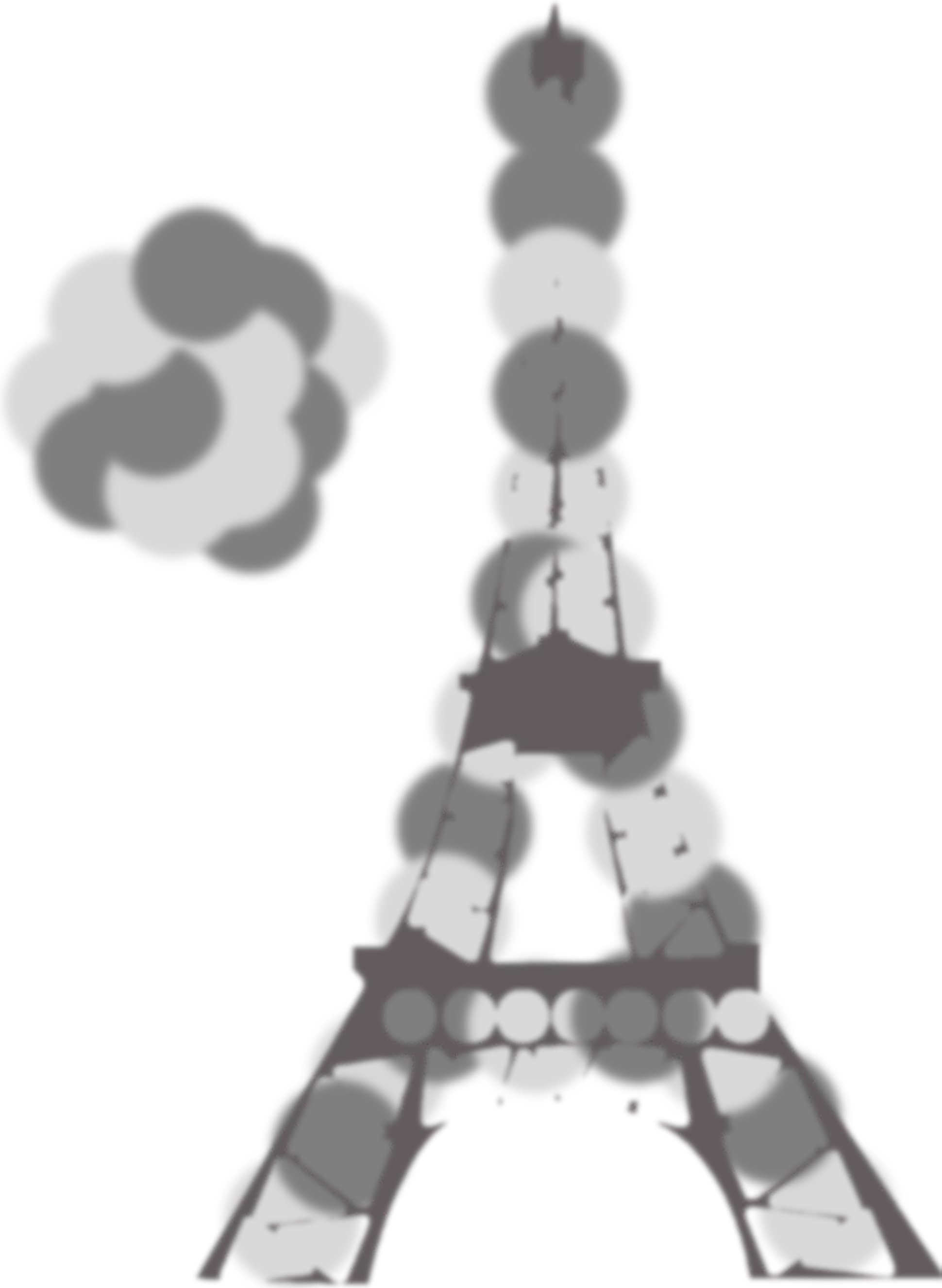
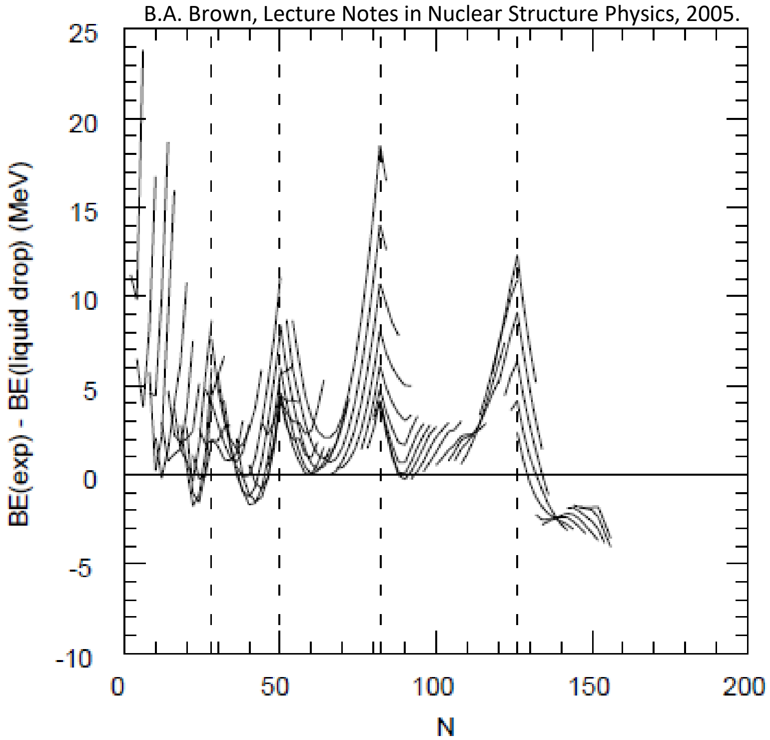
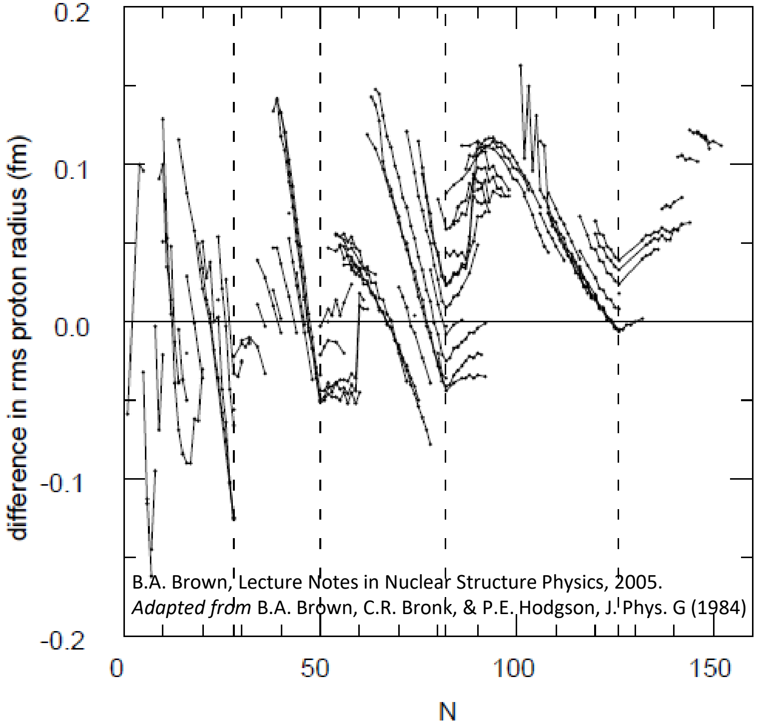
Lecture 3: *Nuclear Structure 1*

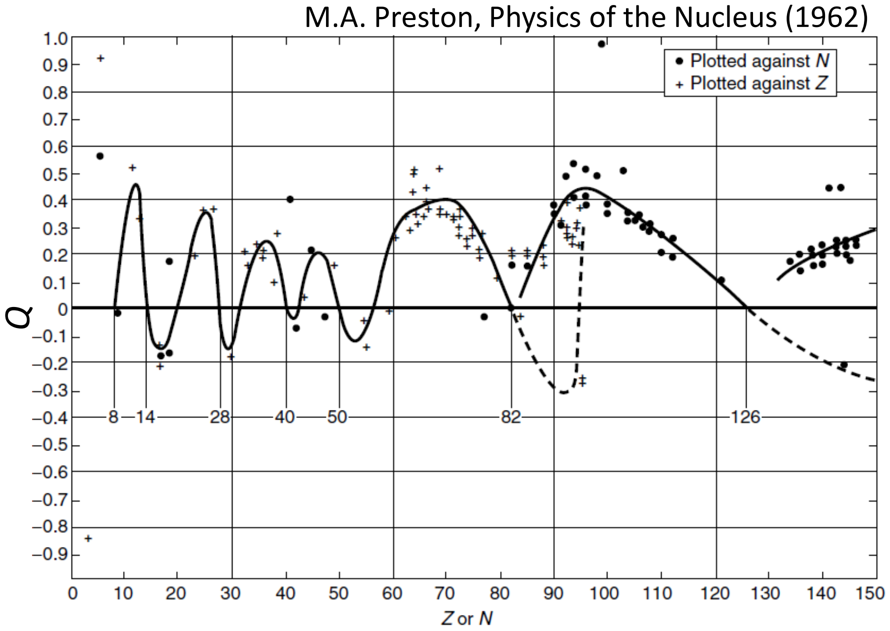
- Why structure?
- The nuclear potential
- Schematic shell model

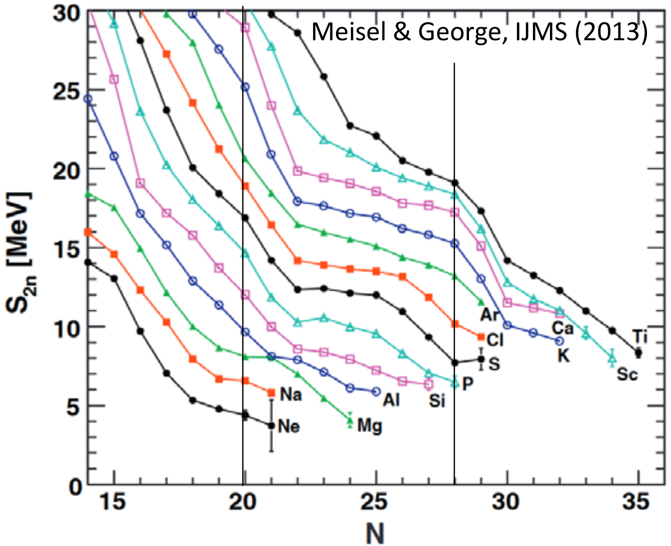


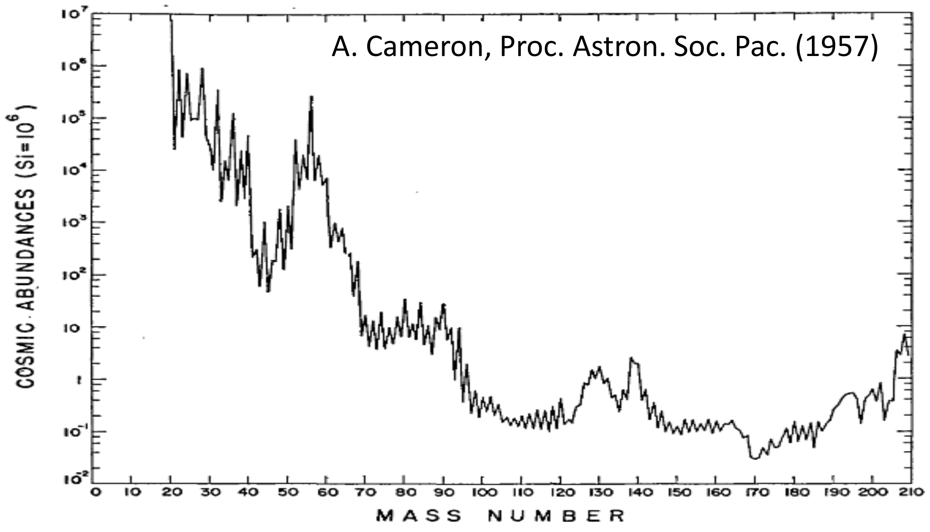
Empirically, several striking trends related to Z,N. *e.g.*

