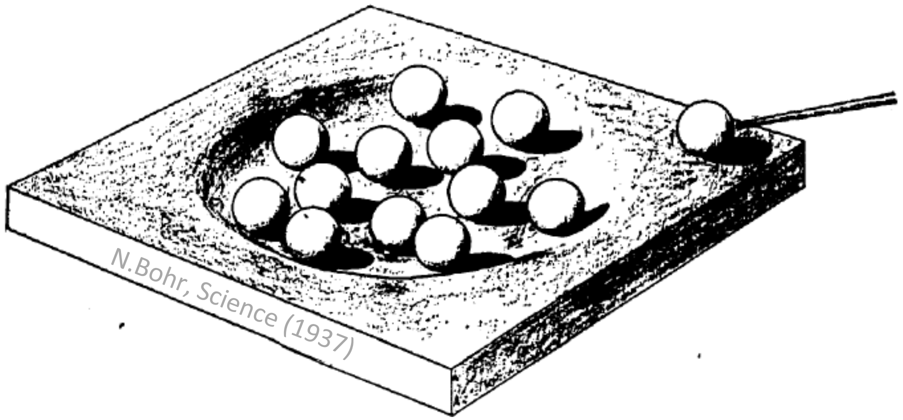
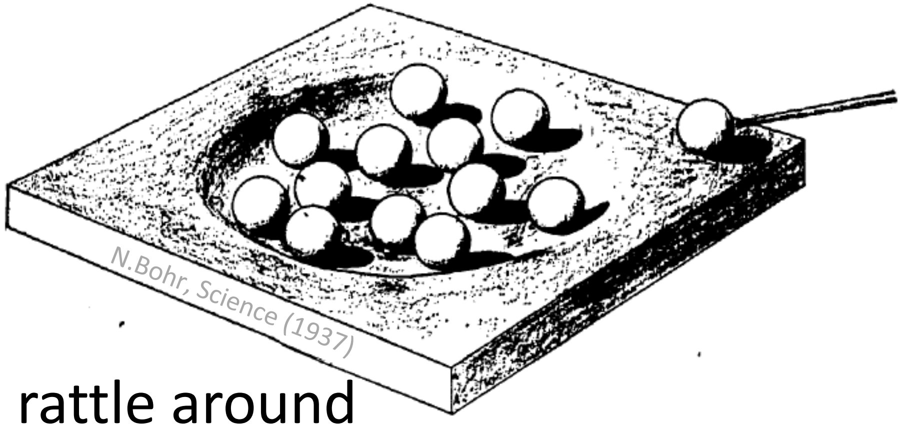
Lecture 17: Statistical Reactions

- Semi-classical picture
- Independence hypothesis ("amnesia")
- Hauser-Feshbach formalism
- Ejectile energy distribution
- Ejectile angular distribution
- Inclusive cross section
- HF applicability



Semi-classical picture



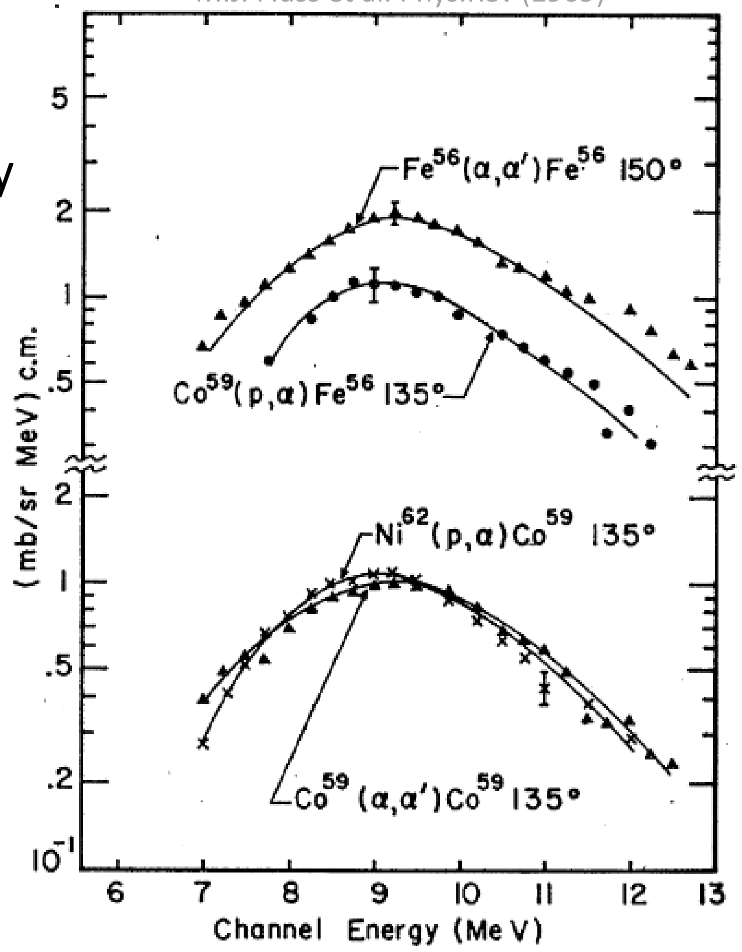
- Consider the case where a projectile fuses with the target, sharing its energy amongst many nucleons in the nucleus, like a billiard ball entering a well and causing several others to rattle around
- The nucleon energies will be distributed statistically and they will scatter with each other until one nucleon happens to pick-up enough of the energy to escape the nucleus (In the analogy, one billiard ball can climb out of the well)
- Adopting this qualitative picture, we expect a few things to result
    - The de-excitation of the compound nucleus is akin to evaporation, meaning the ejectile energy distribution should have a Maxwell-Boltzmann character
    - The multiple collisions occurring with the nucleus erases any signatures left by the initial reaction, so
        - The ejectiles should be isotropic (in the center of mass frame, since momentum still has to be conserved)
        - The de-excitation characteristics for a given compound nucleus excited state energy (e.g. the ejectiles and their energy distributions) shouldn't depend on how the compound nucleus was created

This is termed "amnesia" or "the independence hypothesis"

