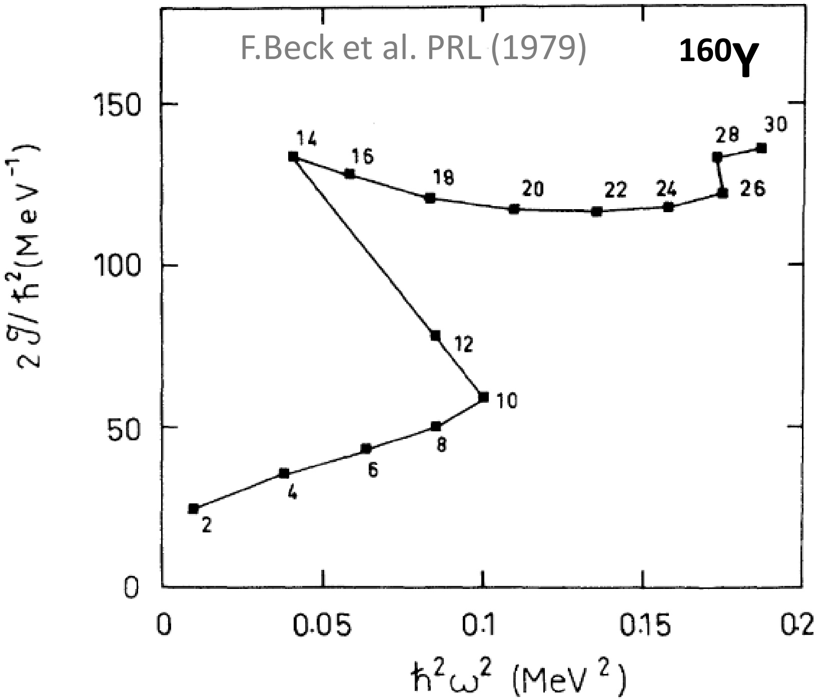
Figure 1: Partial decay scheme of ^{158}Er showing the ground-state, S and 0_2^+ bands, which are labelled 1, 2 and 3, respectively.

Rotational bands: Backbend

- The different I for different rotational bands creates the so-called "backbend"
- This is when we follow the lowest-energy state for a given spin-parity (the "yrast" state) belonging to a given rotational band and plot the moment of inertia and square of the rotational frequency

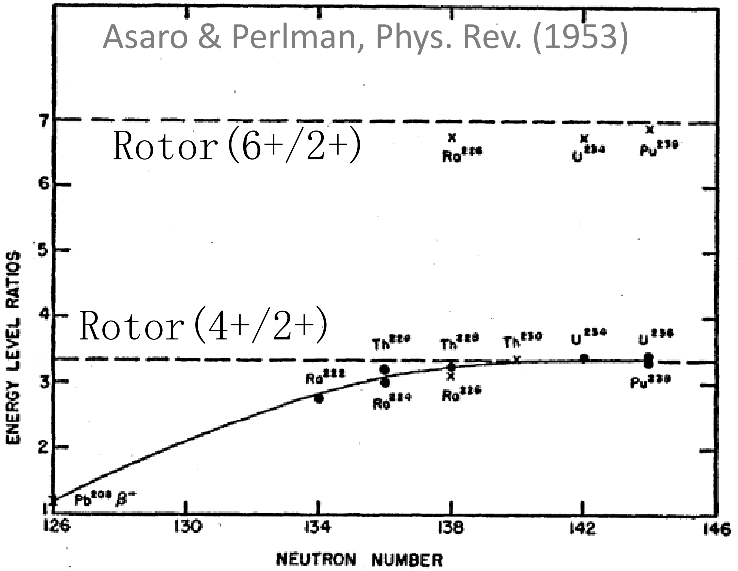


Loveland, Morrissey, Seaborg, Modern Nuclear Chemistry (2006)

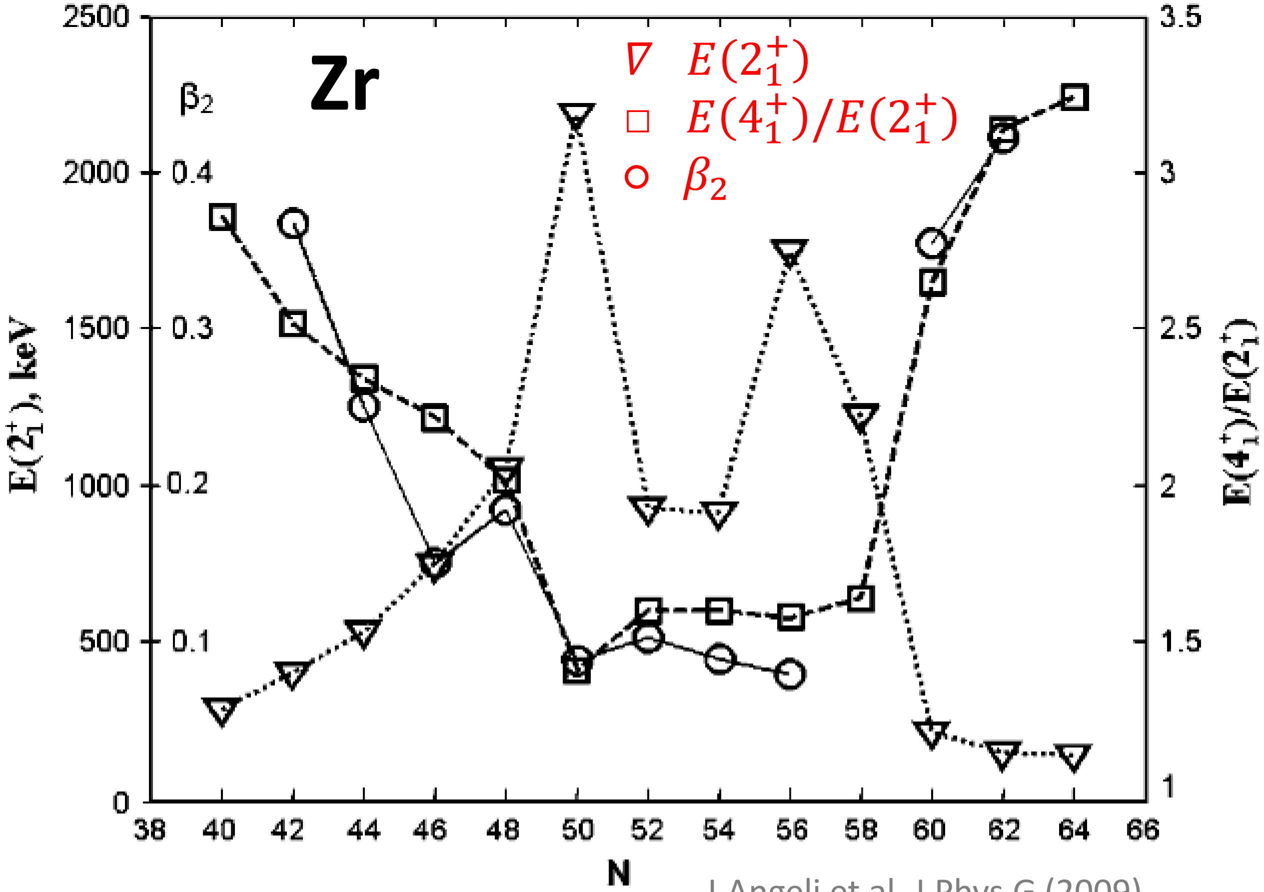


Inferring structure from rotational bands

- Since $E_{rot} = \frac{\hbar^2 j(j+1)}{2I}$ and $I \propto \beta$, we can use rotational bands to probe deformation
- More deformed nuclei have larger β, so excited state energies for the band should be low
    - The 1st 2+ excited state energy is often used to probe this
- The rotor model for rotational bands is validated by the comparison of band excited state energy ratios to the rotor prediction
- The ratio of the yrast 4^+ and 2^+ excited state energies is generally close to the rotor prediction for nuclei far from closed shells



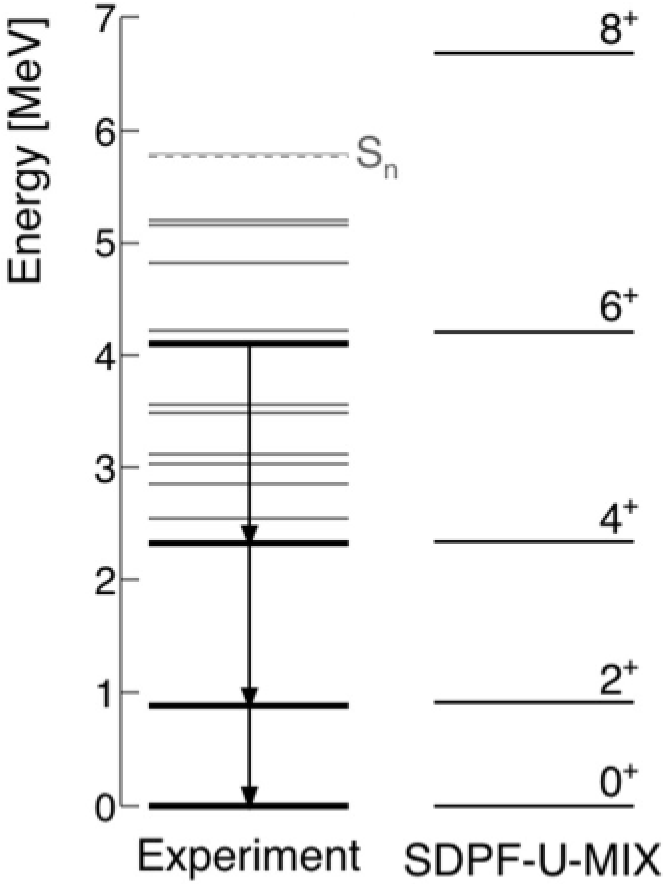
Asaro & Perlman, Phys. Rev. (1953)