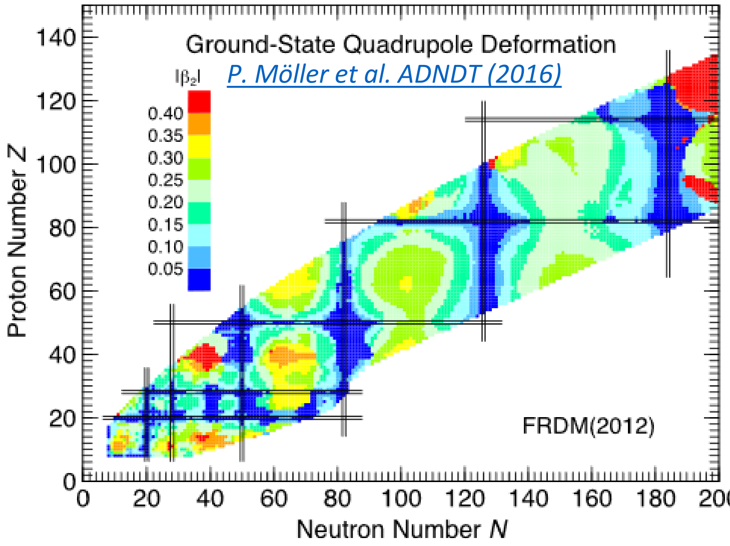








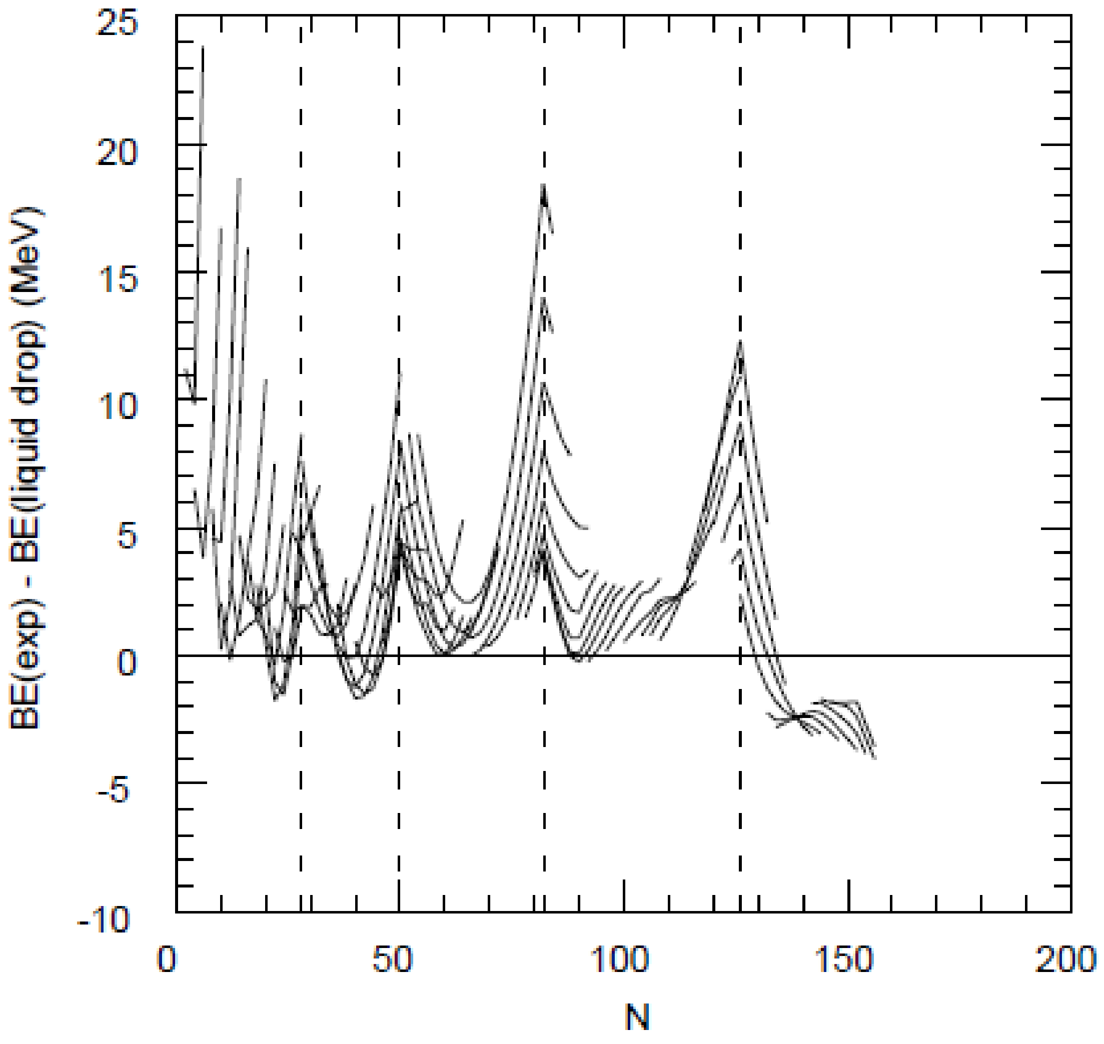


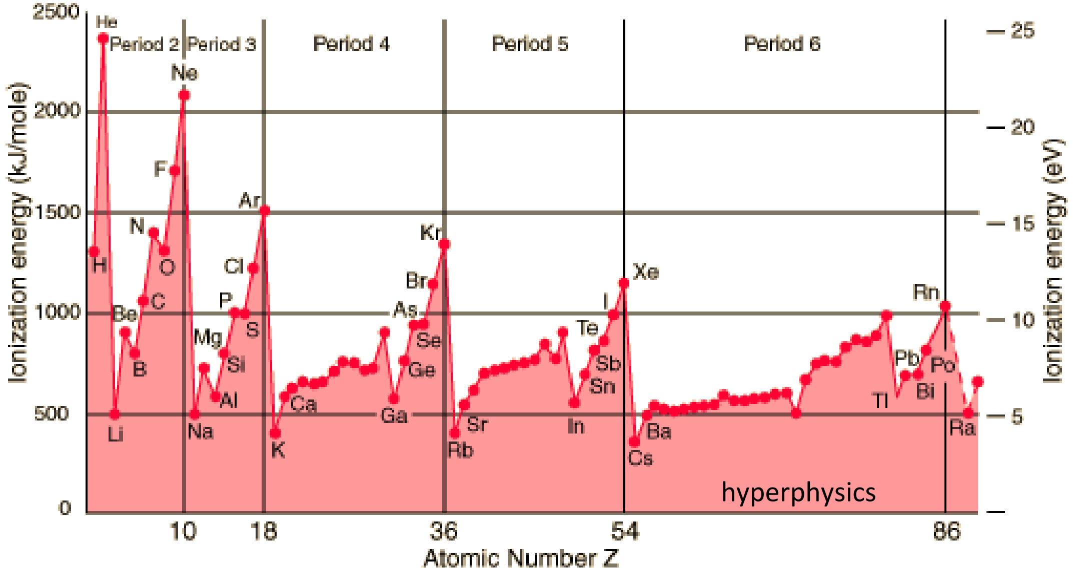


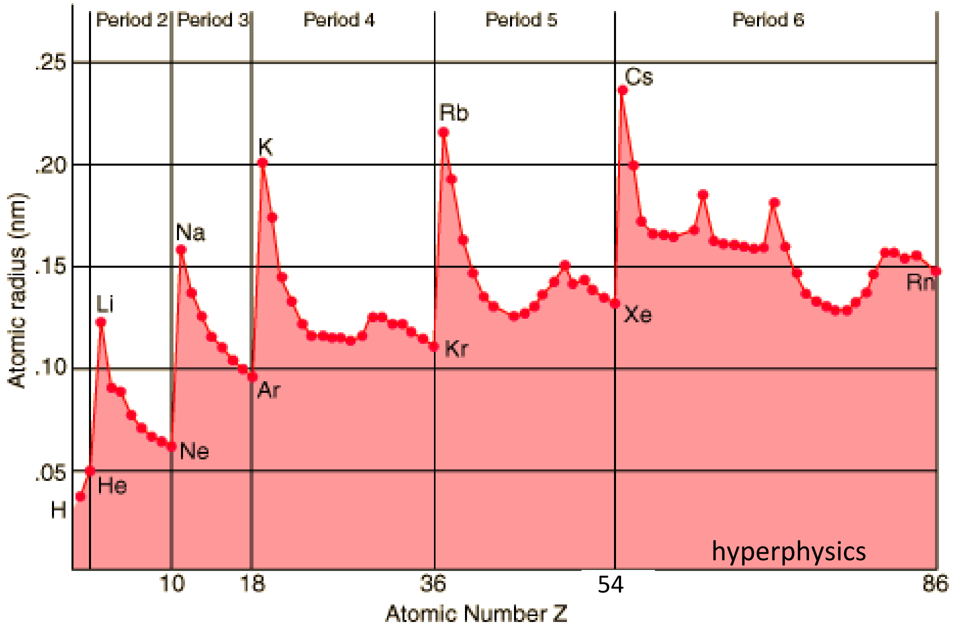
First magic number evidence compilation by M. Göppert-Mayer Phys. Rev. 1948

…reminiscent of atomic structure

B.A. Brown, Lecture Notes in Nuclear Structure Physics, 2005.







Shell Structure

Atomic

- Central potential (Coulomb) generated by nucleus
- Electrons are essentially non-interacting
- Solve the Schrödinger equation for the Coulomb potential and find characteristic (energy levels) shells: *shells at 2, 10, 18, 36, 54, 86*

Nuclear

- No central object
  …but each nucleon is interacted on by the other A-1 nucleons and they're relatively compact together
- Nucleons interact very strongly
  …but if nucleons in nucleus were to scatter, Pauli blocking prevents them from scattering into filled orbitals. Scattering into higher-E orbitals is unlikely. i.e. there is no "weak interaction paradox"
- Can also solve the Schrödinger equation for energy levels (shells) …but obviously must be a different potential: *shells at 2, 8, 20, 28, 50, 82, 126*

…you might be discouraged by points 1 and 2 above, but, remember:
If it's stupid but it works, it isn't stupid.

Calculating eigenstates of the system, a.k.a single particle levels

- The behavior of a quantum-mechanical system is described by the wave function ψ
- For a particle in some potential, we can solve for ψ using the Schrödinger equation,
  - $H\psi = E\psi$ *a.k.a.* $T\psi + V\psi = E\psi$ *a.k.a.* $-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$ (in cartesian coordinates, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$)
- The solutions ψ are the eigenfunctions and their eigenvalues are the corresponding energy E
- As a bonus, when ψ can be expressed in terms of spherical harmonics, $\psi = R(r)Y^l_m(\theta,\phi)$
  we also get the angular momentum for that particular eigenfunction,
  and parity, since the function is either odd or even

- Mathematical challenges aside, to get any traction we obviously need to assume a potential V
- For a single nucleon in the field of a nucleus,
  - V should approximate the mean-field generated by all other nucleons
  - The solutions will be single-particle levels,
    i.e. discrete states the nucleon can occupy

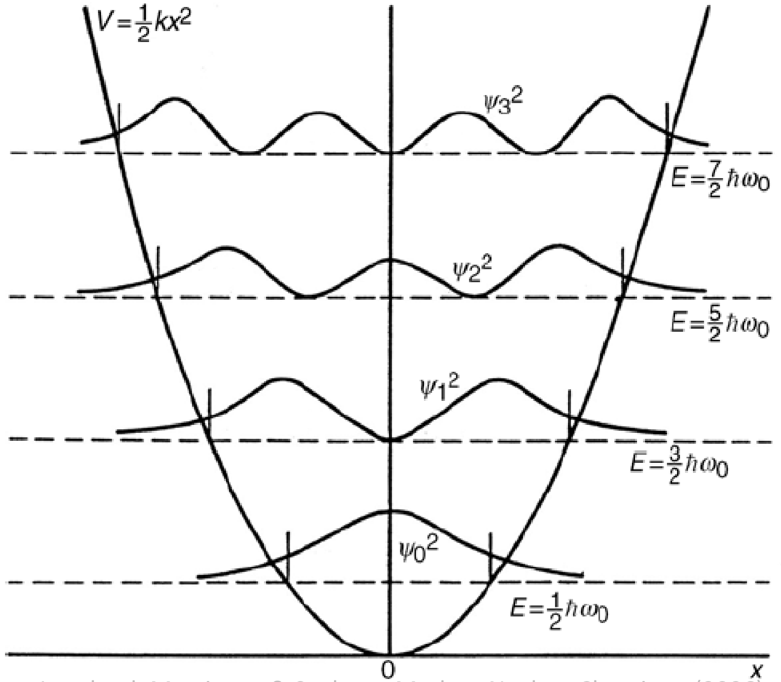
- Since nucleons are indistinguishable, we only need to solve for the single-particle levels for a nucleon and then we can fill those levels (working in terms of increasing E) to generate a model to calculate the properties of our nucleus

First stab at the potential, V: **The Harmonic Oscillator**

- Based on some evidence (and logic) that nuclei aren't perfectly constant in density, Heisenberg (Z. Phys. 1935) posited that a parabolic potential could be assumed, conveniently allowing the adoption of the harmonic oscillator solutions *(one of the few analytically solved systems!)*
- This provides evenly spaced energy levels n, with $E_n = \left(n + \frac{1}{2}\right)h\upsilon$.
- The corresponding angular momenta are $l = n - 1, n - 3, \ldots \geq 0$.
- The number of particles per angular momentum is $2(2l + 1)$ for $2l + 1$ projections & 2 spins
- So, the number of particles per level is:

i.e. only odd or even functions are allowed for each oscillator shell

n	l	# per level	Cumulative
1	0	2(2*0+1) = 2	2
2	1	2(2*1+1) = 6	8
3	0,2	2(2*0+1) = 2 + 2*(2*2+1) = 10 = 12	20
4	1,3	2*(2*1+1) = 6 + 2*(2*3+1) = 14 = 20	40
5	0,2,4	2*(2*0+1) = 2 + 2*(2*2+1) = 10 + 2*(2*4+1) = 18 = 30	70

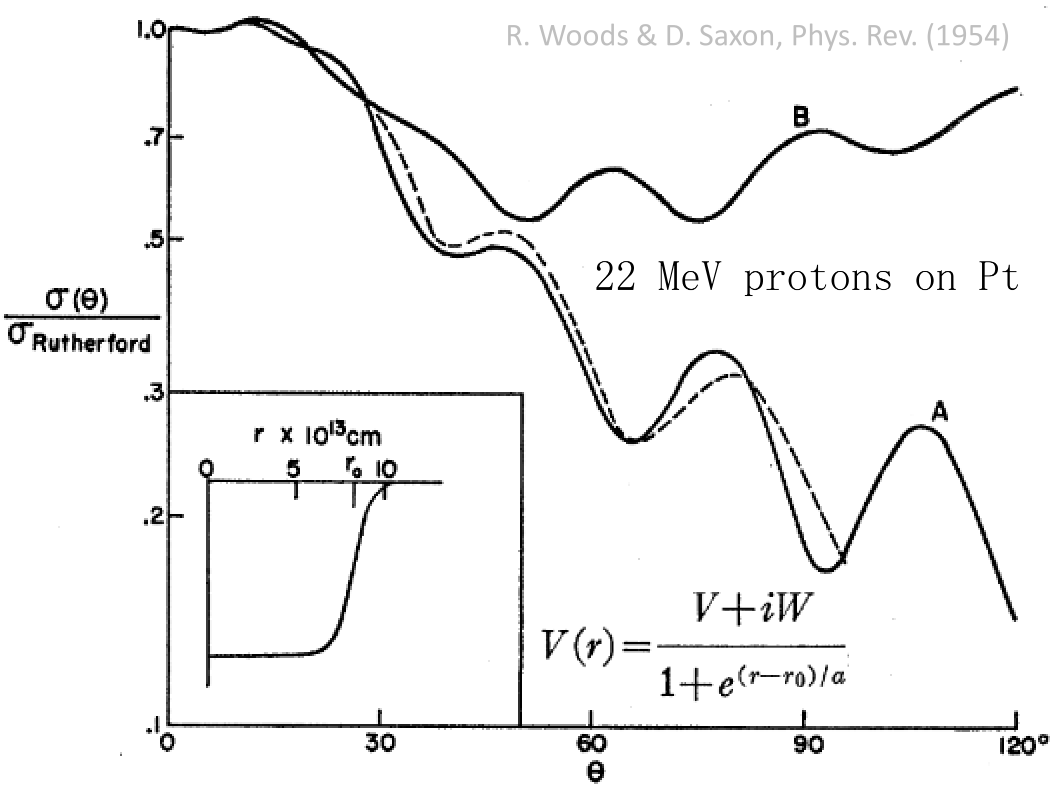


Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

Could the HO potential still be useful for some cases?
...can get the job done for light nuclei (e.g. H. Guo et al. PRC 2017)
...but need to be careful, because can impact results (B.Kay et al arXiv 2017)