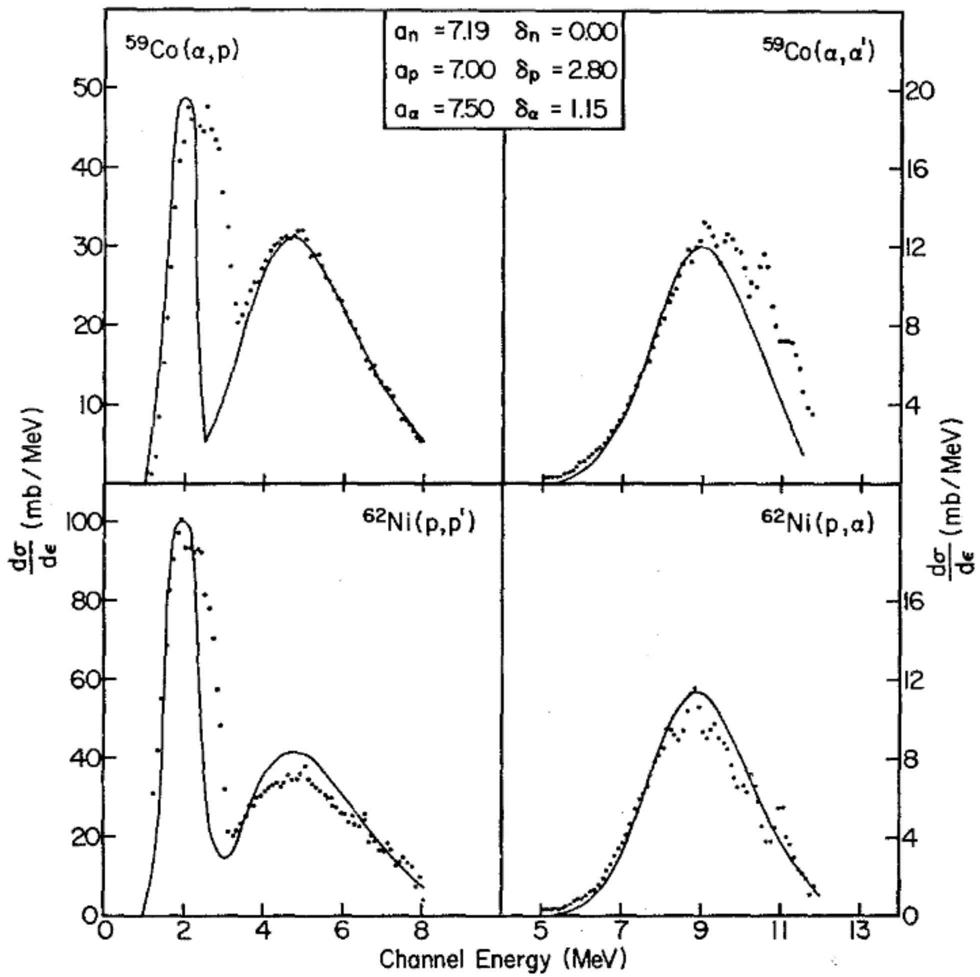
The independence hypothesis (*Amnesia*)

- This statistical reaction picture implies that the compound nucleus forgets how it was formed,
  so the decay properties depend only on the compound nucleus itself
- This is confirmed by decay spectra for nuclei populated by various channels
- The key implication here is that compound nucleus formation and decay probabilities are separated
- One key result which follows is that other probes, e.g. β-decay, can be used to determine key properties needed to understand much harder to measure reactions, e.g. (p,γ)

M.J. Fluss et al. Phys.Rev (1969)

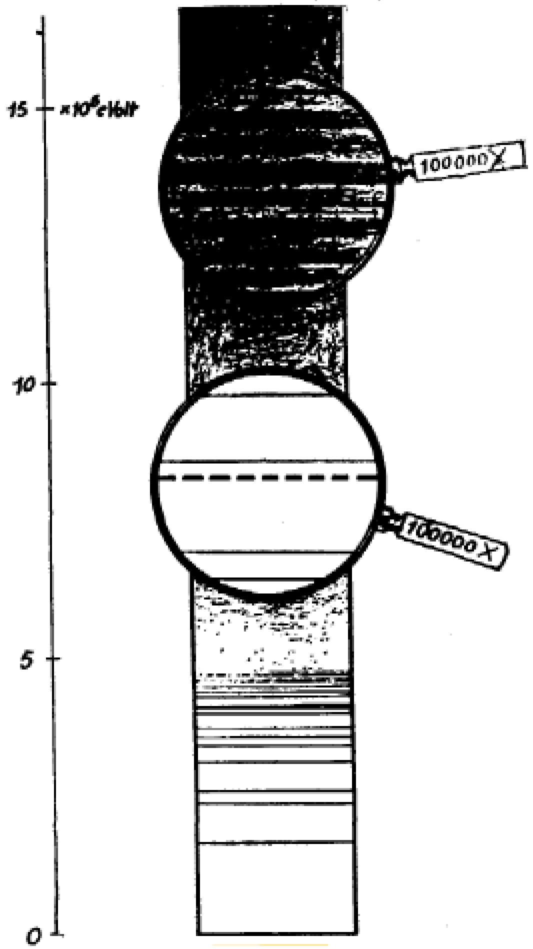


J. Wiley et al., Nuc.Phys.A (1973)



Hauser-Feshbach formalism

N.Bohr, Science (1937)



- The semi-classical picture of nucleons rattling around in the nucleus sampling many configurations until one configuration occurs in which the compound nucleus can de-excite by evaporation corresponds to the case of many nearby resonances
- Anyhow, if you don't buy this, it's true that the characteristics of statistical nuclear reactions [isotropic ejectile emission with a Maxwell-Boltzmann energy distribution] are seen for reactions where high level-density regions are populated in the compound nucleus
- In this picture, many resonances are closely spaced and the projectile will experience an interaction that's the statistical average of said resonances
- Note that Hauser-Feshbach (HF) assumes the level spacing $D \gg \Gamma$.
  It is frequently misstated that HF assumes overlapping resonances ($\Gamma \gg D$), but this is not the case.
  That scenario requires the generalization from Kawai, Kerman, & McVoy (Annals of Physics, 1973)