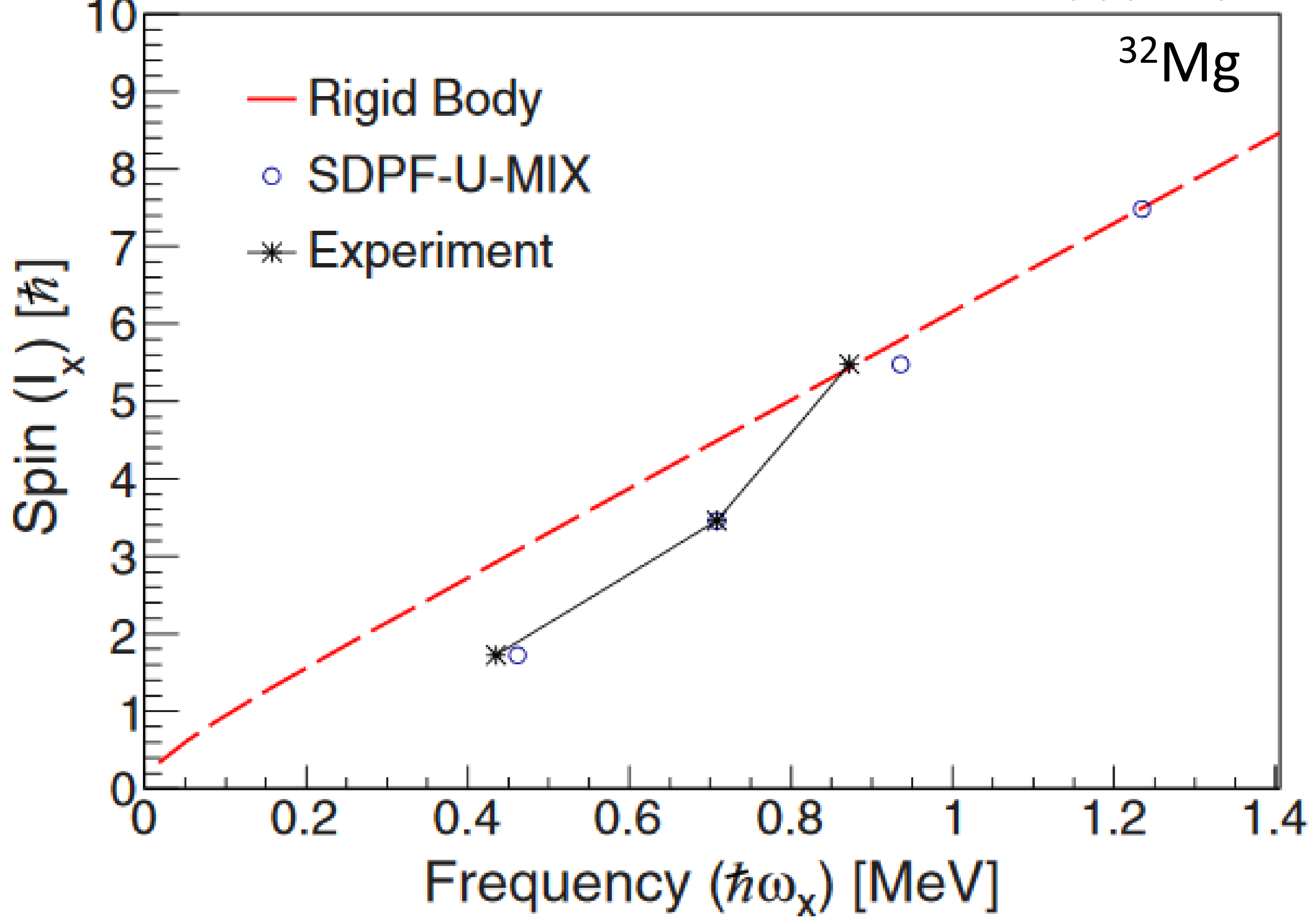
I.Angeli et al. J.Phys.G (2009)

Inferring structure from rotational bands

- To keep life interesting, I can change for a single band, indicating a change in structure, e.g. how particular nucleons or groups of nucleons are interacting



H.Crawford et al. PRC(R) (2016)



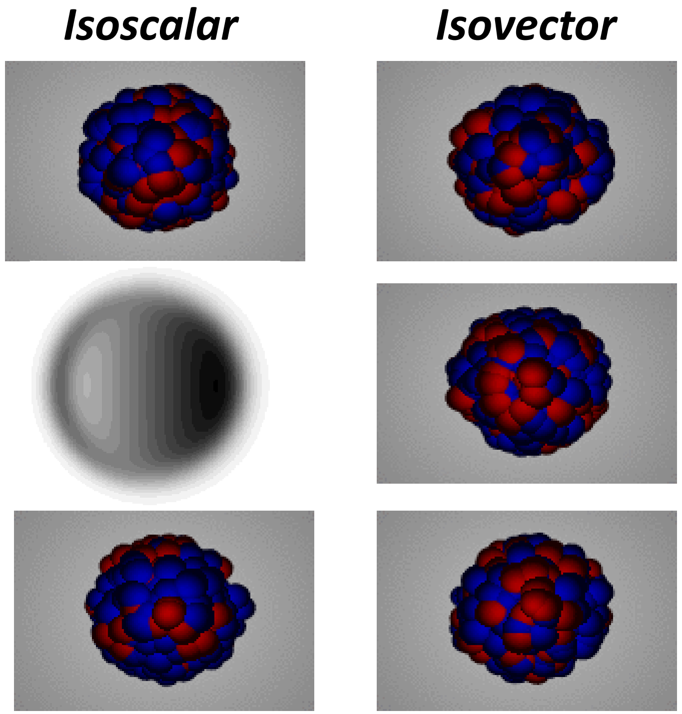
Vibrational modes

- Considering the nucleus as a liquid drop, the nuclear volume should be able to vibrate

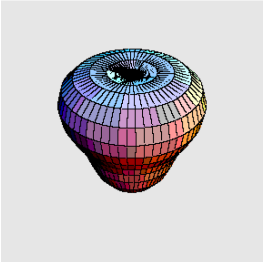
H.J. Wollersheim

- Several multipoles are possible
  - Monopole: in & out motion ($\lambda = 0$)
    - a.k.a the breathing mode

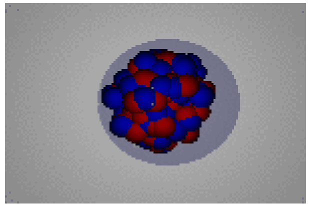
The gifs to the right are for giant resonances, when all protons/neutrons act collectively



  - Dipole: sloshing back & forth ($\lambda = 1$)
    - If all nucleons are moving together, this is just CM motion
  - Quadrupole: alternately compressing & stretching ($\lambda = 2$)
  - Octupole: alternately pinching on one end & then the other ($\lambda = 3$)
  - + Higher



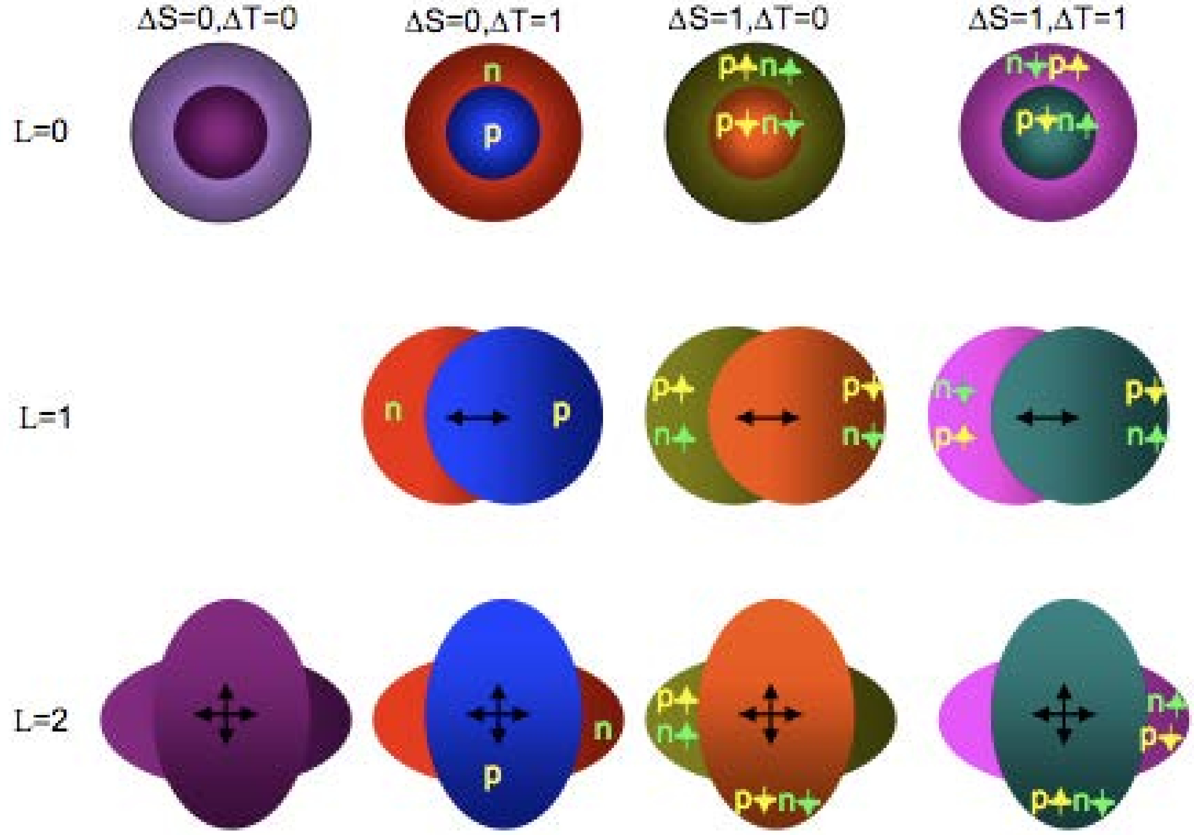
- Protons and neutrons can oscillate separately (“isovector” vibrations)
- All nucleons need not move together
  - e.g. the “pygmy dipole” is the neutron skin oscillation