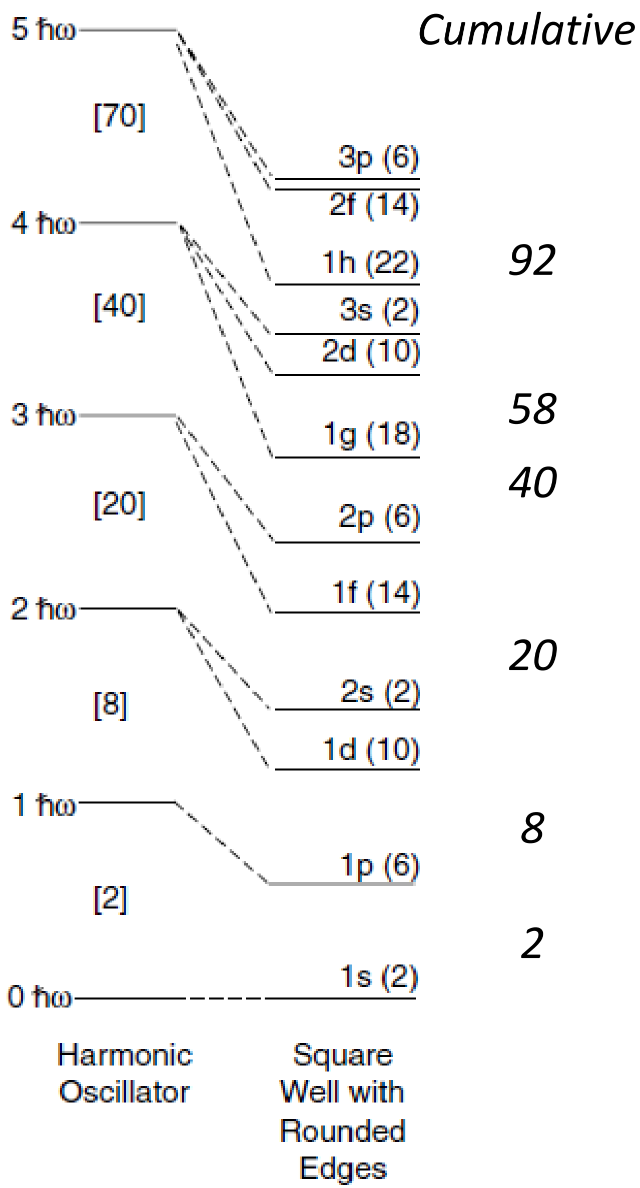
Move to an empirical potential: **Woods-Saxon**

- Since the nuclear interaction is short-range, a natural improvement would be to adopt a central potential mimicking the empirical density distribution
- This is basically a square well with soft edges, as described by the Woods-Saxon potential:



- Using the Woods-Saxon is a good idea because of commitment to reality… but we're no wiser as to the origin of the magic numbers

Was this step completely useless? No! It broke the degeneracy in ℓ

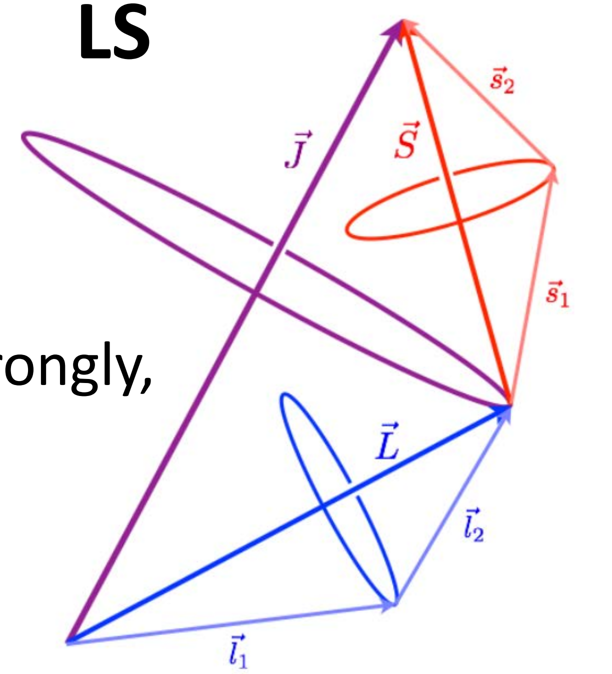


Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

The missing link: **the spin-orbit interaction**

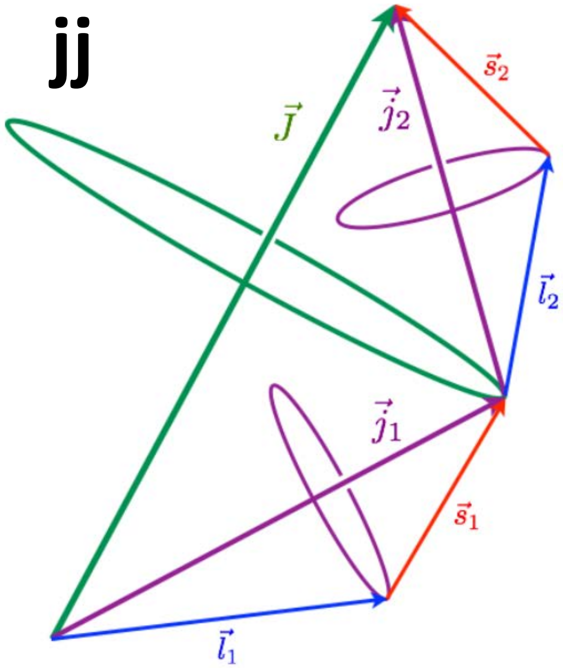
- Due to desperation or genius (or both) Maria Göppert-Mayer
  [Phys. Rev. February 1949] (and nearly simultaneously Haxel, Jensen, & Suess [Phys. Rev. April 1949])
  posited that nucleon spin and orbital angular momentum interacted strongly, making j the good quantum number for a nucleon: $\vec{j} = \vec{l} + \vec{s}$
- Prior to this approach,
  angular momentum was coupled as is typically done for atoms,
  where $\vec{J} = \vec{L} + \vec{S}$, $\vec{L} = \sum_{nucleons} \vec{l}$, and $\vec{S} = \sum_{nucleons} \vec{s}$
  - This is "LS coupling"
- Positing that the spin-orbit interaction is stronger than spin-spin or orbit-orbit means that instead, $\vec{J} = \sum_{nucleons} \vec{j}$ and $\vec{j} = \vec{l} + \vec{s}$
  - This is "jj coupling"

LS



A. Kastberg, Lecture Notes Physique Atomique

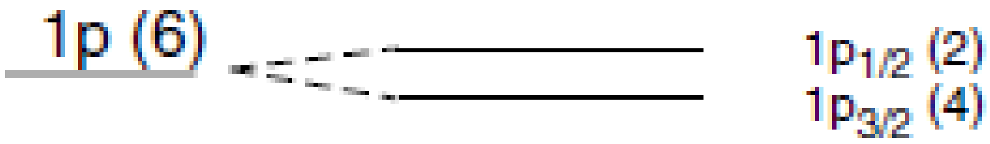
jj



The missing link: **the spin-orbit interaction**

- Now, in considering a valence nucleon, we should calculate its j
- j can only take on values: $|l - s| \leq j \leq |l + s|$ …so for our nucleons, $\left|l - \frac{1}{2}\right| \leq j \leq \left|l + \frac{1}{2}\right|$,
  i.e. l and s are either aligned ($l + s$) or anti-aligned ($l - s$)
  - For $l = 0$: $j = \frac{1}{2}$; $l = 1$: $j = \frac{1}{2}\ or\ \frac{3}{2}$; $l = 2$: $j = \frac{3}{2}\ or\ \frac{5}{2}$; $l = 3$: $j = \frac{5}{2}\ or\ \frac{7}{2}$ … etc.
- Each j has $2j + 1$ projections (a.k.a. # of protons or neutrons, depending which nucleon we're discussing)
  - i.e. 2 states for $j = \frac{1}{2}$, 4 states for $j = \frac{3}{2}$, 6 states for $j = \frac{5}{2}$, 8 states for $j = \frac{7}{2}$ …etc.
- The spin-orbit interaction means there's a j-dependent part of the nuclear potential,
  so the levels corresponding to different j for some l will be split in energy.
- For nucleons, cases with aligned l and j are energetically favored,
  so, for example, $l = 1, j = \frac{3}{2}$ will be lower in energy than $l = 1, j = \frac{1}{2}$

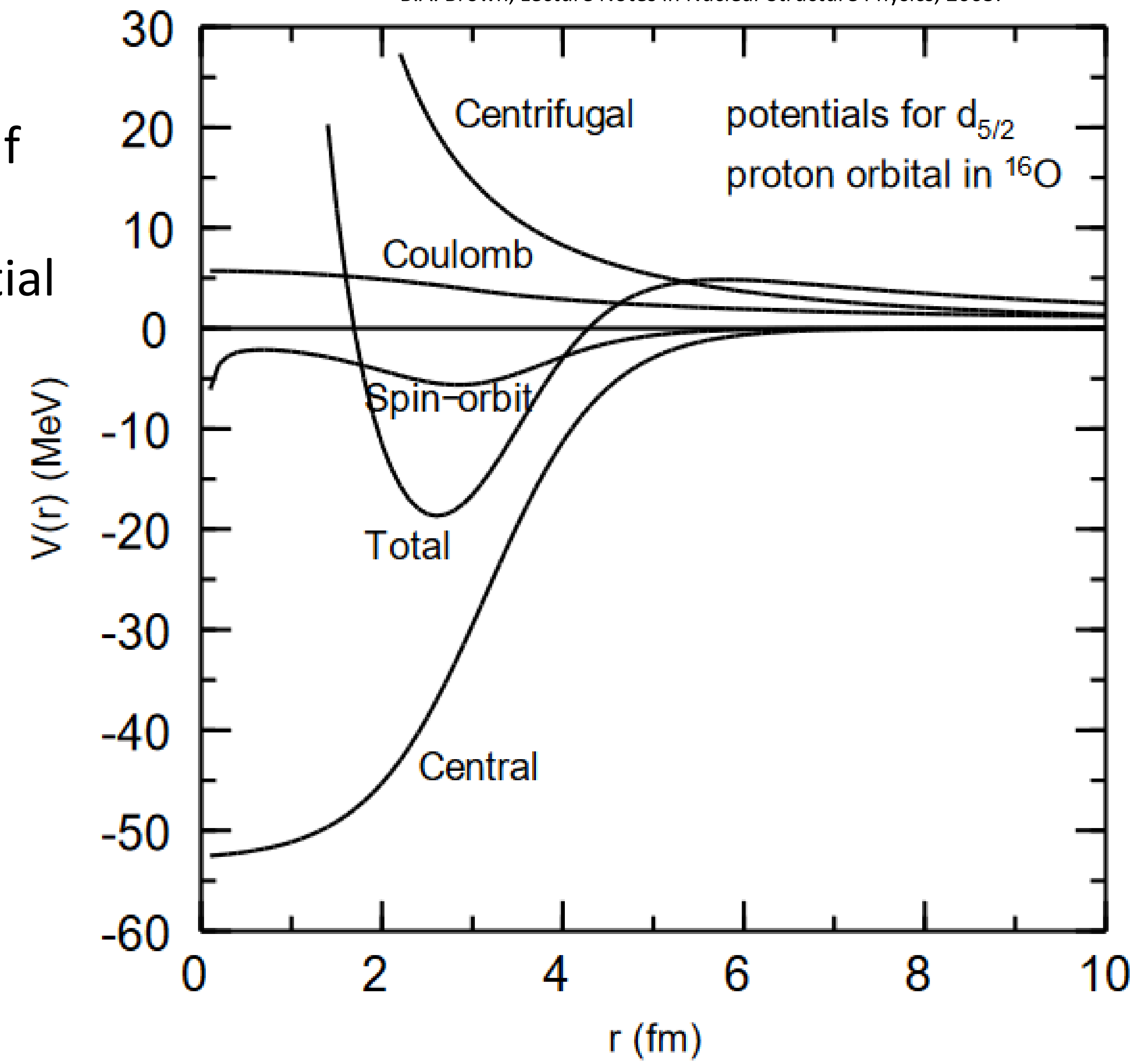
- While we're at it, note the spectroscopic notation:
  - $l = 0, 1, 2, 3, 4, 5, \ldots$ = "s", "p", "d", "f", "g", "h" …