Hauser-Feshbach formalism

- Recall that for a single resonance, $\sigma_{BW,X(a,b)Y}(E) = \pi\left(\frac{\lambda}{\pi}\right)^2 \frac{2J+1}{(2J_a+1)(2J_X+1)} \frac{\Gamma_{aX}(E)\Gamma_{bY}(E)}{(E-E_R)^2+(\Gamma(E))^2/4}$
- Averaging over several resonances in an energy range requires integrating over some energy window dE and normalizing by the level-spacing D
- After brushing up on your residues, you'll recall $\int_0^\infty \frac{\Gamma_{aX}(E)\Gamma_{bY}(E)}{(E-E_R)^2+(\Gamma(E))^2/4} dE = 2\pi \frac{\Gamma_{aX}(E_R)\Gamma_{bY}(E_R)}{\Gamma(E_R)}$
- So the average cross section involving compound nuclear resonances for states with spin J is: $\left\langle \sigma^J_{X(a,b)Y}(E) \right\rangle = \pi\left(\frac{\lambda}{\pi}\right)^2 \frac{2J+1}{(2J_a+1)(2J_X+1)} \frac{2\pi}{D} \left\langle \frac{\Gamma_{aX}\Gamma_{bY}}{\Gamma} \right\rangle$
- We would rather be dealing with average widths, rather than averages of their products and ratios, so we take into account their correlations with a width fluctuation factor W_{ab}, where $\left\langle \frac{\Gamma_{aX}\Gamma_{bY}}{\Gamma} \right\rangle = W_{ab} \frac{\langle\Gamma_{aX}\rangle\langle\Gamma_{bY}\rangle}{\langle\Gamma\rangle}$
- So, the cross section is: $\sigma_{X(a,b)Y} = \sum_J \left\langle \sigma^J_{X(a,b)Y}(E) \right\rangle = \pi\left(\frac{\lambda}{\pi}\right)^2 \sum_J \frac{2J+1}{(2J_a+1)(2J_X+1)} \frac{2\pi}{D} W_{ab} \frac{\langle\Gamma_{aX}\rangle\langle\Gamma_{bY}\rangle}{\langle\Gamma\rangle}$

Note: Calculating W_{ab} can be complicated [See the Talys manual]
… but luckily it is ≈1 if multiple channels are open [which is usually the case].
For elastic scattering, the entrance and exit are obviously correlated; W_{ab} ≈2-3.

Hauser-Feshbach formalism

- The sum over resonances $\sigma_{X(a,b)Y}(E) = \sum_J \left\langle \sigma^J_{X(a,b)Y}(E) \right\rangle = \pi \left(\frac{\lambda}{\pi}\right)^2 \sum_J \frac{2J+1}{(2J_a+1)(2J_X+1)} \frac{2\pi}{D} W_{ab} \frac{\langle \Gamma_{aX} \rangle \langle \Gamma_{bY} \rangle}{\langle \Gamma \rangle}$ works fine in principle …but in practice we don't have all of this information about all of the levels within the compound nucleus. Also, it turns out we don't need it.
- Consider the reaction cross section (i.e. summing over all ejectile channels), for compound nuclear states of spin J: $\sigma_{X+a}(E) = \sum_b \sigma_{X(a,b)}(E) = \pi \left(\frac{\lambda}{\pi}\right)^2 \frac{2J+1}{(2J_a+1)(2J_X+1)} \frac{2\pi}{D} \langle \Gamma_{aX} \rangle$
- Recall from our semi-classical cross section discussion in the Reactions lecture, the reaction cross section for a given orbital angular momentum transfer l is: $\sigma_l(E) = \pi \left(\frac{\lambda}{\pi}\right)^2 gT_l$ where g is the statistical factor (for our spinless particle from before: $2l+1$) and T_l is the transmission coefficient for the entrance channel calculated using the optical model
- By comparing the two, we can see the transmission coefficient for some reaction channel is related to that channel's average width by $T_{chan} = 2\pi \frac{\langle \Gamma_{chan} \rangle}{D}$

You could calculate T from an optical potential, get <Γ>, and Monte Carlo the Porter-Thomas distribution to estimate a realistic resonance partial width

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- Since $0 \leq T \leq 1$, sometimes (by considering a Taylor expansion) this is enforced by $T_{chan} = 1 - \exp\left(-2\pi \frac{\langle \Gamma_{chan} \rangle}{D}\right)$, so that any choice of $\langle \Gamma \rangle / D$ will result in a reasonable T

Hauser-Feshbach formalism

- Now, by swapping-in the transmission coefficients T, which we can get using the optical model, for the average resonance widths Γ, which we generally don't know, we get the Hauser-Feshbach cross section

$$\sigma_{X(a,b)Y}^{HF} = \sum_J \left\langle \sigma_{X(a,b)Y}^{J}(E) \right\rangle = \pi \left(\frac{\lambda}{\pi}\right)^2 \sum_J \frac{2J+1}{(2J_a+1)(2J_X+1)} W_{ab} \frac{T_{aX} T_{bY}}{\sum_{chan} T_{chan}}$$

- The full sum requires taking into account angular momentum conservation, parity conservation, and energy conservation (to determine which outgoing channels are possible)
- For exit channels, we need to take into account the number of discrete states that are available for such a decay. Naturally, more final states leads to a higher probability for that type of decay. As such, really T_{bY} is $\sum_{n_b} T_{bY}$ and T_{chan} is $\sum_{n_{chan}} T_{chan}$.