Vibrational modes

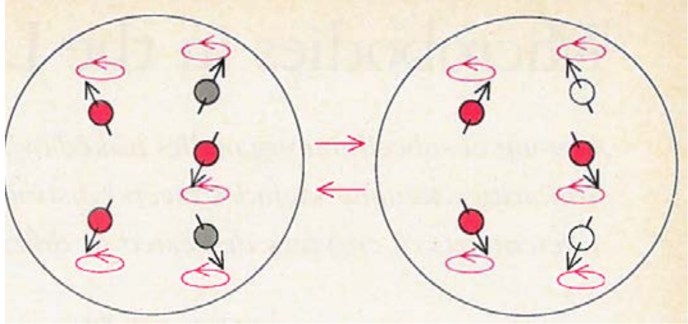
- Additionally, oscillations can be grouped by spin

  …leaving a pretty dizzying range of possibilities

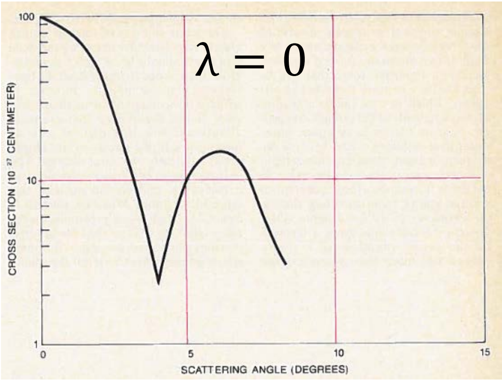


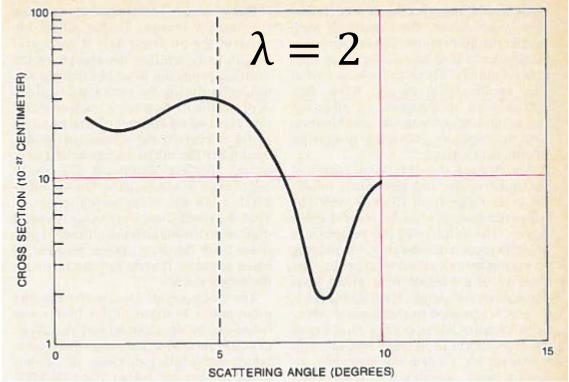
TAMU

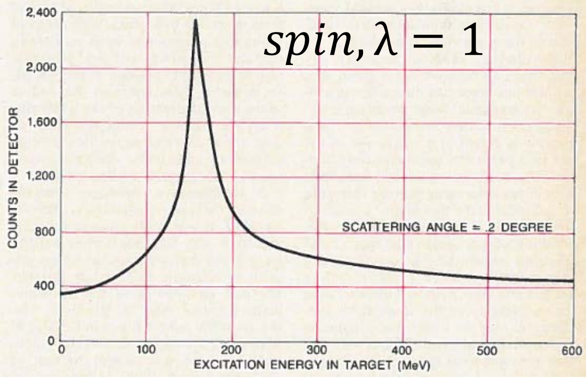
G.Bertsch, Sci.Am. (1983)



Scattering experiments are key to identifying the vibrational properties of particular excited states because one obtains characteristic diffraction patterns.







The myriad of possible nuclear vibrations are discussed in a friendly manner here:
[Vibrations of the Atomic Nucleus, G. Bertsch, Scientific American (1983)]

Rough energetics of vibrational excitations

- In essence, a nuclear vibration is like a harmonic oscillator
- There is some oscillating deviation from a default shape and a restoring force attempts to return the situation to the default shape
- The restoring force differs for each mode and so therefore do the characteristic frequencies ω, which have a corresponding energy $\hbar\omega$
  - Nuclear matter is nearly incompressible, so the monopole oscillation takes a good bit of energy to excite.

    For even-even nuclei, the monopole oscillation creates a 0^+ *state at* $\approx 80A^{-1/3}MeV$
  - Neutrons and protons are relatively strongly bound together, so exciting an isovector dipole also takes a good bit of energy

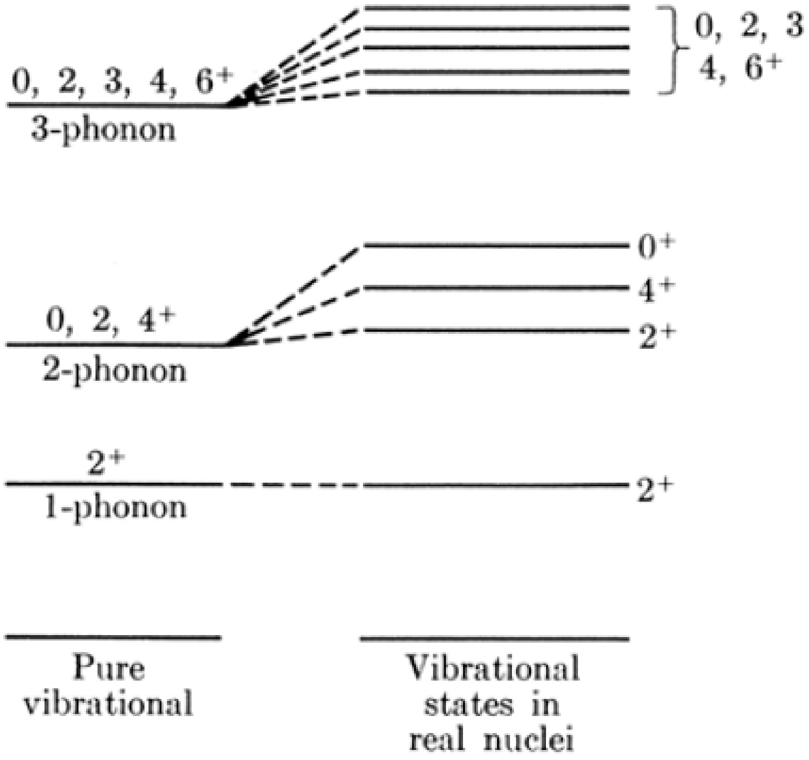
    For even-even nuclei, the dipole oscillation creates a 1^- *state at* $\approx 77A^{-1/3}MeV$
  - The squishiness of the liquid drop is more amenable to quadrupole excitations, so these are the lowest-energy excitations

    For even-even nuclei, the quadrupole oscillation creates a 2^+ *state at ~1-2MeV.*
    The giant quadrupole oscillation is at $\approx 63A^{-1/3}MeV$
  - Similarly, octupolar shapes can also be accommodated

    For even-even nuclei, the octupole oscillation creates a 3^- *state at ~4MeV*

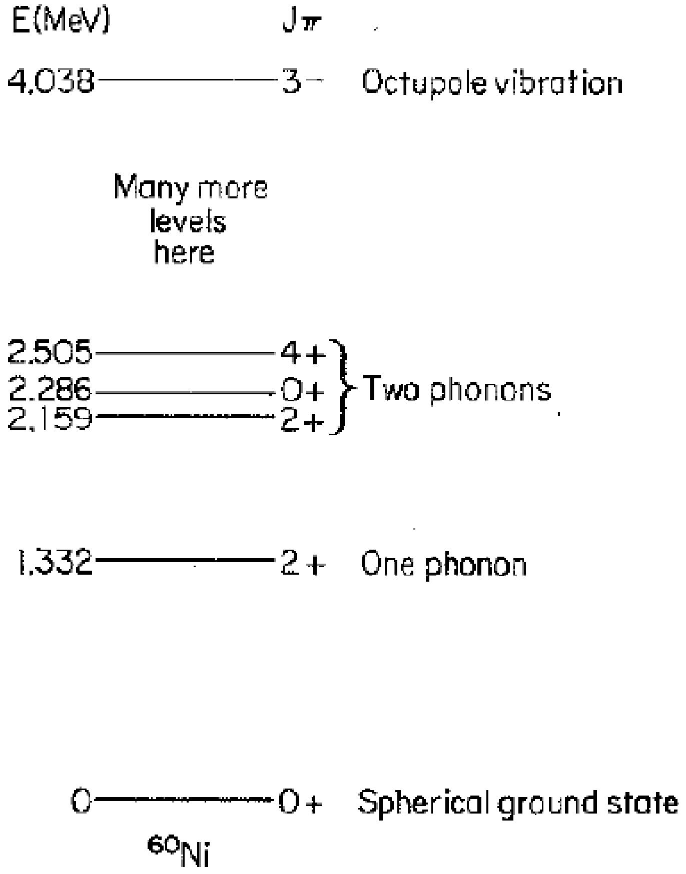
Vibrational energy levels

- Just as the quantum harmonic oscillator eigenvalues are quantized, so too will the energy levels for different quanta (*phonons*) of a vibrational mode.
- Similarly, the energy levels have an even spacing, $E_n = (n + \frac{1}{2})\hbar\omega$
- Even-even nuclides have 0+ ground states, and thus, for a $\lambda = 2$ vibration, $n = 2$ excitations will maintain the symmetry of the wave-function (i.e. $n = 1$ excitations would violate parity)
- Therefore, the 1st vibrational state will be 2^+
- We can excite an independent quadrupole vibration by adding a second phonon
- The second phonon will build excitations on the first, coupling to either 0^+,2^+, or 4^+
- Employing a nuclear potential instead winds up breaking the degeneracy for states associated with a given number of phonons