Result: the **nuclear potential**

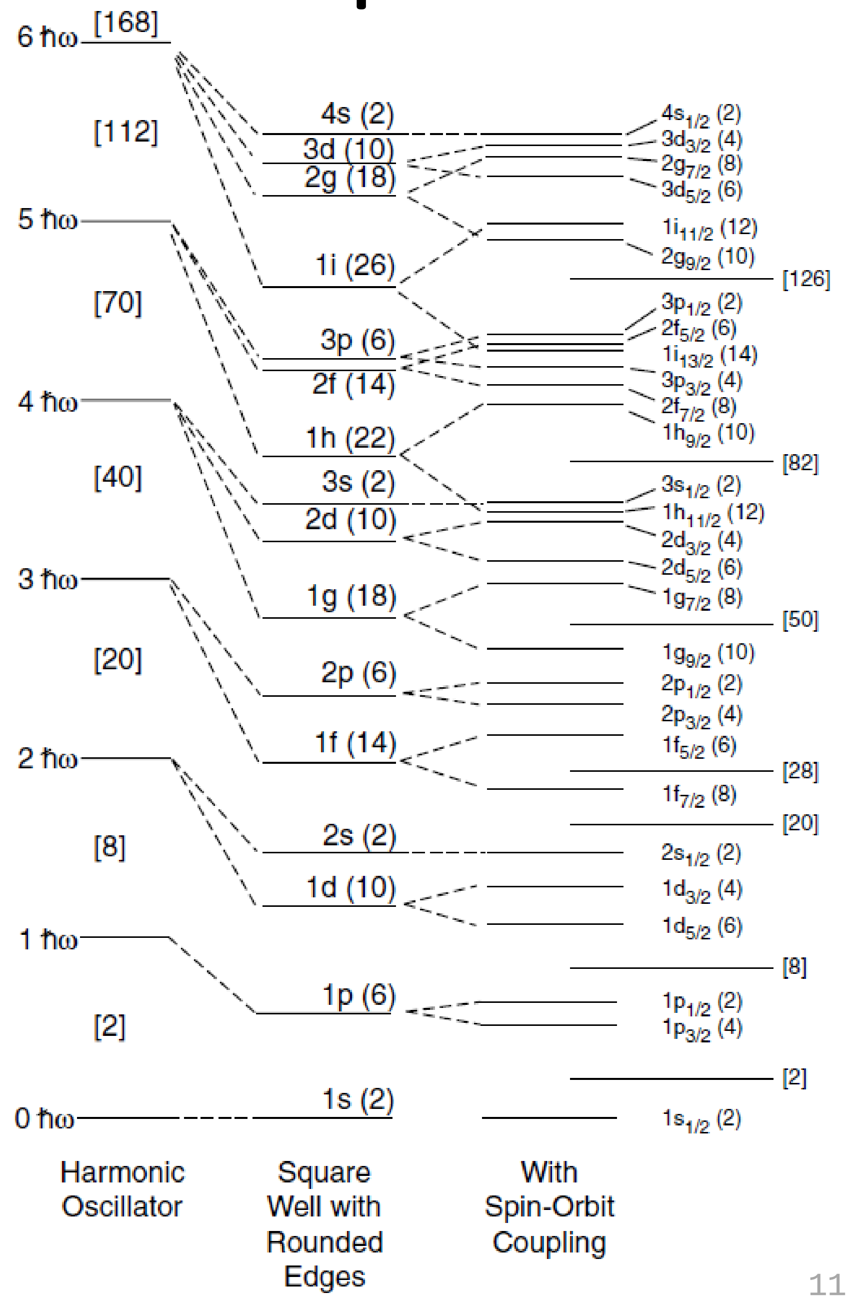
- Nucleons within a nucleus can be treated as if they are
  - **Attracted** by a Woods-Saxon central potential
  - **Repelled** by a Coulomb potential from a charged sphere (if proton)
  - **Attracted *or* Repelled** if l and s are parallel or anti-parallel by the spin-orbit force (Peaks at surface)
  - **Repelled** by a centrifugal barrier (if the nucleon were to exit the nucleus, carrying away angular momentum $l > 0$)

B.A. Brown, Lecture Notes in Nuclear Structure Physics, 2005.



Putting it all together: **“shells” from the nuclear potential**

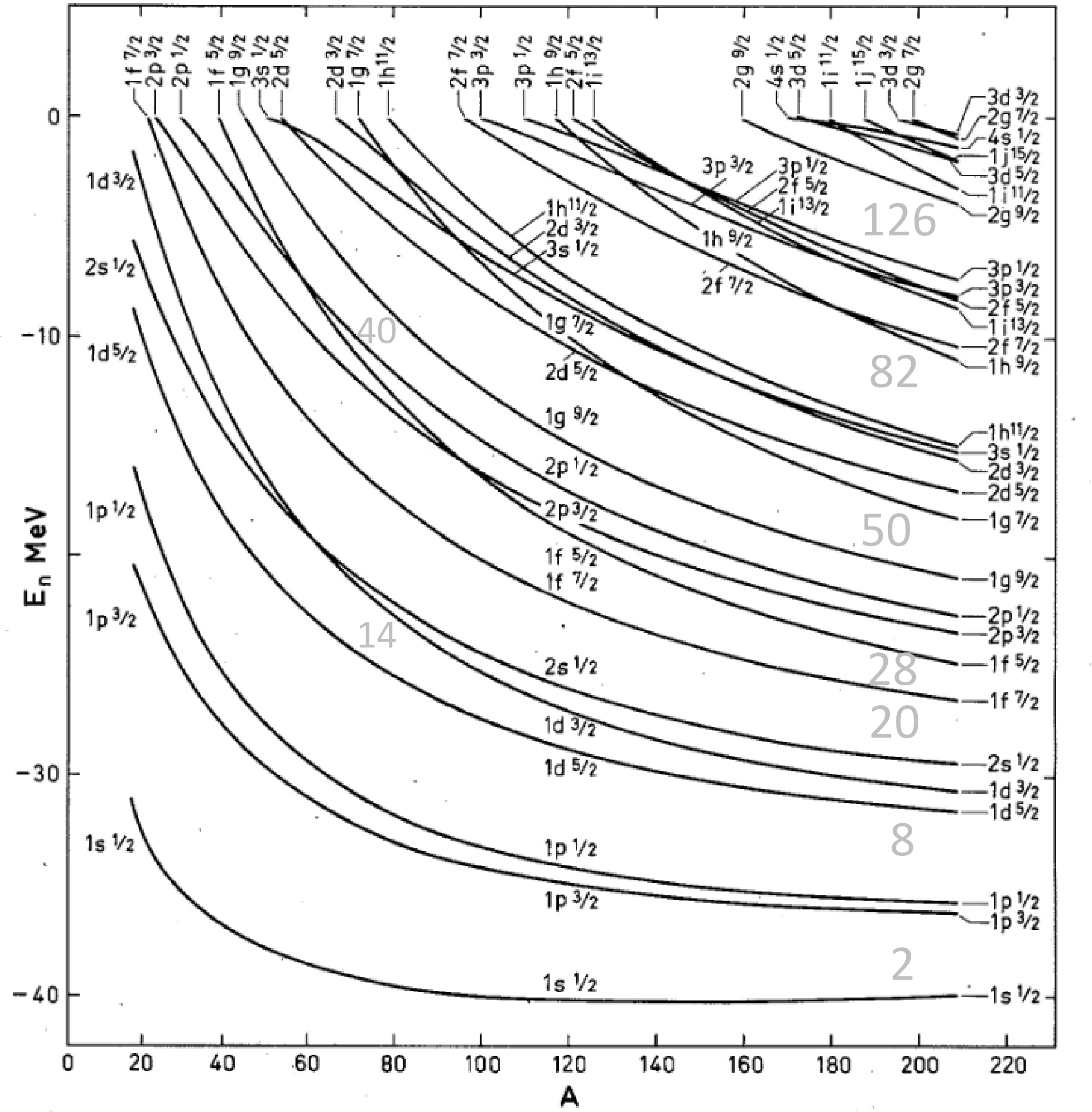
- Considering the nucleus as nucleons interacting in a mean-field potential, generated by the spatial distribution of all other nucleons, and each nucleon having a strong interaction between its orbital & spin angular momentum, properly predicts the magic numbers.
- Note that **neutrons and protons are considered separately**.
- When adding neutrons or protons to a nucleus, the lowest energy state will (generally) consist of filling each orbital as you go upward.
- The regions between the large gaps in nucleon energy are referred to as “shells”.
  - E.g. Between 8 and 20 neutrons (or protons) is the “sd-shell”, between 28 and 40 neutrons (or protons) is the “fp-shell”.
  - More exotic neutron-rich nuclides exist, so typically people are talking about the neutron shell
  - Nucleons can get excited into higher-lying states, so states above the ground-state are relevant in calculations



Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

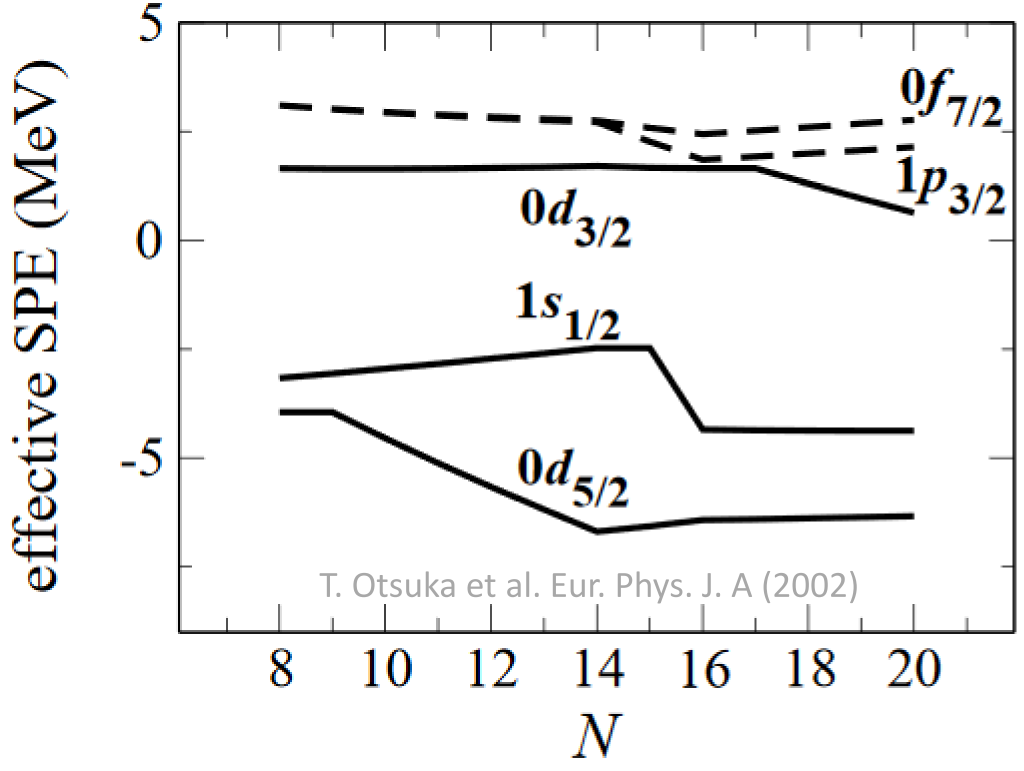
As a heads-up, level ordering doesn't follow a fixed set of rules

For stable, spherical nuclides:



Bohr & Mottelson, Nuclear Structure, Vol. I (1969)

For, e.g, n-rich O isotopes:



T. Otsuka et al. Eur. Phys. J. A (2002)

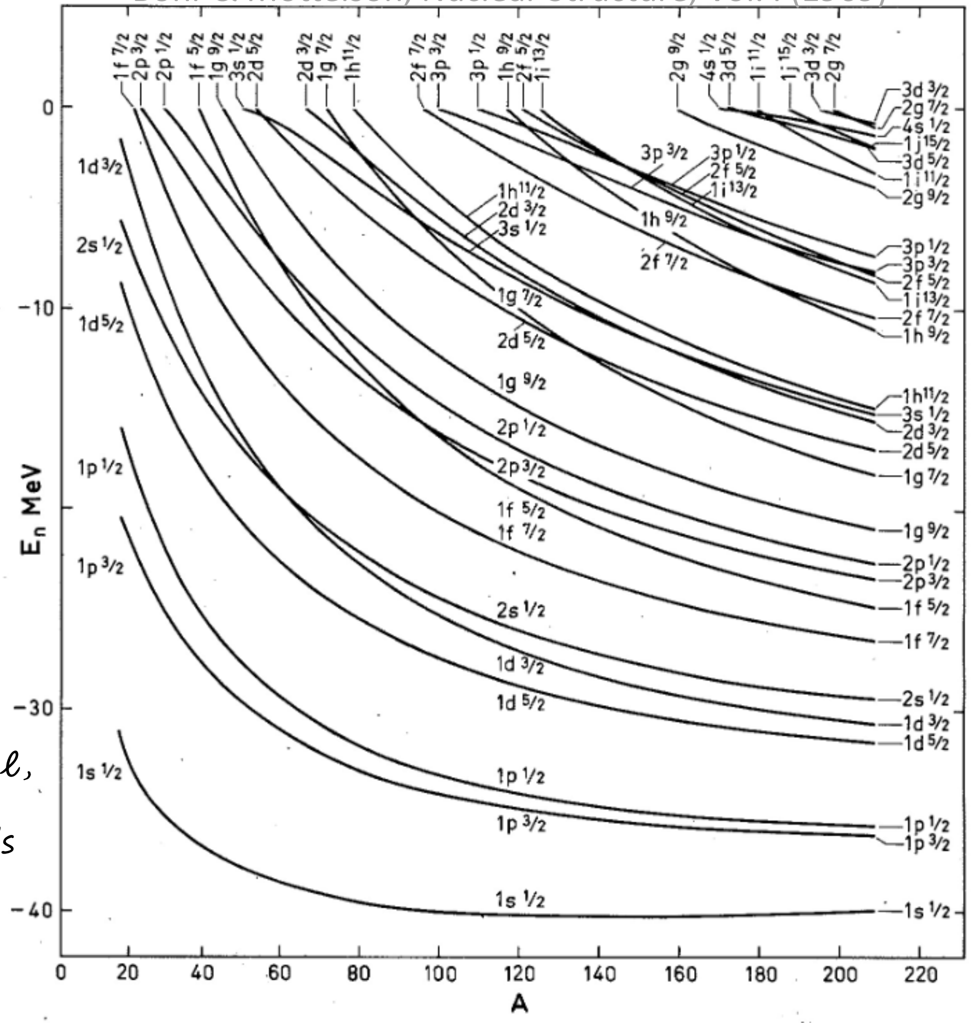
Magic numbers "break down" and new ones can appear for exotic nuclides

Common form for the **nuclear potential**

- $V(r) = V_{central}(r) + V_{spin-orbit}(r)\vec{l}\cdot\vec{s} + V_{Coulomb}(r) + V_{centrifugal}(r)$
  - $V_{central}(r) = V_{ce}\left(\frac{1}{1+\exp\left(\frac{r-R_{ce}}{a_{ce}}\right)}\right)$
    - $V_{ce,protons} = V_0 + \frac{N-Z}{A}V_1$ or $V_{ce,neutrons} = V_0 - \frac{N-Z}{A}V_1$
  - $V_{spin-orbit}(r) = V_{so}\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{1+\exp\left(\frac{r-R_{so}}{a_{so}}\right)}\right)$
  - $V_{Coulomb}(r) = \begin{cases} \frac{Ze^2}{r} & \text{for } r \geq R_c \\ \frac{Ze^2}{R_c}\left(\frac{3}{2} - \frac{r^2}{2R_c^2}\right) & \text{for } r \leq R_c \end{cases}$
  - $V_{centrifugal}(r) = \frac{\hbar^2}{2\mu_{reduced}}\frac{l(l+1)}{r^2}$

  This is only relevant when a nucleon is removing/adding orbital angular momentum ℓ, so you would not use it to calculate single particle levels

Bohr & Mottelson, Nuclear Structure, Vol. I (1969)