E.g. $\mathcal{T}_\gamma(E,J,\pi) = \sum_{\nu=0}^{\nu_r} \mathcal{T}_\gamma^\nu(E,J,\pi,E_r^\nu,J_r^\nu,\pi_r^\nu) + \int_{E_r^{\nu_r}}^{E} \sum_{J_r,\pi_r} \mathcal{T}_\gamma^\nu(E,J,\pi,E_r^\nu,J_r^\nu,\pi_r^\nu) \cdot \rho(E_r,J_r,\pi_r) \mathrm{d}E_r$

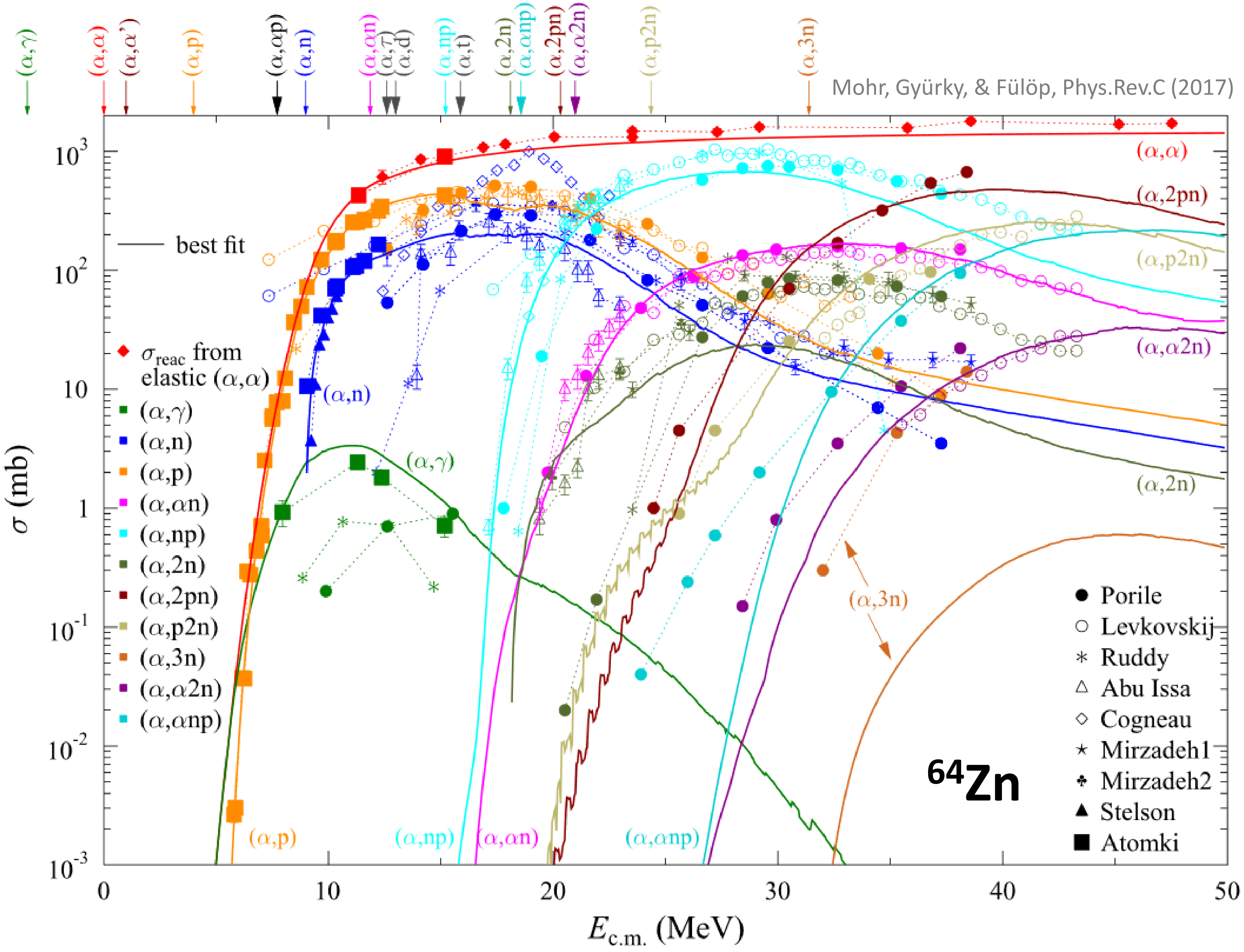
- So, the key ingredients to calculating the Hauser-Feshbach cross section are
  - The transmission coefficient for the entrance channel *… from an optical potential*
  - The transmission coefficient for all exit channels *… from optical potentials*
  - The number of available levels (and their energies) for all exit channels *… from level density models and spin-cutoff parameters*

**For γ-rays, instead of an OMP, need a γ-strength function (γSF)*

See the Scattering and Alpha-Decay lectures for T-coeffs from OMPs, the Nuclear Structure 3 for level-densities and spin-cutoff parameters, and the Gamma-decay for gamma-strength functions

Comparison to cross section measurements

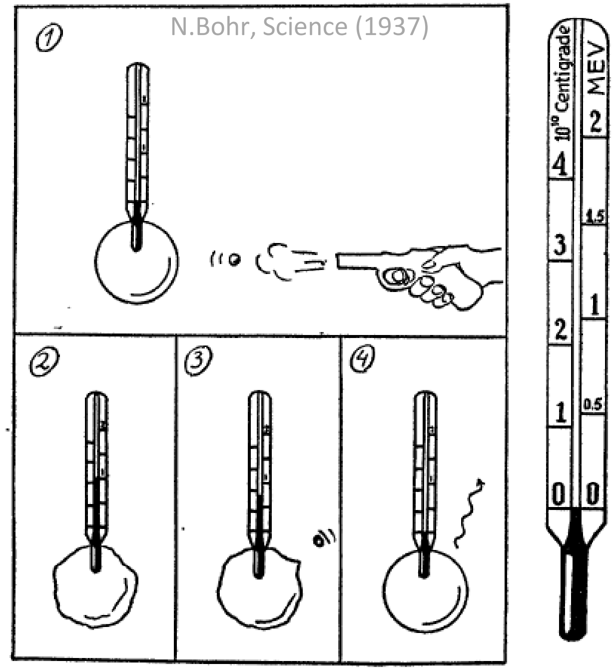
- HF calculations do a great job of reproducing data, when applicable (we'll discuss applicability in a minute)
- Given reasonable input choices, HF predictions sometimes only vary as much as a factor of a few, but can occasionally vary up to a factor of 100 (10 is more typical)