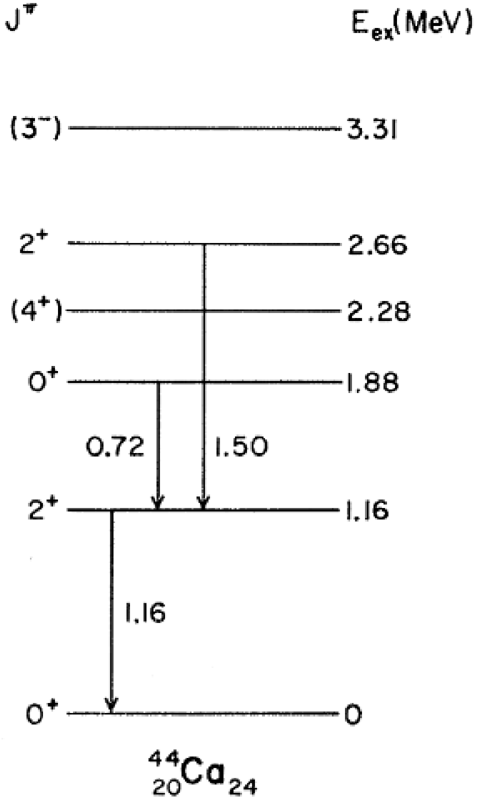


Loveland, Morrissey, Seaborg, Modern Nuclear Chemistry (2006)

Vibrational energy levels are pretty obvious for spherical nuclei

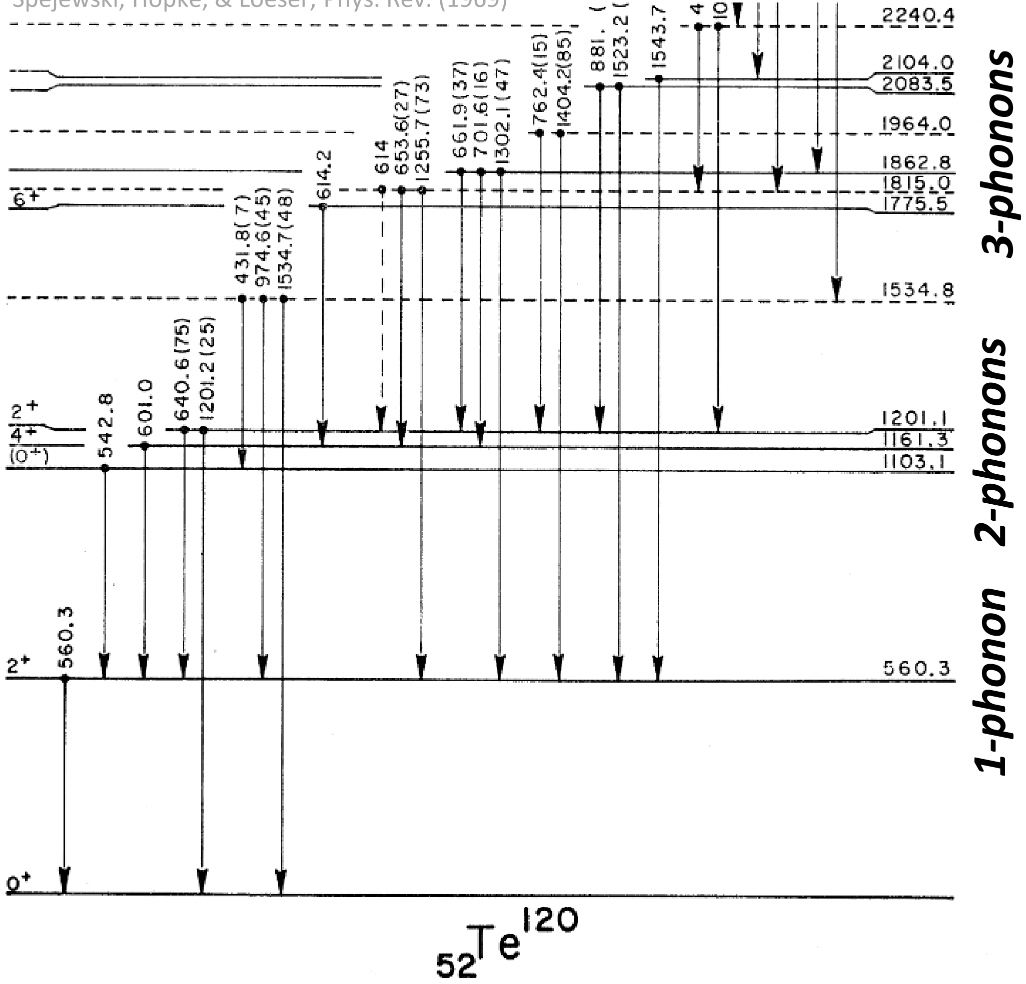


G. Harvey, Introduction to Nuclear Physics and Chemistry (1962)



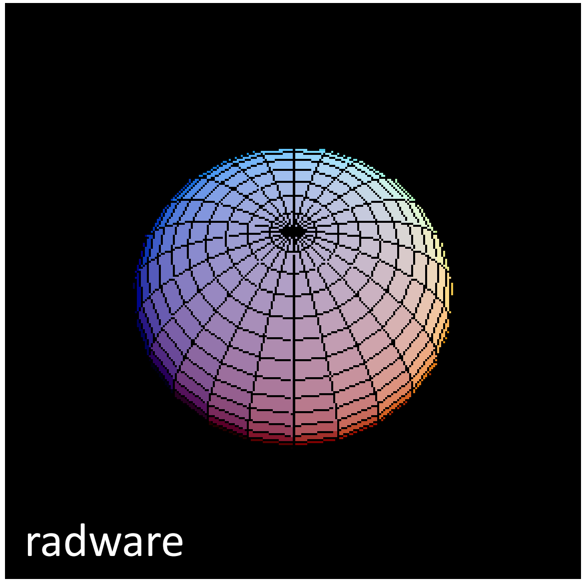
Matin, Church, & Mitchell Phys. Rev. (1966)

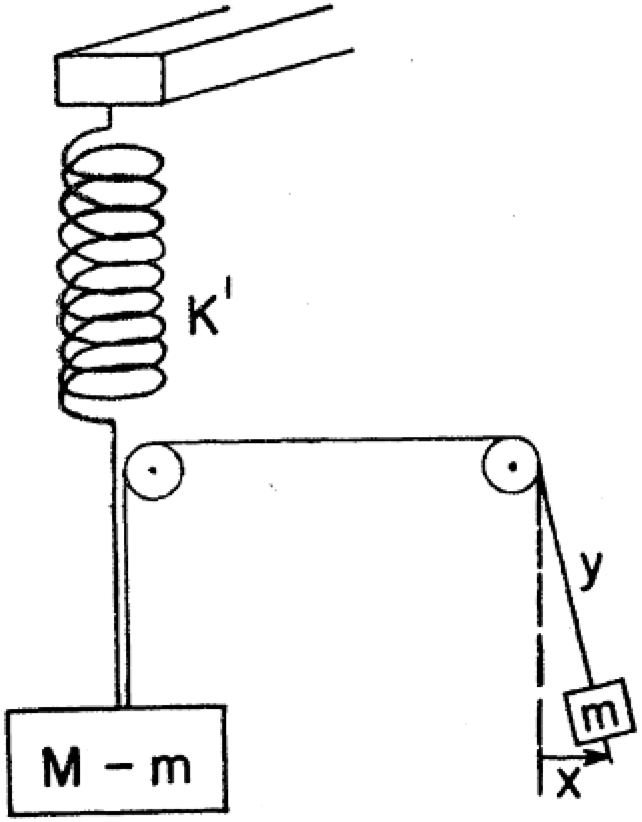
Spejewski, Hopke, & Loeser, Phys. Rev. (1969)



- The typical signature for vibrating spherical nucleus is $E(4_1^+)/E(2_1^+) \approx 2$
  …though that obviously won't be the case for a deformed nucleus