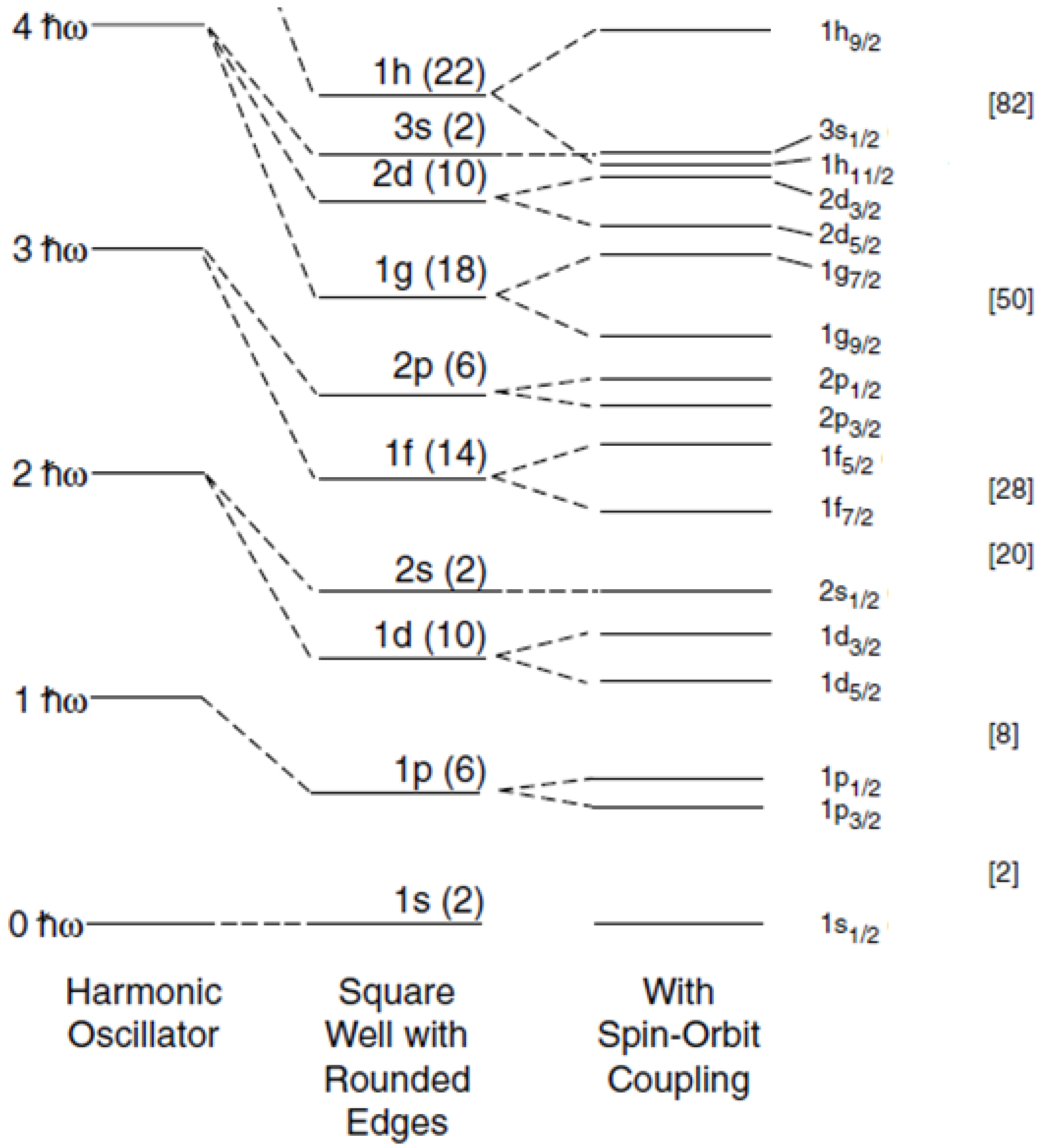
- Typically, $R_{ce} = R_{so} = R_c = r_0 A^{1/3}$
- For $r_0 = 1.27fm, a_{ce} = a_{so} = 0.67fm,$

$V_0 = -55MeV, V_1 = -33MeV, V_{so} = -0.44V_{ce}$, get neutron single-particle energies above

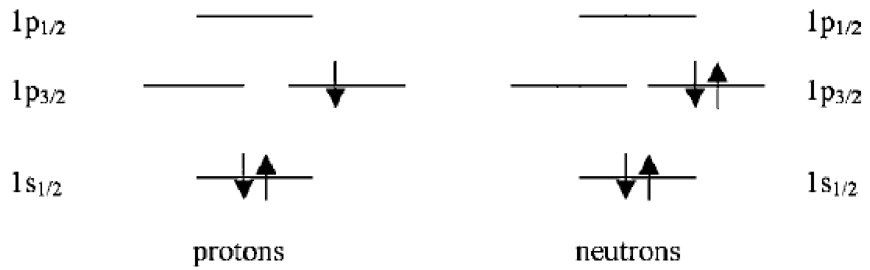
Filling the shells

Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

- We can construct a nucleus using our “shell model”:
  - A nucleon will go in the lowest-energy level which isn’t already filled, i.e.
    - the largest angular momentum, j
    - for the lowest orbital angular momentum, l
    - for the lowest oscillator shell, n
  - $2j + 1$ protons or neutrons are allowed per level
  - Each level is referred to by its nlj
    - n by the # for the oscillator shell (convention either starts with 0 or 1)
    - l by spectroscopic notation (s=0,p=1,d=2,f=3,…)
    - j by the half-integer corresponding to the spin

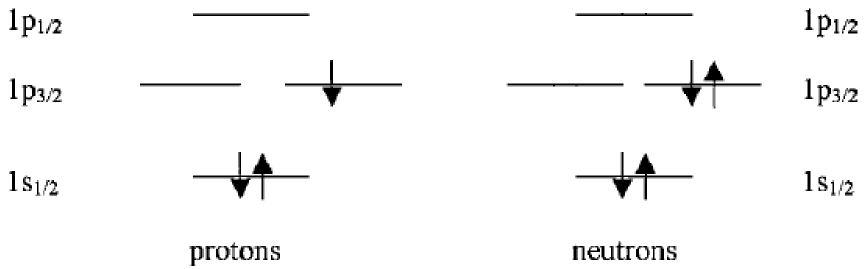


- For example: ^{7}Li (Z=3, N=4)



Basic properties from the shell model: J^π

- Recall that, from the pairing hypothesis, nucleons pair & cancel spins.
- So, the unpaired nucleons determine the properties of a nucleus.
  Unpaired nucleons sum to determine the spin & multiply to determine the parity
- Revisiting ^{7}Li:

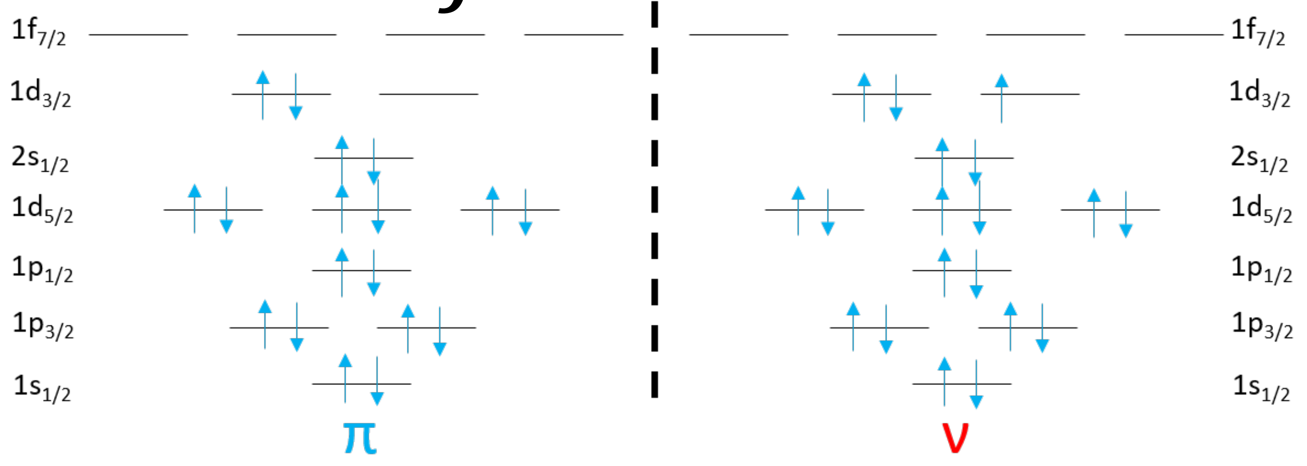


  - The only nucleon without a dance partner is the $1p_{3/2}$ proton; i.e. $J = 3/2,\ \pi = (-1)^1$
    So, the ^{7}Li ground-state should be $J^\pi = \frac{3}{2}^-$
  - What's the lowest energy excitation possible? (note pairing is strong)
    Moving the $p_{3/2}$ proton up to $p_{1/2}$
  - So, the first excited state of ^{7}Li should be $J^\pi = \frac{1}{2}^-$
  - Compare to data:

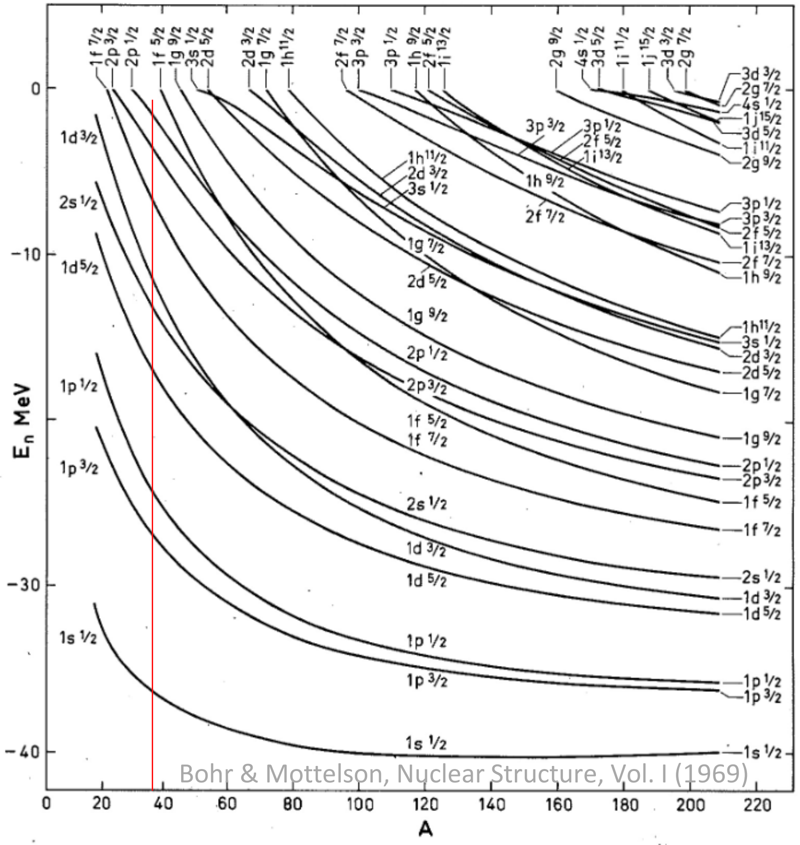
Basic properties from the shell model: J^{π}

- Now that we're feeling fat & sassy, let's try another case: **^{37}Ar**
- Based on our shell-model, we expect the ground-state to be $3/2^{+}$

  …and it is!

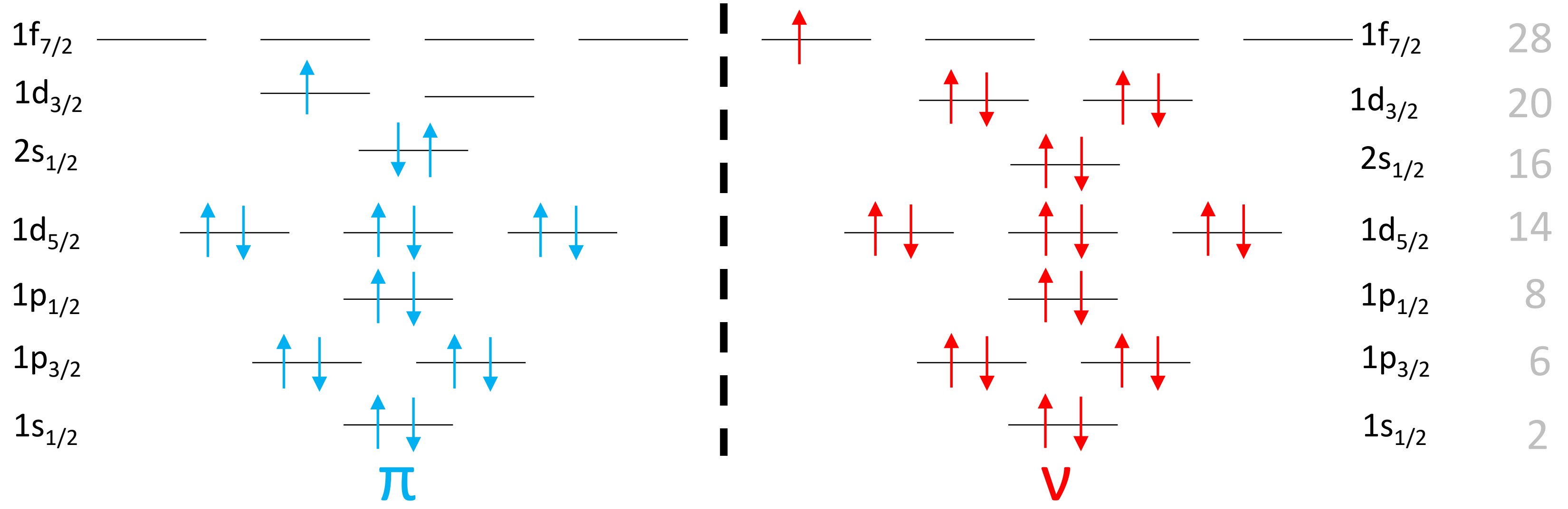


- Now for the first excited state, a logical thought would be the odd $d_{3/2}$ neutron would pop up to the $f_{7/2}$ level, creating a state with $7/2^{-}$
  - …but the first excited state is $1/2^{+}$ (the 2nd x.s. is $7/2^{-}$)
- What happened?
  - We have to keep in mind pairing & energy-costs
  - The $2s_{1/2}$-$1d_{3/2}$ gap is smaller than the $1d_{3/2}$-$1f_{7/2}$ gap (for low A)
  - And, **pairing energy increases with the l of the level**



Basic properties from the shell model: J^{π}

- Looking at a more complicated case, ^{38}Cl (Z=17, N=21)



- 2 valence nucleons: one $d_{3/2}$ proton and one $f_{7/2}$ neutron
- Allowed couplings are $|j_1 - j_2| \leq J \leq |j_1 + j_2|$
- So for this case: $J = 2, 3, 4, 5$
- How do we decide which combination has the lowest energy?