Ejectile energy distribution

- We want the energy distribution $P(\varepsilon)$ for ejectile b for a decay from a compound nucleus C excited to energy E_C^* to a residual nucleus with excitation energy $E_R^* = E_C^* - S_{b,C} - \varepsilon$, where $S_{b,C}$ is the separation energy for ejectile b from C
- Following Weisskopf (Phys.Rev. 1937), from time reversal symmetry ("detailed balance" in kinetics), $C \rightarrow b + R$ should have the same rate (a.k.a. probability) as $b + R \rightarrow C$, once accounting for statistical factors: $P_{C\rightarrow b+R}(\varepsilon) \propto P_{b+R\rightarrow C}(\varepsilon)$
- Each probability should be normalized by the density of available final states: $\frac{P_{C\rightarrow b+R}(\varepsilon)}{\rho_R(E_R^*)\rho_b(\varepsilon)} = \frac{P_{b+R\rightarrow C}(\varepsilon)}{\rho_C(E_C^*)}$
- The rate for forming the compound nucleus $P_{b+R\rightarrow C}(\varepsilon)$ for particle b moving with velocity v within the nuclear volume of R given the intrinsic cross section σ_{b+R} is: $P_{b+R\rightarrow C}(\varepsilon) = \sigma_{b+R}\, {}^{v}/_{V_R}$
- So, $P_{C\rightarrow b+R}(\varepsilon) = \sigma_{b+R} \frac{\sqrt{2\varepsilon/m}}{V_R} \frac{\rho_R(E_R^*)}{\rho_C(E_C^*)} \rho_b(\varepsilon)$
- The number of states b can occupy to result in residual nucleus excitation energy E_R^* scales as $V_R\sqrt{\varepsilon}$, as we saw back in the lecture on level density (Nuclear Structure 3)
- In that same lecture, we found that in the constant-temperature approximation, level densities scale as $\rho_{nuc}(E) \propto \frac{1}{T} e^{E^*/T}$, where T is the nuclear temperature

Ejectile energy distribution



N.Bohr, Science (1937)

- Putting those pieces together, $P_{C\to b+R}(\varepsilon) = \sigma_{b+R}\frac{\sqrt{2\varepsilon/_m}}{V_R}\frac{\rho_R(E_R^*)}{\rho_C(E_C^*)}\rho_b(\varepsilon)$,

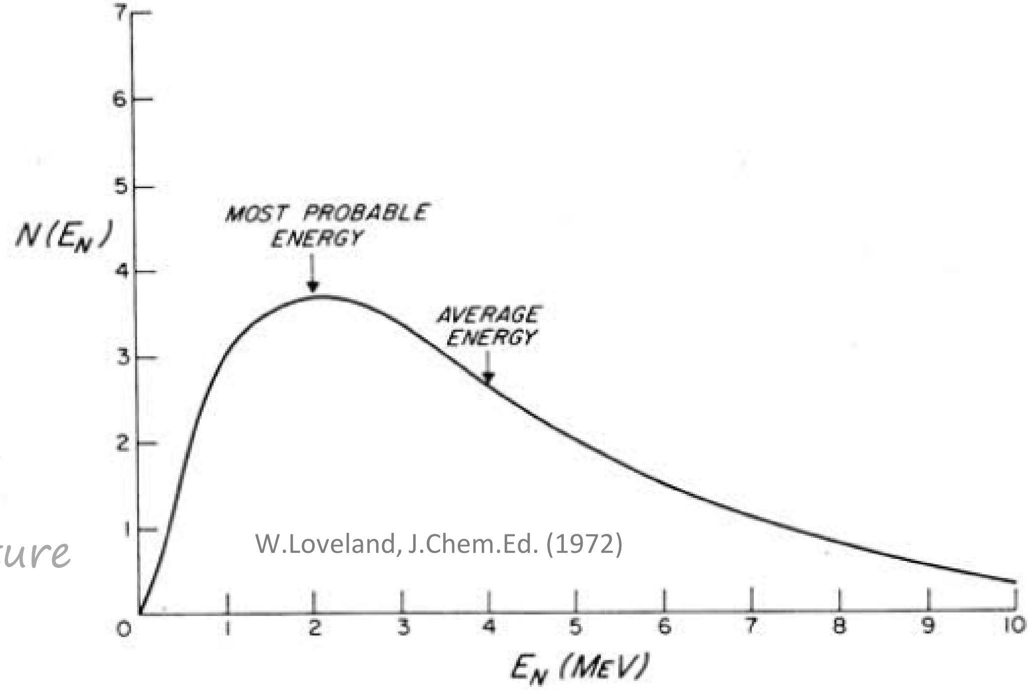
  becomes $P_{C\to b+R}(\varepsilon) \propto \varepsilon\frac{\exp\left(\frac{E_R^*}{T_R}\right)}{\exp\left(\frac{E_C^*}{T_C}\right)} = \varepsilon\frac{\exp\left(\frac{E_C^* - S_{b,C} - \varepsilon}{T_R}\right)}{\exp\left(\frac{E_C^*}{T_C}\right)} \propto \varepsilon\exp\left({}^{-\varepsilon}/_{T_R}\right)$

- Thus, we've arrived at the promised Maxwell-Boltzmann distribution for the ejectile energy distribution, confirming the picture of a "heated" nucleus "evaporating" nucleons to "cool"

- The distribution is more commonly written as $N_b(\varepsilon) = \varepsilon\exp\left({}^{-\varepsilon}/_{T_R}\right)$, since the other factors wind up being ≈ 1

- To arrive at useful numbers, recall the nuclear temperature is related to excitation energy by $T \approx \sqrt{E^*/_a}$ where (empirically) $a \approx \frac{A}{8}MeV^{-1}$

For more accurate calculations, see the Talys manual for empirical Temperature and level-density parameter