Yo dawg, I heard you like collective excitations



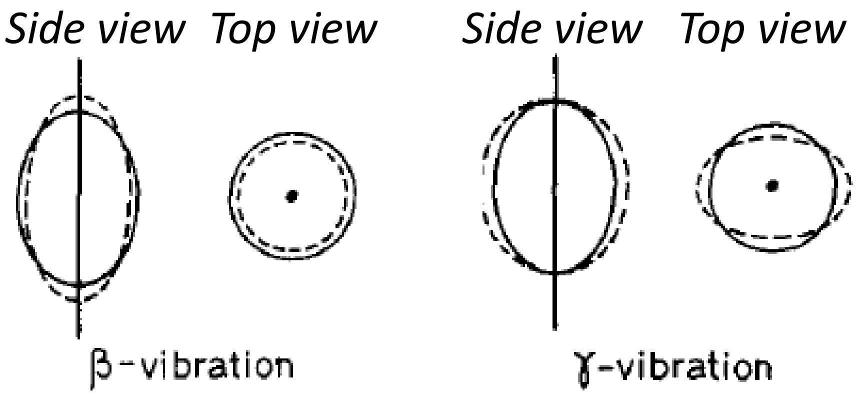


D. Inglis, Phys. Rev. (1955)

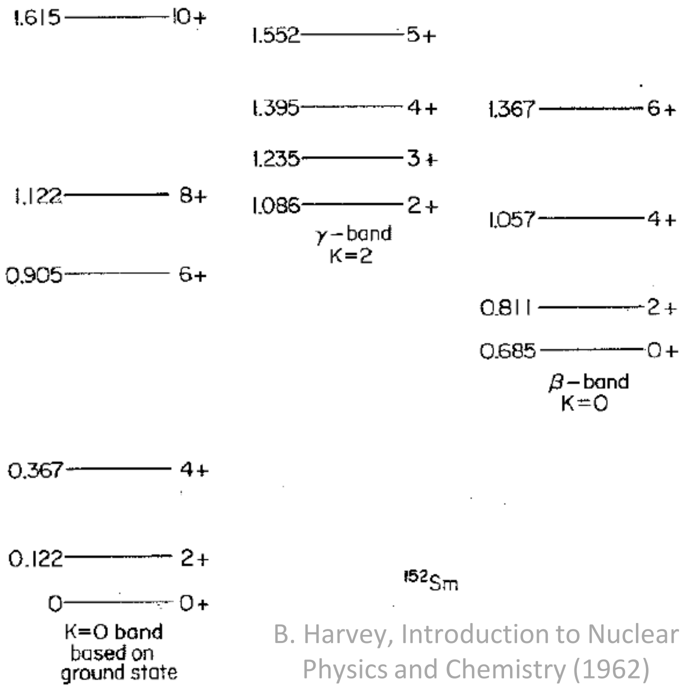
so I put some rotations on your vibrations so you can oscillate while you rotate

Rotational bands can build on vibrational states

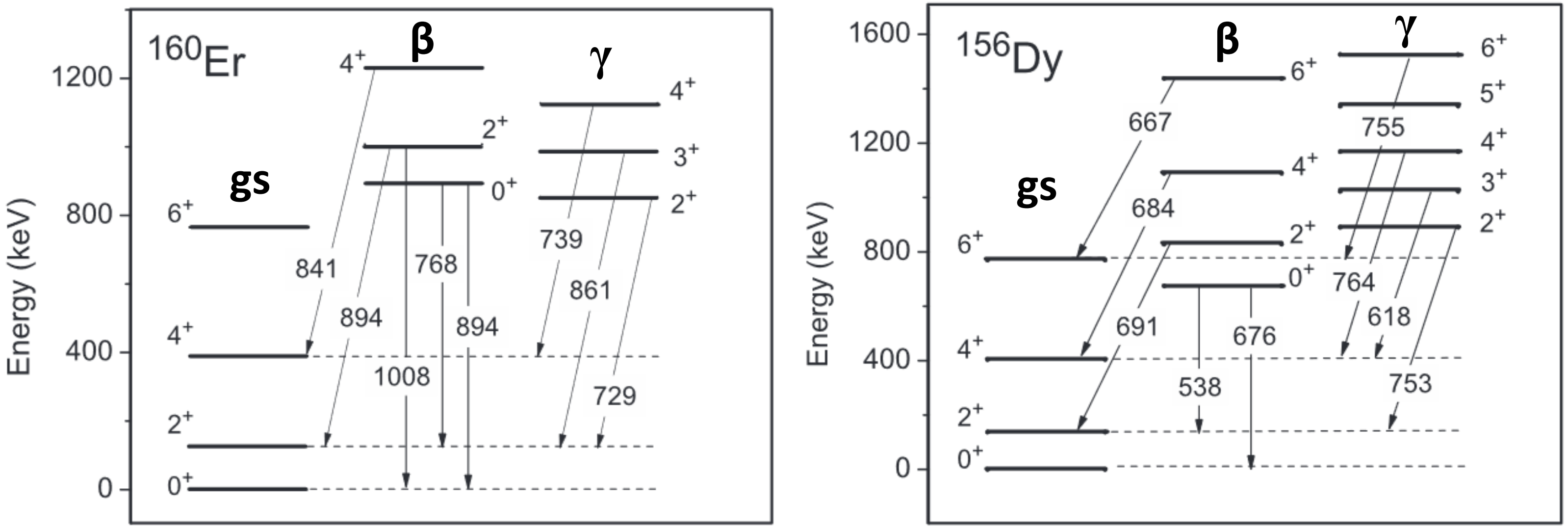
- Deformed nuclei can simultaneously vibrate and rotate
- The coupling depends on whether the vibration maintains axial symmetry or not
- The two types, β and γ, are in reference to how the vibration deforms the shape in terms of Hill-Wheeler coordinates (Hill & Wheeler, Phys. Rev. (1953))
- Exemplary spectra:



Bohr & Mottelson, Nuclear Structure Volume II (1969)



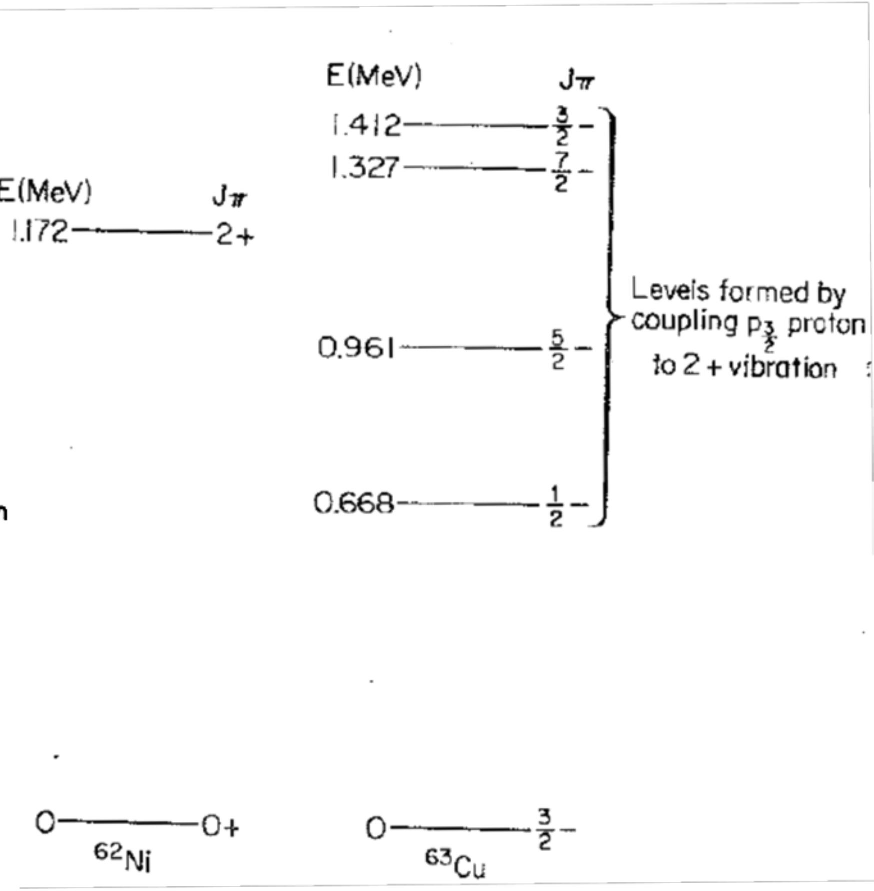
B. Harvey, Introduction to Nuclear Physics and Chemistry (1962)



N. Blasi et al. Phys.Rev.C (2014)

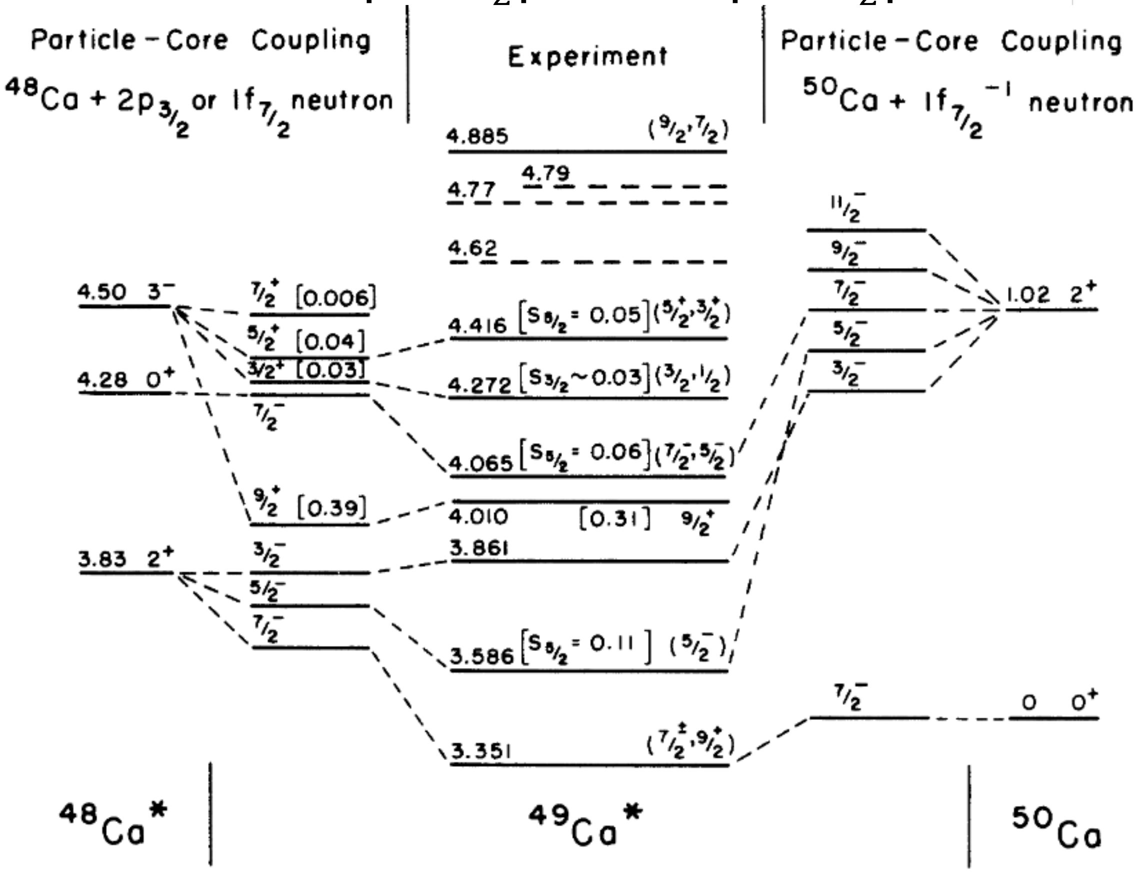
Single-particle states can build on vibrational states