Using the descriptively named: "jj coupling rules for Odd-Odd nuclei" from Brennan & Bernstein (Phys. Rev. 1960)

Basic properties from the shell model: J^{π} for odd-odd

- For Odd-Z, Odd-N nuclides, need a method to determine which jj-coupling is the lowest energy
- An empirically-based set of rules was developed by Brennan & Bernstein (Phys. Rev. 1960)
- They noticed that, when coupling j, $j_1 = l_1 \pm s_1$ and $j_2 = l_2 \pm s_2$,
  - **Rule 1**: If ($j_1 = l_1 + s_1$ and $j_s = l_2 - s_2$) <u>or</u> ($j_1 = l_1 - s_1$ and $j_s = l_2 + s_2$), then $J = |j_1 - j_2|$
    - e.g. for a $d_{3/2}$ proton $\left(j_p = 2 - \frac{1}{2} = \frac{3}{2}\right)$ and a $f_{7/2}$ neutron $\left(j_n = 3 + \frac{1}{2} = \frac{7}{2}\right)$, $J = \frac{7}{2} - \frac{3}{2} = 2$
      For this case, $\pi = \prod \pi_i = (-1)^2 * (-1)^3 = -$

      Ex: ^{38}Cl
  - **Rule 2**: If ($j_1 = l_1 + s_1$ and $j_s = l_2 + s_2$) <u>or</u> ($j_1 = l_1 - s_1$ and $j_s = l_2 - s_2$), then $J = |j_1 \pm j_2|$
    - e.g. for a $d_{5/2}$ proton $\left(j_p = 2 + \frac{1}{2} = \frac{5}{2}\right)$ and a $d_{5/2}$ neutron $\left(j_n = 2 + \frac{1}{2} = \frac{5}{2}\right)$, $J = \frac{5}{2} + \frac{5}{2} = 5$
      For this case, $\pi = \prod \pi_i = (-1)^2 * (-1)^2 = +$

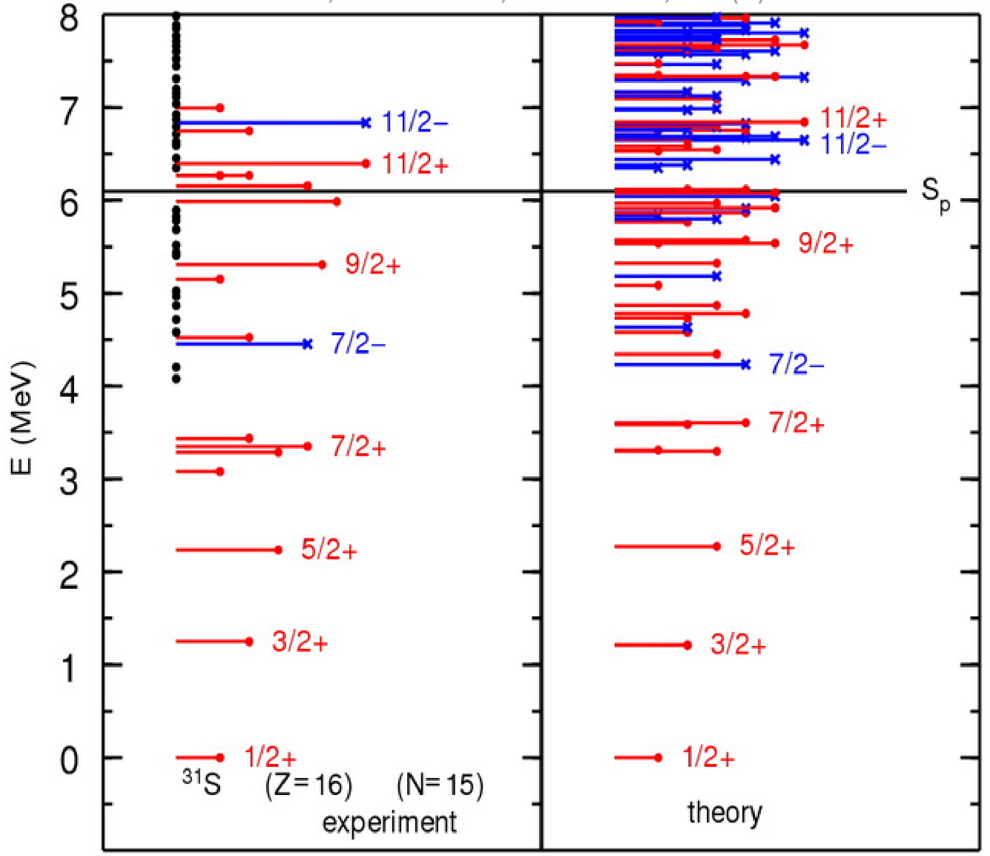
      Ex: ^{26}Al
  - **Rule 3**: If one odd nucleon has been promoted (e.g. to an s-orbital to pair with a nucleon), leaving behind a "hole", and the other odd nucleon stays a particle, then $J = j_1 + j_2 - 1$
    - e.g. for a $d_{3/2}$ proton hole $\left(j_p = \frac{3}{2}\right)$ and a $f_{7/2}$ neutron $\left(j_n = \frac{7}{2}\right)$, $J = \frac{7}{2} + \frac{3}{2} - 1 = 4$
      For this case, $\pi = \prod \pi_i = (-1)^2 * (-1)^3 = -$

      Ex: ^{40}K

*These don't always work…but when they don't, this can tell you something:
Either there's more than a single-particle level interaction going on,
or your particle(s)/hole(s) don't occupy the levels we naïvely assumed.
(e.g. S. Liddick et al. Phys. Rev. C 2004)*

Shell-model is pretty good at predicting J^{π} (among other things)

B. Brown, W.A. Richter, & C. Wrede, PRC(R) 2014

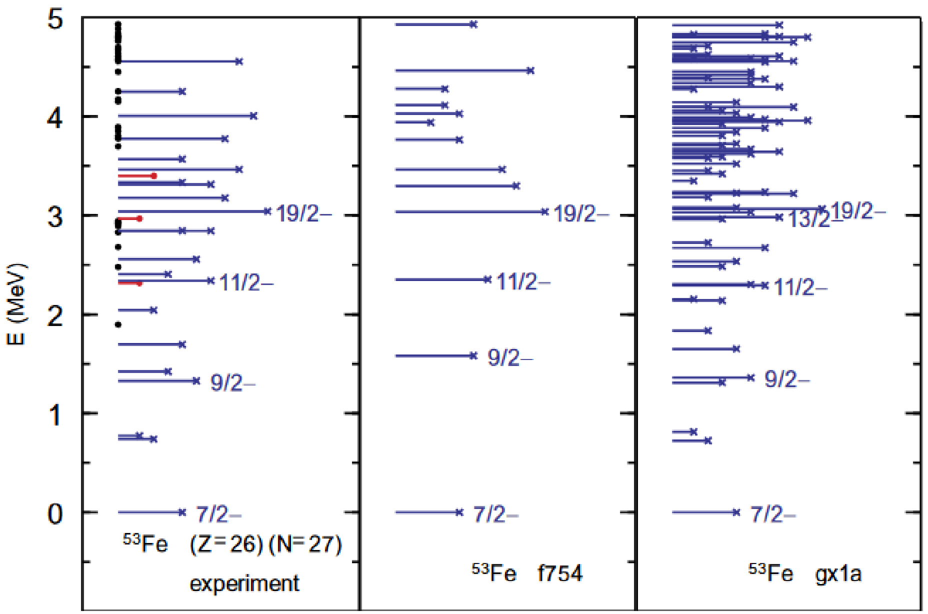


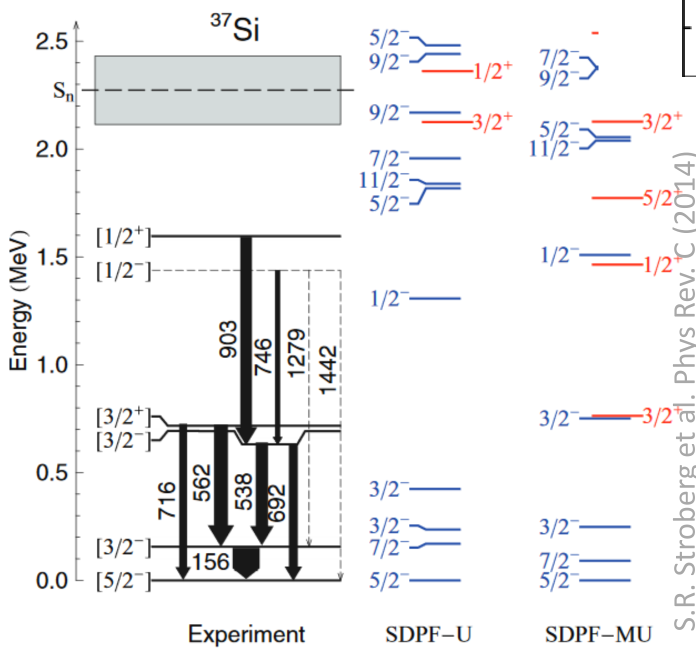
…some major challenges are

including all the relevant levels in the calculation

and

B. Brown & W. Rae, Nuc. Dat. Sheets (2014)





S.R. Stroberg et al. Phys Rev. C (2014)

choosing a good interaction between nucleons

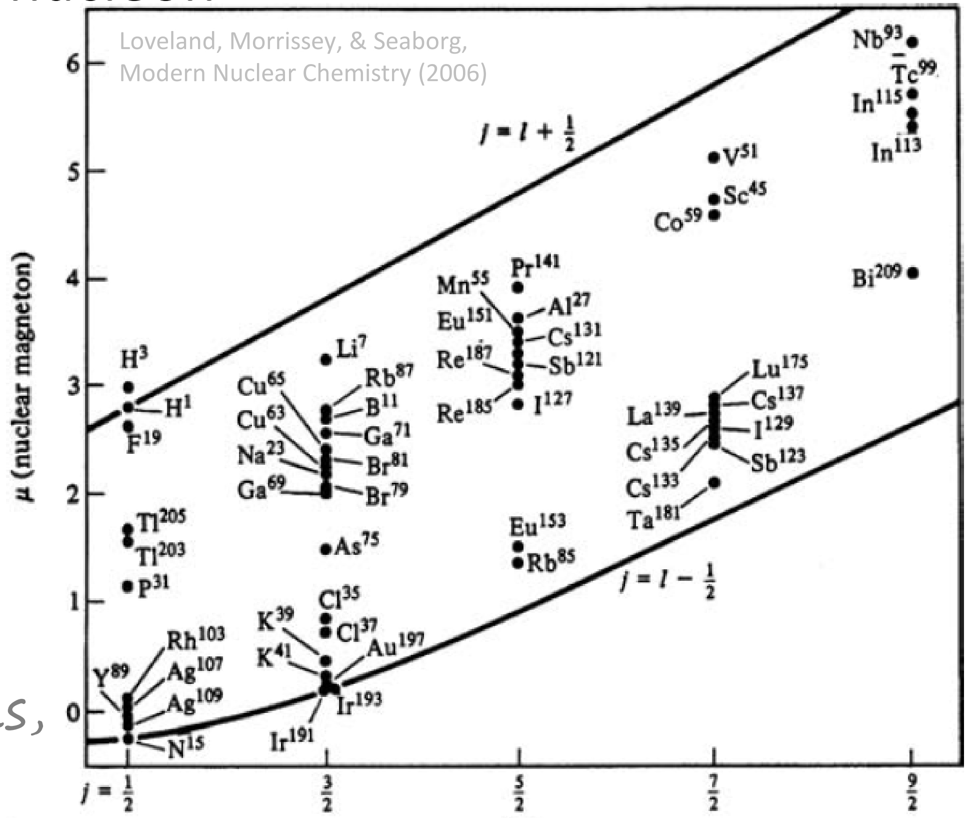
What else are J^π predictions good for? Magnetic dipole moments

- Recall that for a single particle, the magnetic dipole moment is: $\mu = j g_j \mu_N$
- After some fancy footwork, it can be shown that the Landé g-factor can be expressed as:
  - $g_j = \left(\frac{j(j+1)+l(l+1)-s(s+1)}{2j(j+1)}\right) g_l + \left(\frac{j(j+1)-l(l+1)+s(s+1)}{2j(j+1)}\right) g_s$
- Since spins cancel for paired nucleons, we might expect the magnetic dipole moment of a nucleus with 1-unpaired nucleon to be determined by that nucleon

  Expected values of μ are therefore:
  - $\mu = j g_j = l g_l + \frac{1}{2} g_s$ for $j = l + \frac{1}{2}$
  - $\mu = j g_j = j\left(1 + \frac{1}{2l+1}\right) g_l - j\left(\frac{1}{2l+1}\right) g_s$ for $j = l - \frac{1}{2}$
  - Protons: $g_l = 1, g_s = 5.6$, Neutrons: $g_l = 0, g_s = -3.8$
  - These boundaries are the "Schmidt lines"

    (Th. Schmidt, Z.Phys. (1937))