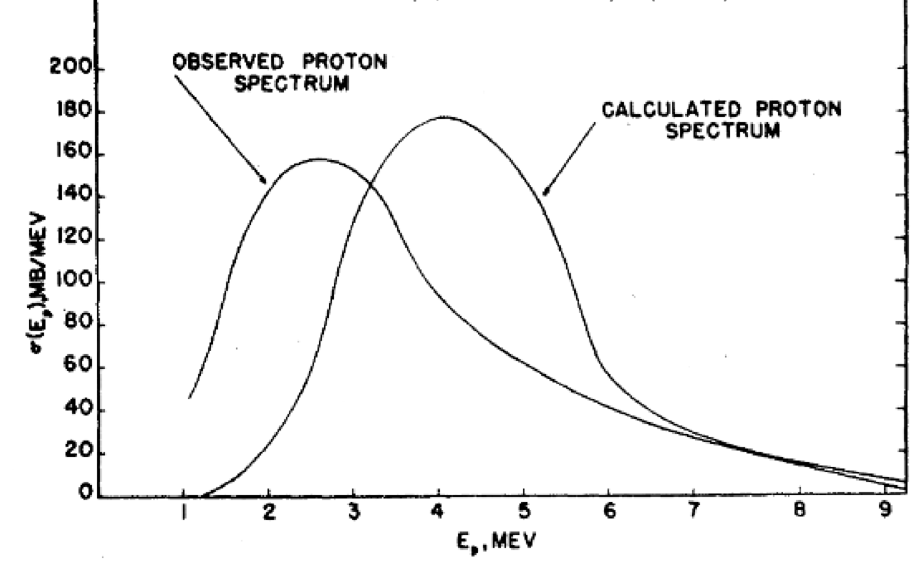


W.Loveland, J.Chem.Ed. (1972)

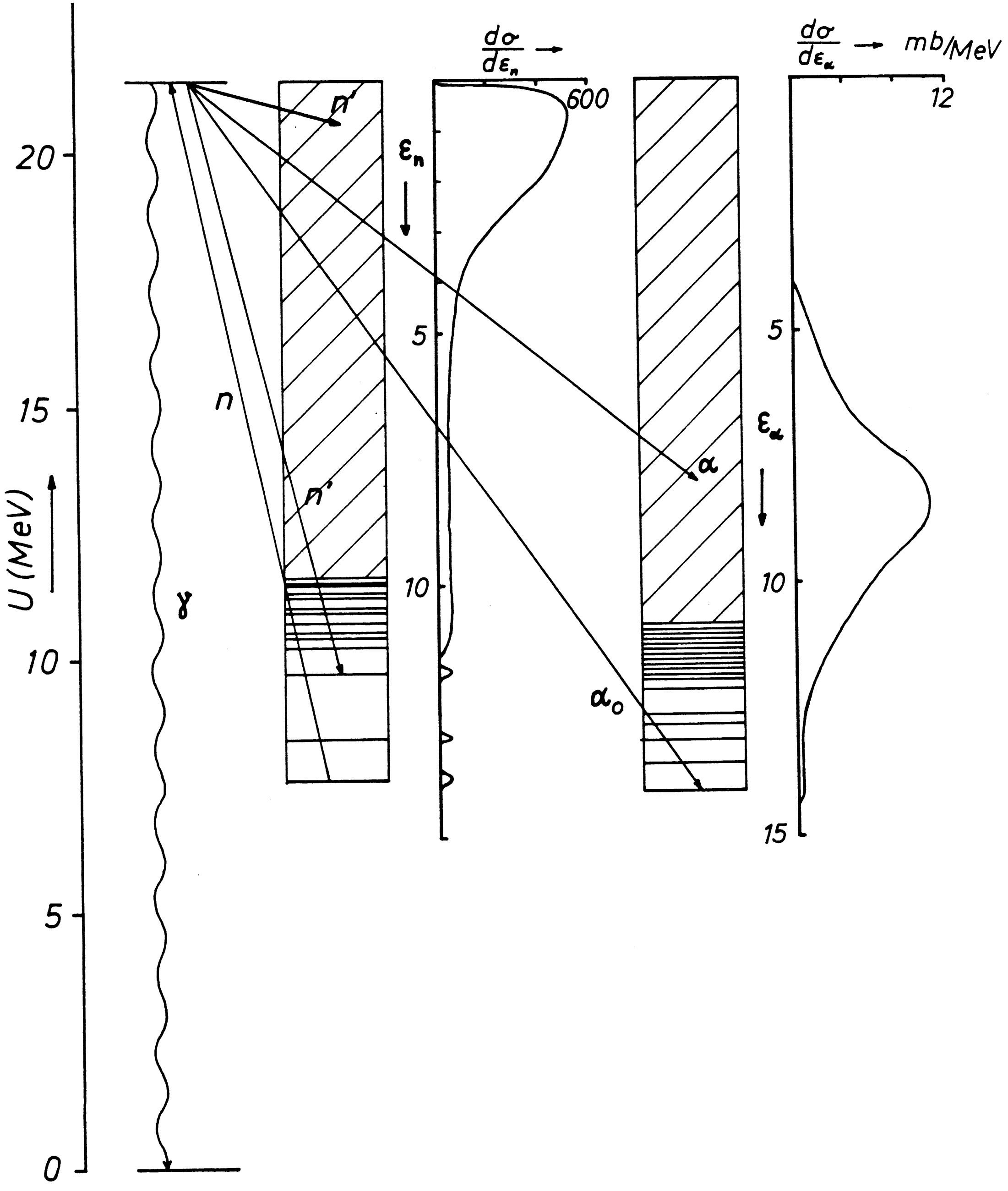
Ejectile energy distribution

- An important correction exists for charged particles, since a charged ejectile has to tunnel through the Coulomb barrier
- For those cases, in the MB distribution, $\varepsilon \rightarrow\approx \varepsilon - V_{coul}$, which shifts the distribution peak to higher energies
- However, since it's the surface nucleons that are usually emitted, V_{coul} is an overestimate

V.Weisskopf, Rev.Mod.Phys. (1957)