- For some odd-A nuclei, excited states appear to result from the unpaired nucleon to a vibrational phonon
- For ^{63}Cu, the ground state has an unpaired $p_{3/2}$ nucleon
- Coupling this to a 2$^+$ state allows $\left|2-\frac{3}{2}\right| \le j \le \left|2+\frac{3}{2}\right|$, i.e. $\frac{1}{2}^-, \frac{3}{2}^-, \frac{5}{2}^-, \frac{7}{2}^-$



B. Harvey, Introduction to Nuclear Physics and Chemistry (1962)

- Another example:



Canada, Ellegaard, & Barnes, Phys.Rev.C (1971)

Recap of basic structure models discussed thus far

- Schematic shell model
  - Great job for ground-state J^{π}
  - Decent job of low-lying excited states for spherical nuclei, particularly near closed shells
  - Miss collective behavior that arises away from shell closures
- Collective model
  - Rotational excitations explain several J^{π} for deformed, **even-even** nuclei
    These are "mid shell" nuclei, because they're not near a shell closure
  - Vibrational excitations explain several J^{π} for spherical, **even-even** nuclei
    These are "near shell" nuclei, because they're near a shell closure
  - Miss single-particle behavior that can couple to collective excitations

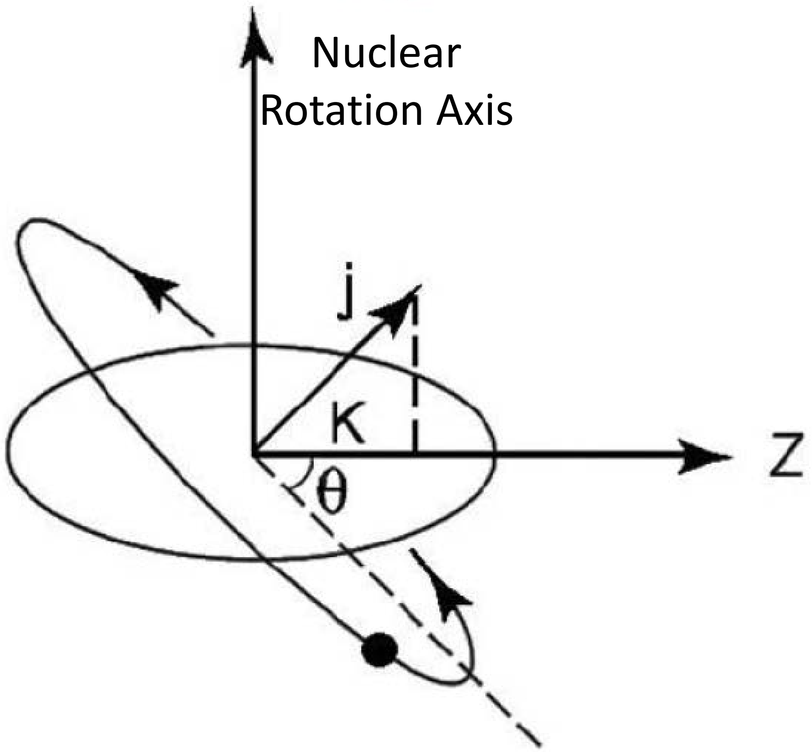
What do we do for collective behavior for odd-A nuclei?
the Nilsson model (a.k.a. deformed shell model)

Nilsson Model: combining collective & single-particle approaches

- Our schematic shell model was working perfectly fine until you threw it away like a cheap suit because of a little deformation!

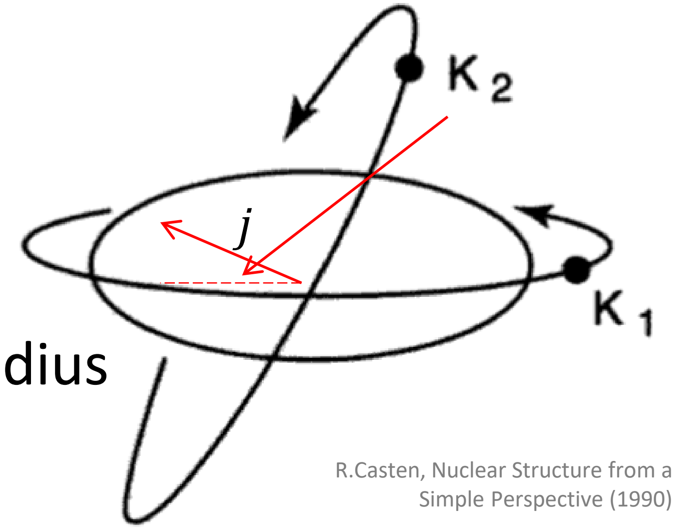
Luckily, Mottelson & Nilsson (Phys. Rev. 1955) weren't so hasty.

- Consider a deformed nucleus with axial symmetry that has a single unpaired nucleon orbiting the nucleus
- *We're sticking to axial symmetry because*
  - *Most deformed nuclei have this property (mostly prolate)*
  - *The math, diagrams, and arguments are easier*
- The nucleon has some spin, j, which has a projection onto the axis of symmetry, K
- j has $\frac{2j+1}{2}$ possible projections K
- Therefore, each single particle state from our shell model now splits into multiple states, identified by their K, each of which can contain two particles (spin up and spin down) for that given projection



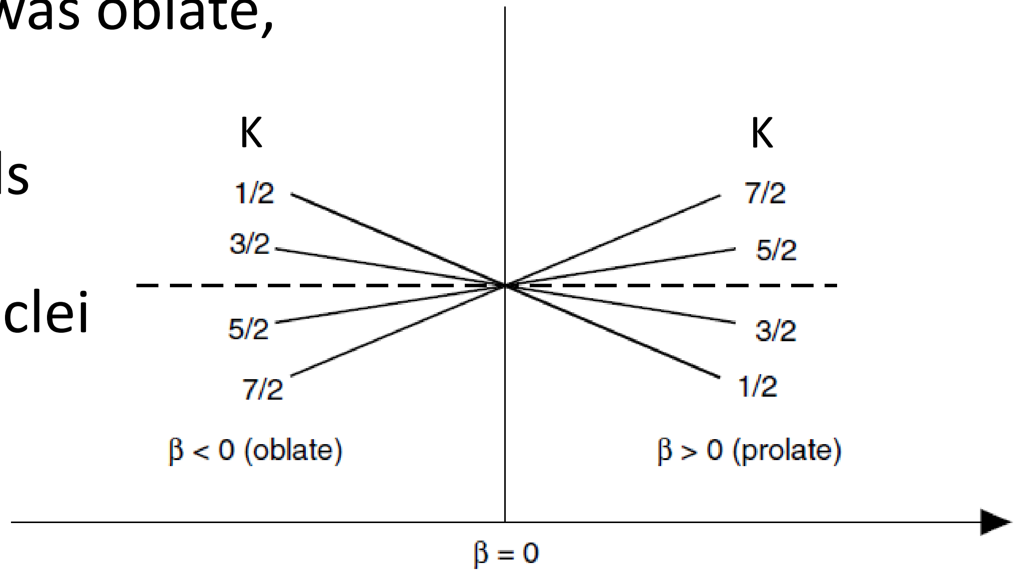
R.Casten, Nuclear Structure from a Simple Perspective (1990)

Nilsson model: single-particle level splitting



R.Casten, Nuclear Structure from a Simple Perspective (1990)

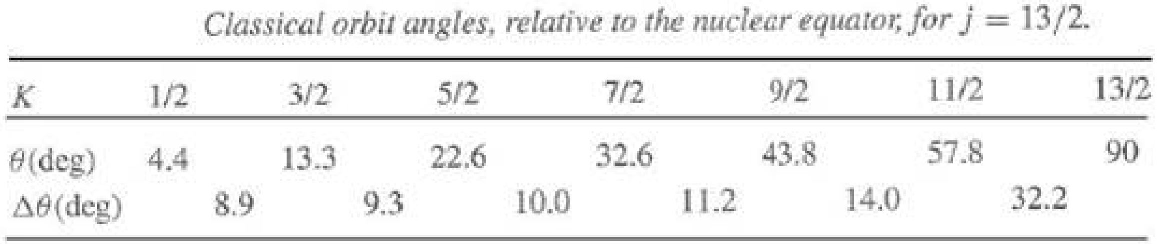
- Consider the options for our nucleon's orbit around the nucleus
- Orbits with the same principle quantum number will have the same radius
- Notice that the orbit with the smaller projection of j (K_1) sticks closer to the bulk of the nucleus during its orbit
- Since the nuclear force is attractive,
  the K_1 orbit will be more bound (i.e. lower energy) than the K_2 orbit
- The opposite would be true if the nucleus in our picture was oblate, squishing out toward the K_2 orbit
- Therefore, for prolate nuclei, lower K single-particle levels will be more bound (lower-energy),
  whereas larger K states will be more bound for oblate nuclei