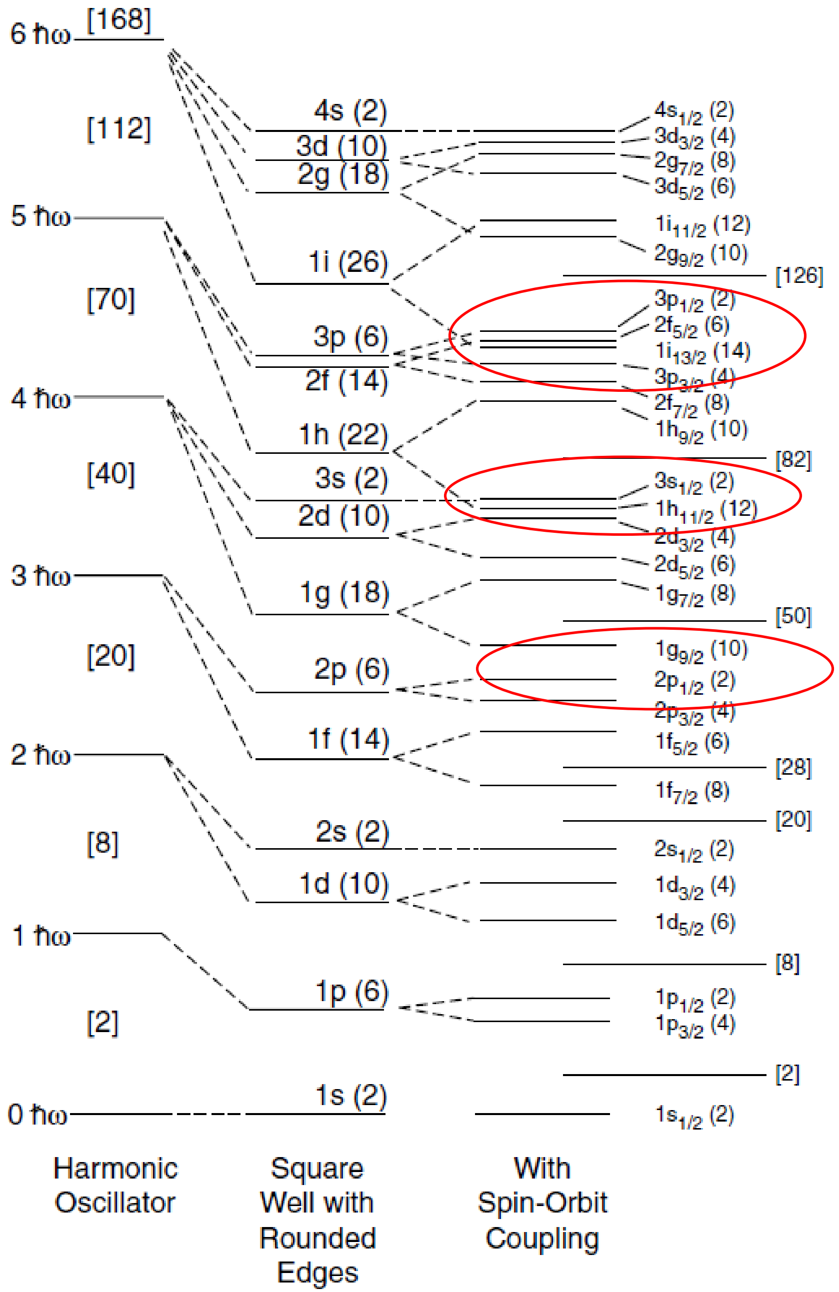
    and nearly all measured g_j fall between these



As with excited state J^π's, deviation between experiment & shell model predictions tell us something interesting is going on. E.g. mixing between single-particle occupations, polarization of the core, corrections to meson-exchange mediating the nuclear force [(G. Neyens, Rep. Prog. Phys. (2003)](), J. Booten et al. Phys.Rev.C (1991))

What else are J^{π} predictions good for? **Isomers** (long-lived x.s.)

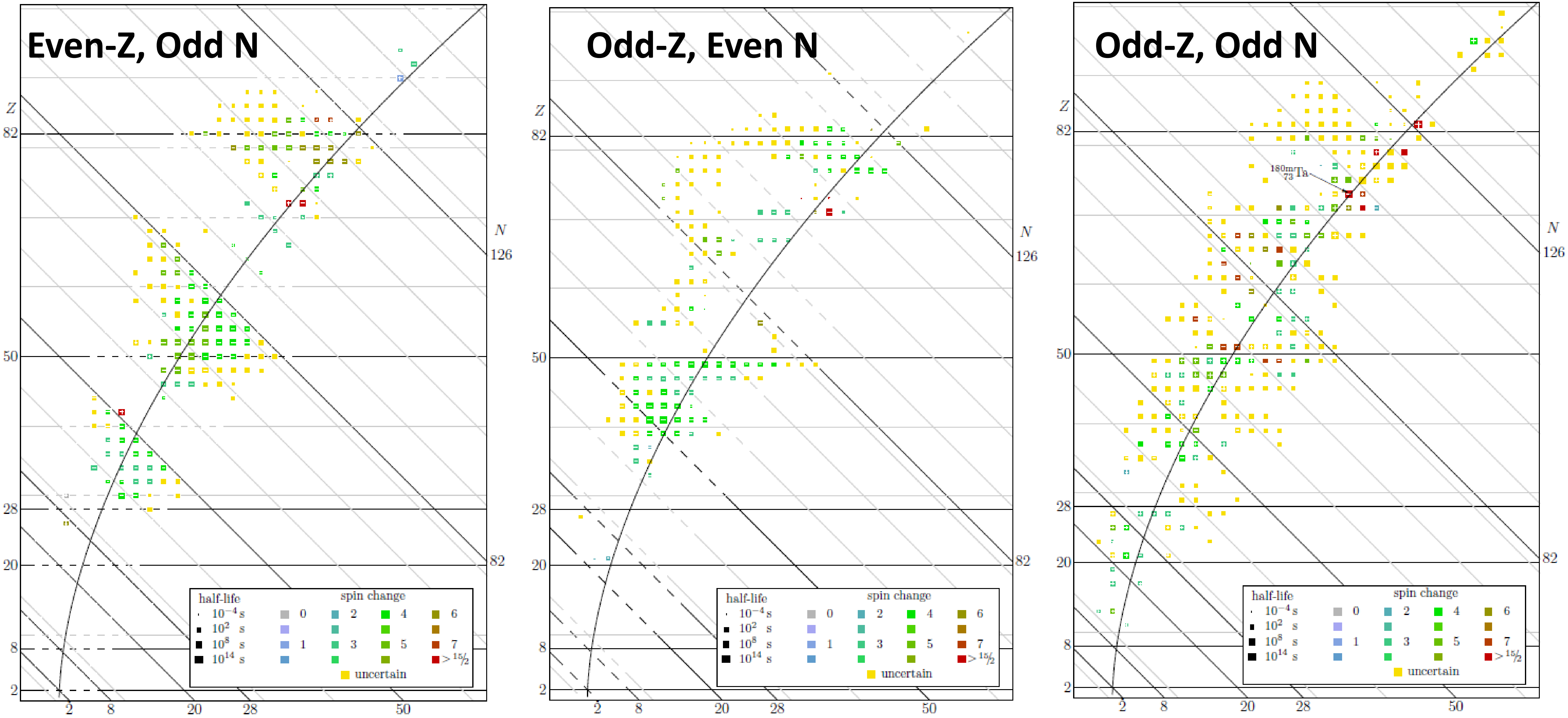
- Most excited states decay via γ-emission in a matter of femto-seconds, but some stick around for many nanoseconds, milliseconds, seconds, or even universe lifetimes.
- These are meta-stable states, a.k.a. isomers
- The reason is γ-emission is suppressed, since it would require large angular momentum transfer
- So, where do we expect low-lying high-j excited states?
    - Where a large Δj exists between neighboring levels (thanks to the spin-orbit interaction) that are near the last single particle orbit.
        - Namely, below magic #'s 50, 82, 126
        - For these cases, we expect a parity change
    - Where multiple j are possible for the ground-state (but one is favored by the Brennan-Bernstein rules) and high-j single-particle levels are involved
        - Namely, odd-odd nuclei
        - For these cases, we don't expect a parity change



Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

Isomers on the Nuclear Chart

L. van Dommelen, Quantum Mechanics for Engineers (2012)

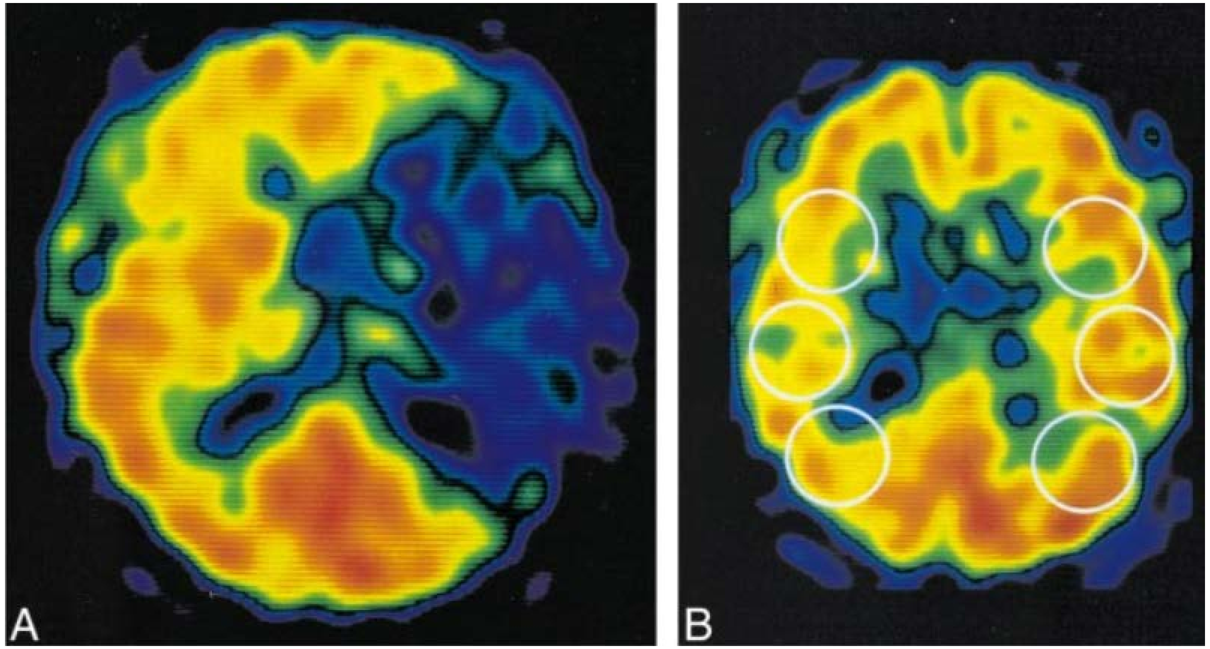


Special cases exist (mostly for higher-A nuclides) where even-even nuclei have isomers
(e.g. M. Müller-Veggian et al., Z.Phys.A (1979))

Impact of Isomers (selected examples)

Medical Imaging

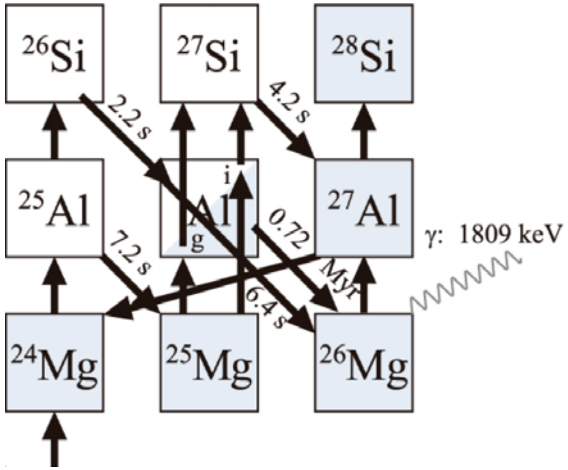
e.g. mapping blood flow in the brain with SPECT using ^{99m}Tc

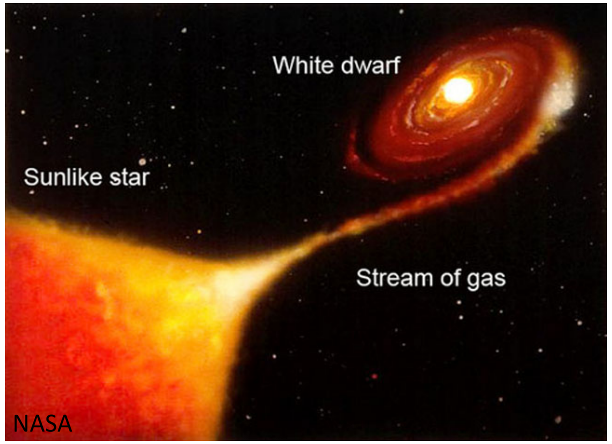


K. Ogasawara et al. American Journal of Neuroradiology (2001)

Nuclear Astrophysics

e.g. ^{26m}Al complicating nova nucleosynthesis calculations





J. José, Stellar Explosions (2015)

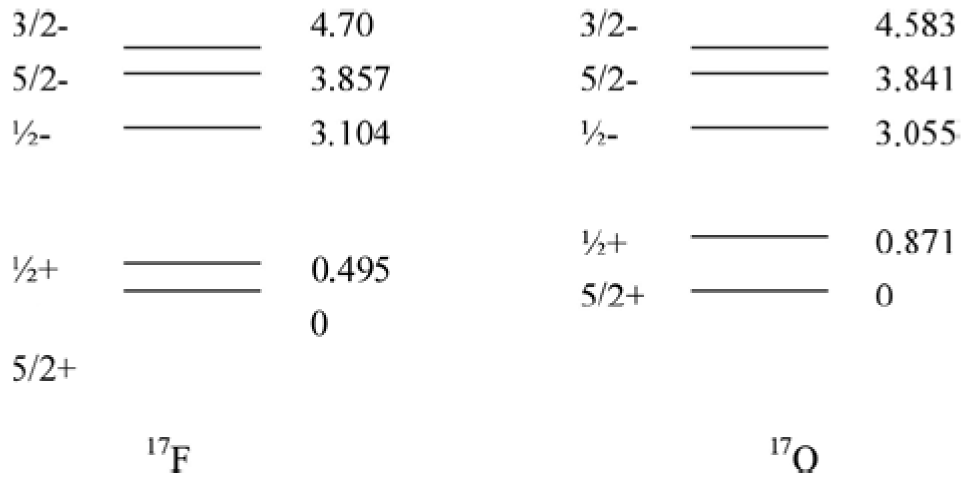
Nuclear Energy Storage

Controlled energy storage and release using isomers and lasers was a hot topic for a while
…but it turns out to be exceedingly difficult, to the point of maybe not being possible

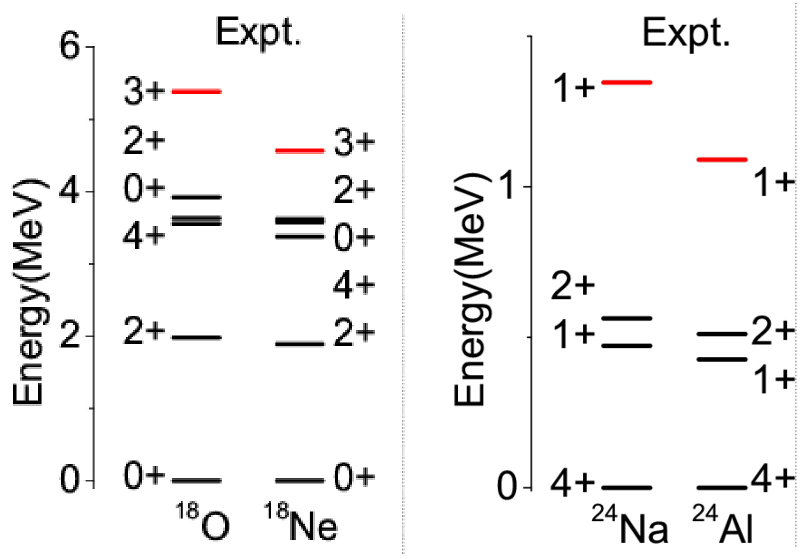
P. Walker & J. Carrol, Physics Today (2005)

What else are J^{π} predictions good for? Mirror Nuclei

- Note that our methods to determine J^{π} didn't depend on whether we were working with protons or neutrons
- If we interchange N for Z, we will get the same answer
- Such pairs are called "mirror nuclei"
- When we examine the levels for mirror nuclei, correcting for the different Coulomb energy, we see a remarkable similarity
- E.g.



Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

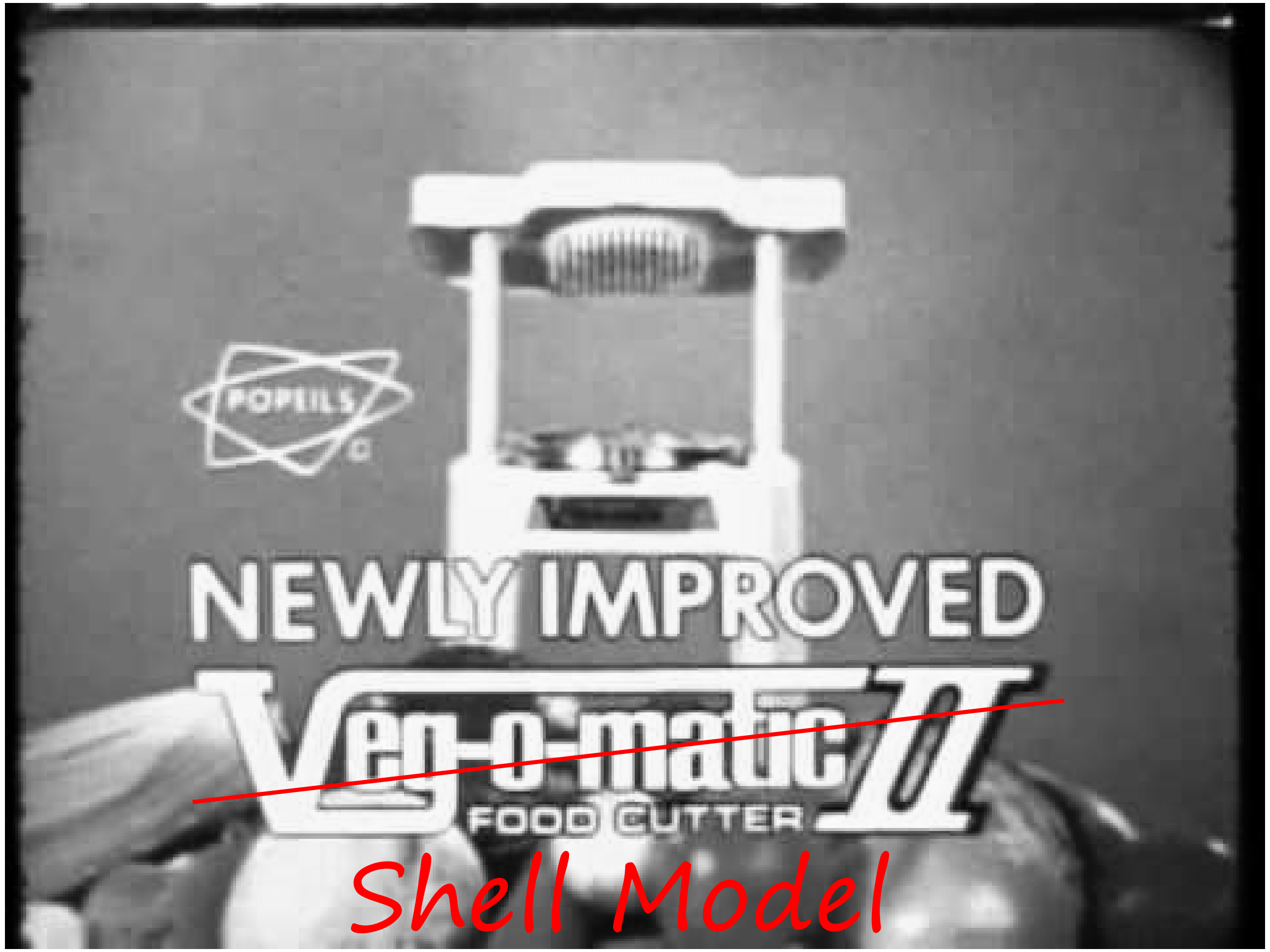


C. Yuan et al., Phys. Rev. C (2014)

- Such mirror symmetry is evidence for charge-independence of the nuclear force and a justification for the concept of isospin

It's also handy when you need estimates for a nucleus you can't access when you can access its mirror (e.g. C.Akers et al. PRC 2016)

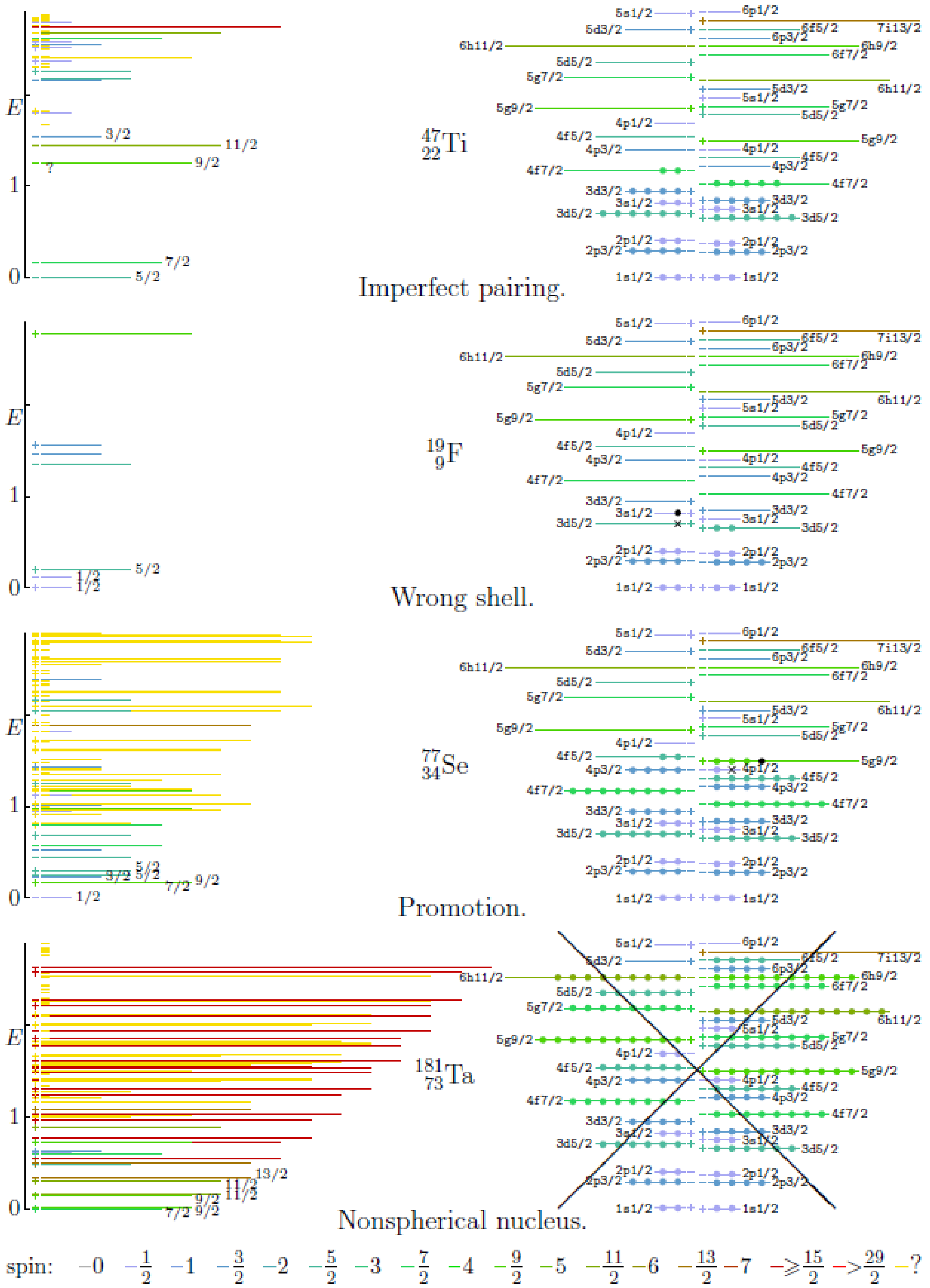
The Shell Model:
It slices, it dices, it makes julienne fries!
What can't it do!?



Shell Model Limitations:

- You'll often find the shell model description isn't as good as you would hope
- The detailed reasons for failures are varied, however, they mostly indicate the basic premise of the calculation is incorrect
- Shell Model calculations (generally speaking) assume
  - Minimally-interacting nucleons (i.e. mostly independent, except pairing)
  - Spherical inert cores of nucleons
- Solutions to these problems are:
  - Modify the shell-model calculation to get the added level of realism
  - Use a different model

L. van Dommelen, Quantum Mechanics for Engineers (2012)

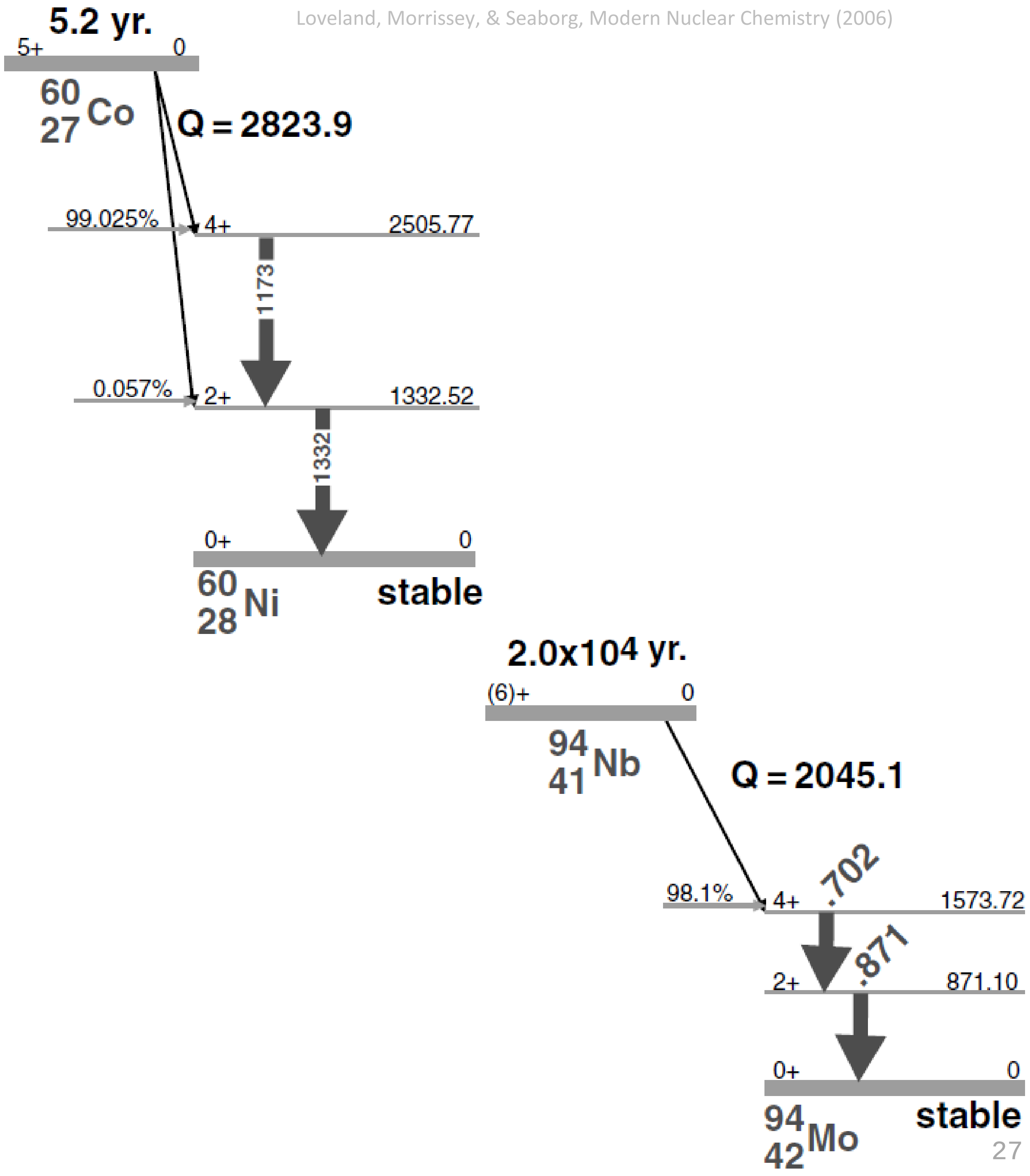


Shell Model Limitations:

- For example, nuclei have series of states that are spaced in energy and linked via transitions that can be described by a collective rotation/vibration of the nucleus.
- The rotational bands of even-even nuclei link the ground state to 2+,4+,6+,8+, etc., excited states

- Another example is the inability to predict nuclear masses.
- Shell model potentials must be adjusted to reproduce the ground-state binding energy
- Some other approach, such as a collective model (e.g. the liquid drop model) is needed instead

Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)



Further Reading

- Chapter 6, Appendix E: Modern Nuclear Chemistry (Loveland, Morrissey, Seaborg)
- Chapters 7: Nuclear & Particle Physics (B.R. Martin)
- Chapters 6-8: Lecture Notes in Nuclear Structure Physics (B.A. Brown)
- Chapter 14, Section 12: Quantum Mechanics for Engineers (L. van Dommelen)
- Chapter 1, Section 6: Nuclear Physics of Stars (C. Iliadis)
- Chapter 11: The Atomic Nucleus (R. Evans)