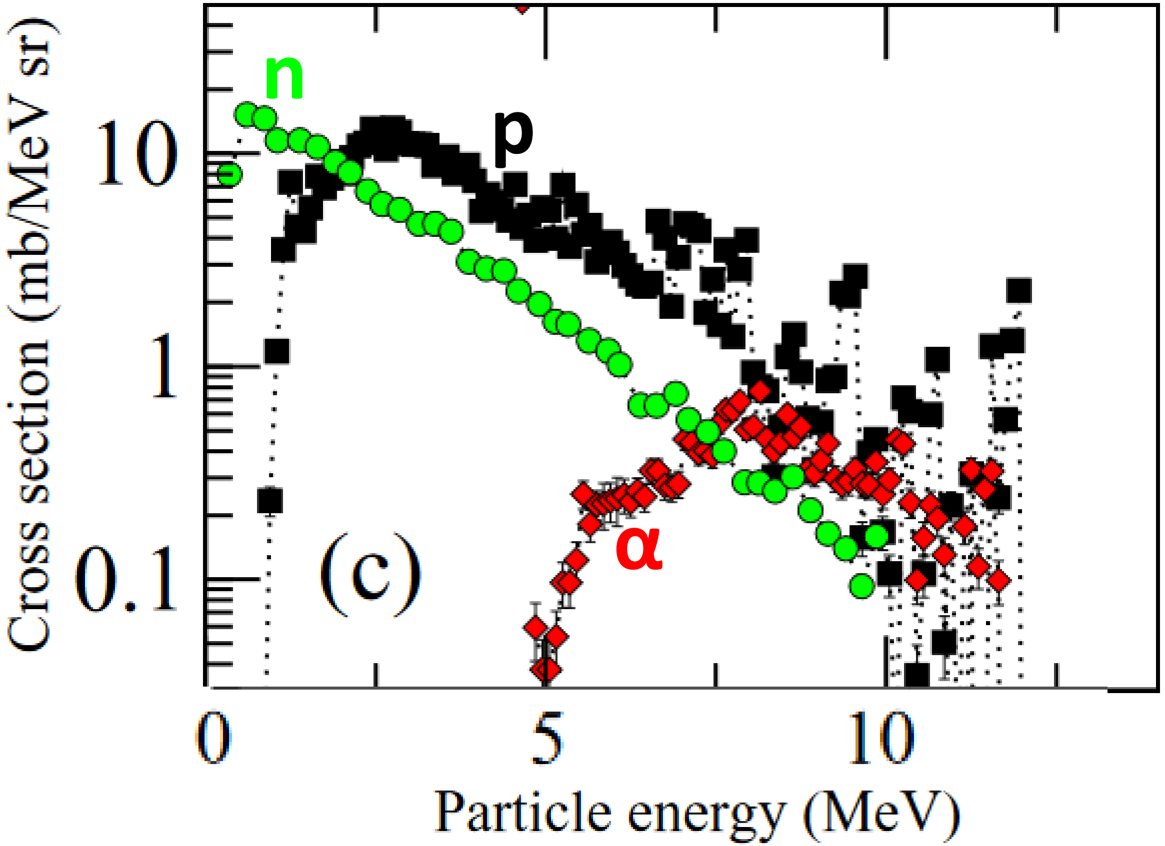
B.V.Zhuravlev? (In their talk …but can't find original source)

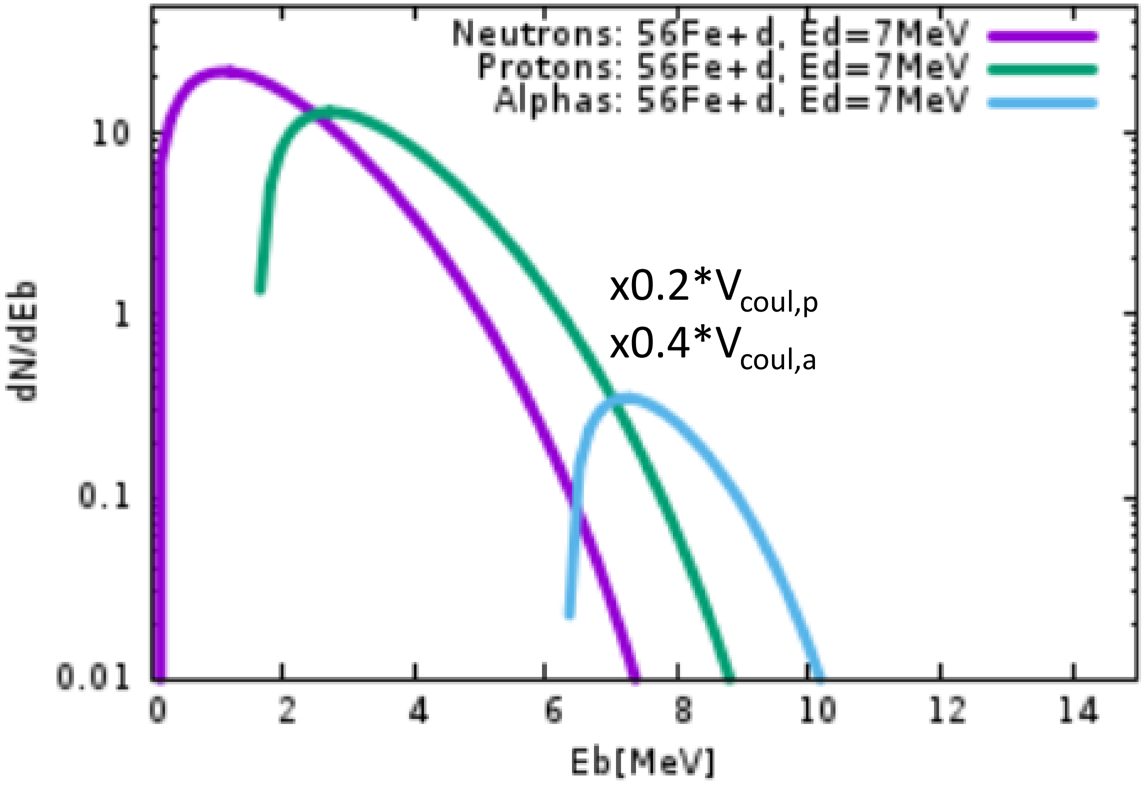


Ejectile energy distribution

- Example: ^{56}Fe+d for E_d=7MeV: $E_C^* = Q_{56Fe(d,\gamma)} + E_{cm} \approx 19.1MeV, S_{n,58Co} \approx 8.6MeV$
- Applying an arbitrary normalization (x50) results in the purple distribution [compare to green]
- Performing a similar calculation for protons and alphas yields comparable spectra, but ***cheat alert***: alternative normalizations had to be chosen, as did reduction factors for V_{coul}