Loveland, Morrissey, Seaborg, Modern Nuclear Chemistry (2006)

Nilsson model: single-particle level splitting

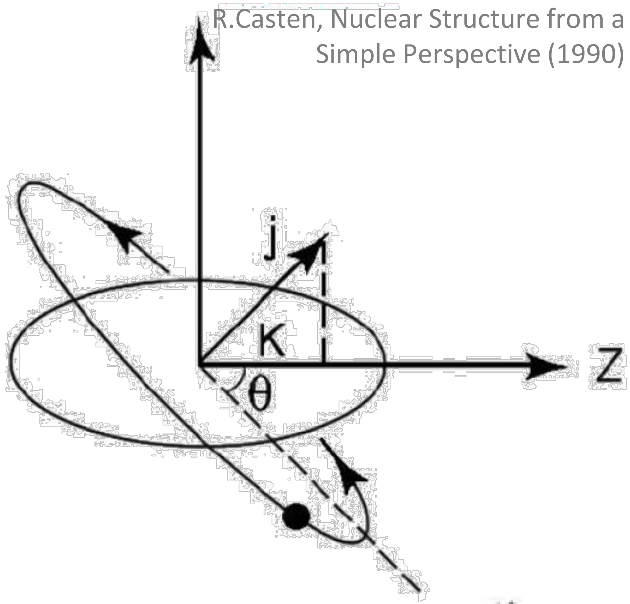
- Continuing with our schematic picture, we see that the proximity of the orbiting nucleon to the nucleus isn't linear with K, since $\sin\theta \sim \frac{K}{j}$



Classical orbit angles, relative to the nuclear equator, for $j = 13/2$.

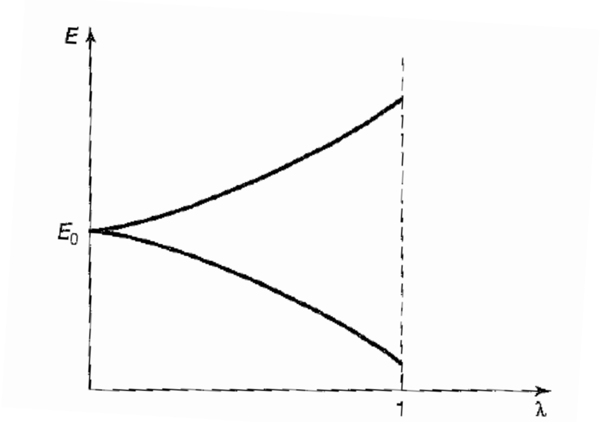
K	1/2		3/2		5/2		7/2		9/2		11/2		13/2
θ(deg)	4.4		13.3		22.6		32.6		43.8		57.8		90
$\Delta\theta$(deg)		8.9		9.3		10.0		11.2		14.0		32.2	

R.Casten, Nuclear Structure from a Simple Perspective (1990)

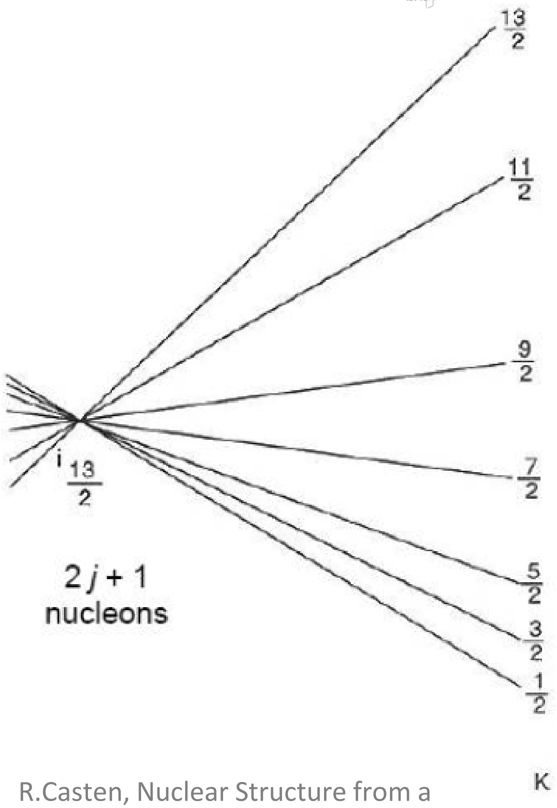


R.Casten, Nuclear Structure from a Simple Perspective (1990)

- So the difference in binding for $\Delta K = 1$ increases as K increases
- Now, considering the fact that single particle levels of different j can have the same projection K, we arrive at a situation that is essentially the two-state mixing of degenerate perturbation theory, where it turns out the perturbation breaks the degeneracy and causes the states to repel each other in a quadratic fashion where the strength of the deflection depends on the proximity of the states in energy



D.Griffiths, Introduction to Quantum Mechanics (1995)

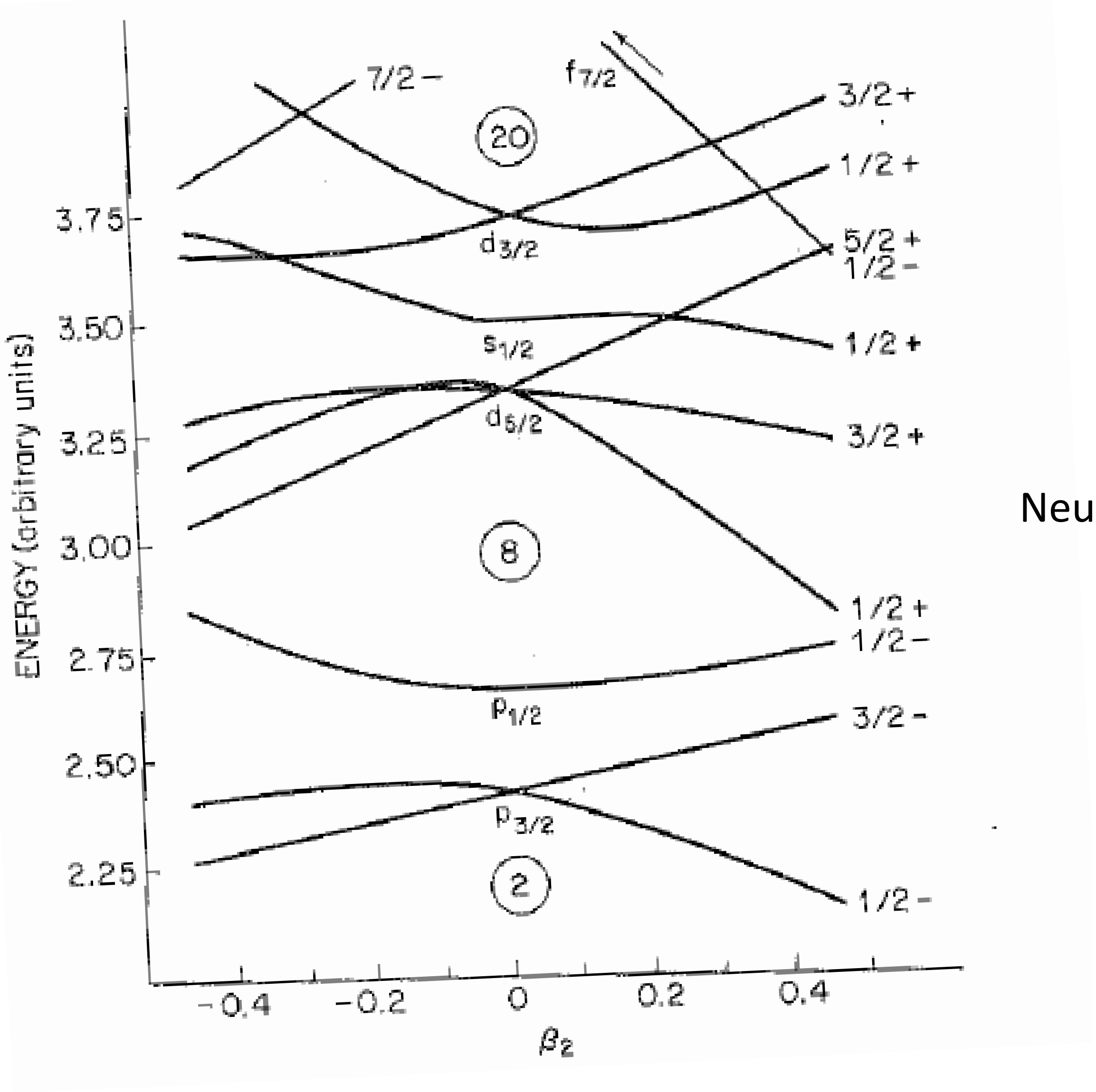


R.Casten, Nuclear Structure from a Simple Perspective (1990)

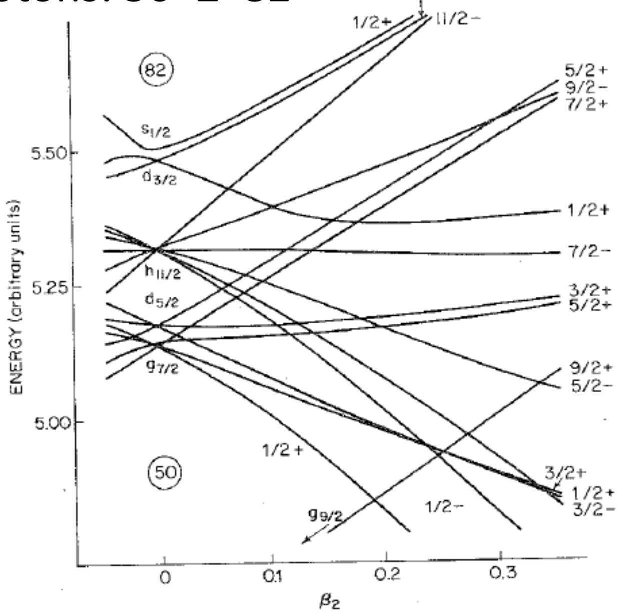
Nilsson Model: single particle levels vs β

B. Harvey, Introduction to Nuclear Physics and Chemistry (1962)

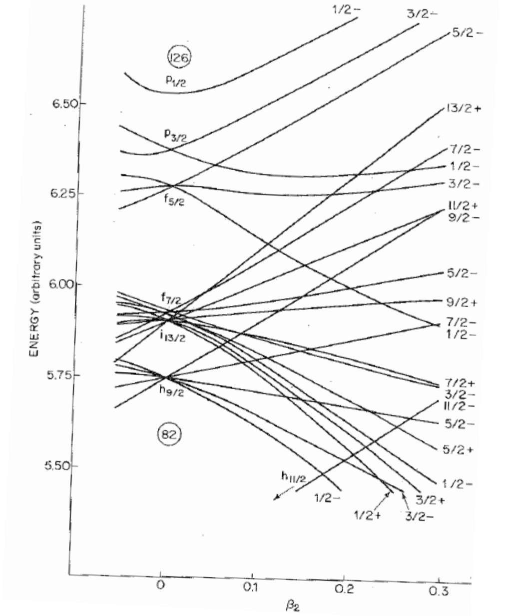
Nucleons: $8 < Z < 20$ *or* $8 < N < 20$



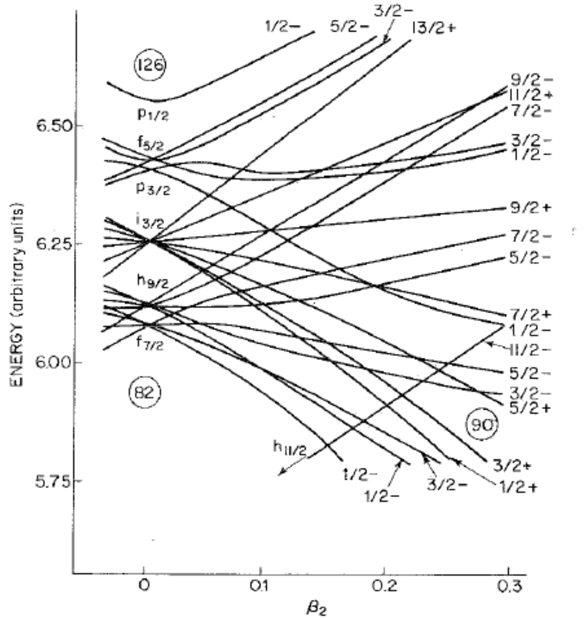
Protons: 50<Z<82



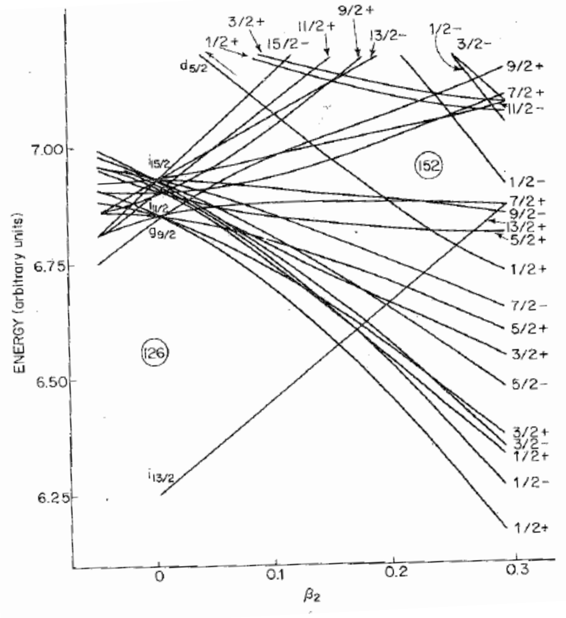
Protons: Z>82



Neutrons: 82<Z<126



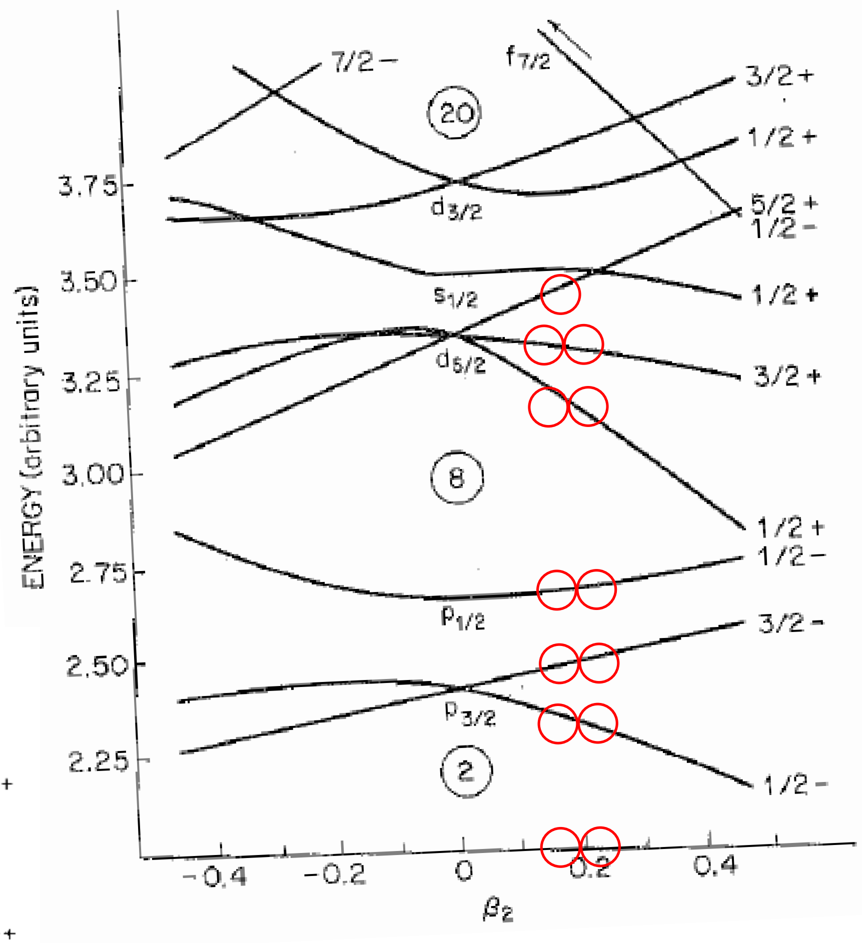
Neutrons: N>126

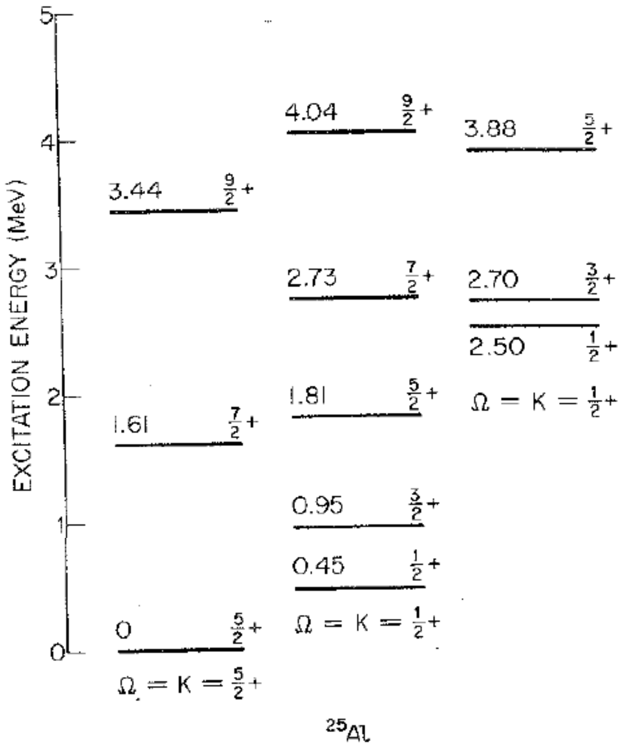


Nilsson Model: Example

- Consider ^{25}Al, for which we expect $\beta_2 \approx 0.2$, like ^{27}Al
- There are 13 protons and 12 neutrons, so the unpaired proton will be responsible for J^{π}
- Filling the single-particle levels,
  - We place two protons in the 1$s_{1/2}$ level, which isn't shown
  - Then two more in 1/2$^-$, two more in 3/2$^-$, two more in 1/2$^-$, two more in the 1/2$^+$, two more in the 3/2$^+$
  - And the last one winds up in the 5/2$^+$ level
- So, we predict $J^{\pi}_{g.s} = {}^{5}/_{2}{}^{+}$
- For the first excited sate, it seems likely the proton will hop up to the nearby 1/2$^+$ level
- Agrees with data
- Since ^{25}Al is deformed, we should see rotational bands with states that have (integer)+j and $\propto j(j+1)$ spacing

B. Harvey, Introduction to Nuclear Physics and Chemistry (1962)





Further Reading

- Chapter 6: Modern Nuclear Chemistry (Loveland, Morrissey, Seaborg)
- Chapter 7: Nuclear & Particle Physics (B.R. Martin)
- Chapter 14, Section 13: Quantum Mechanics for Engineers (L. van Dommelen)
- Chapter 5, Section G: Introduction to Nuclear Physics & Chemistry (B. Harvey)
- Chapter 8: Nuclear Structure from a Simple Perspective (R. Casten)