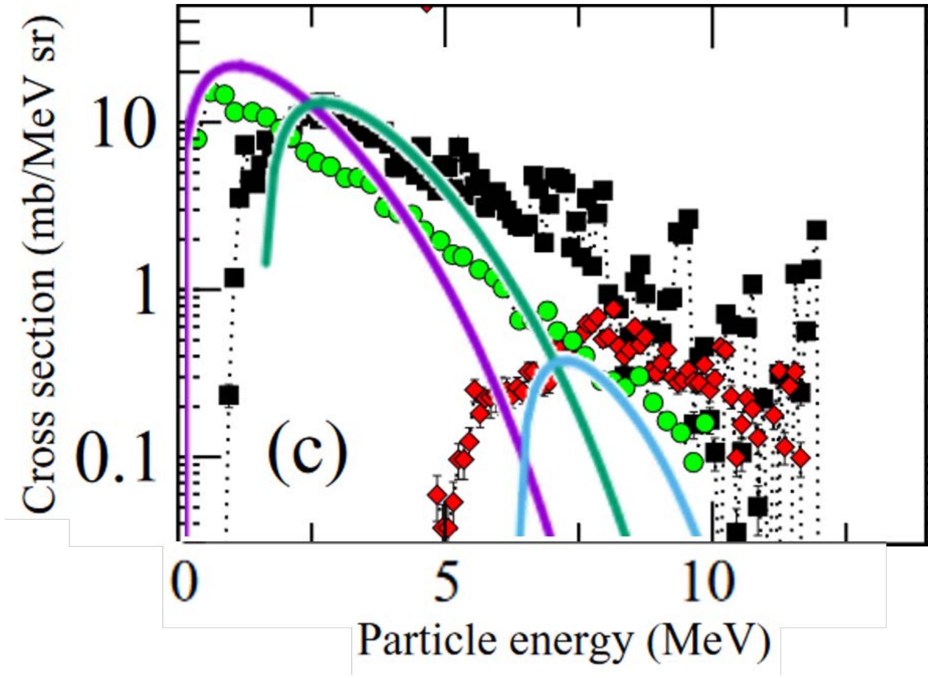
A.P.D.Ramiriez et al. Phys.Rev.C (2015)





Ejectile energy distribution

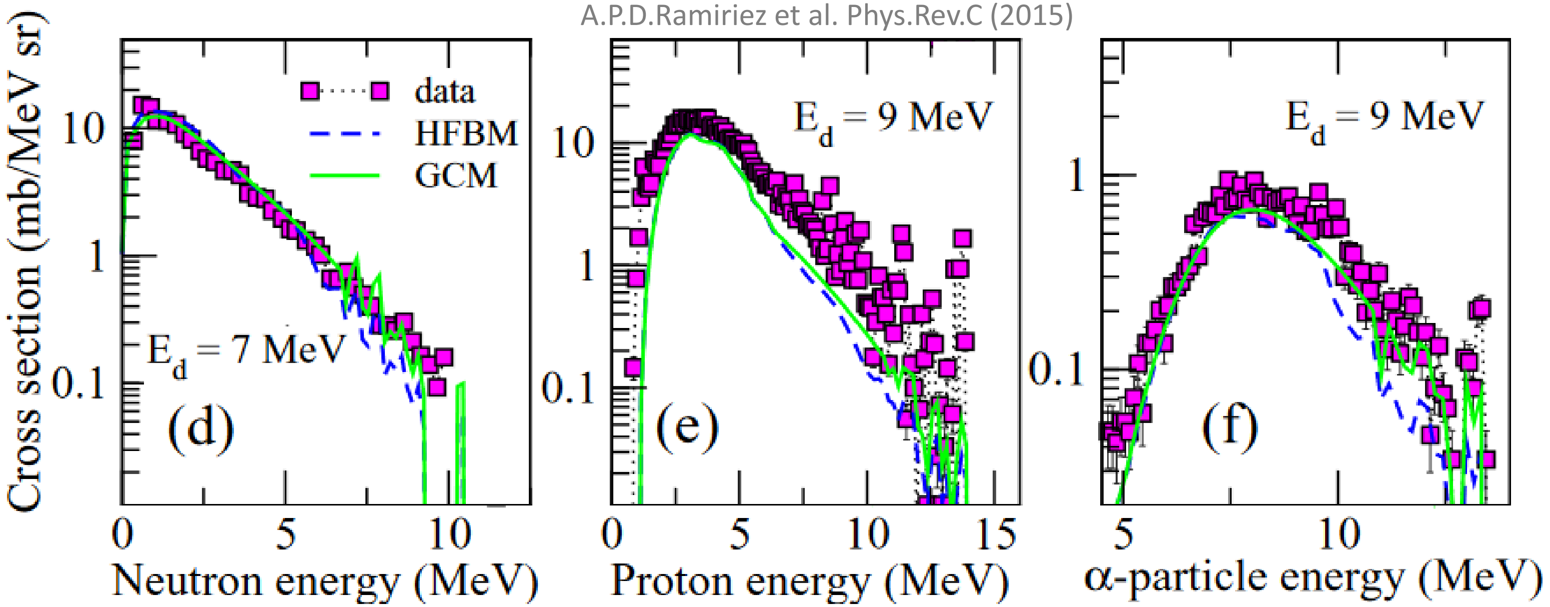
A.P.D.Ramiriez et al. Phys.Rev.C (2015)



- Given that sub-par comparison, how is the ejectile energy distribution useful?
  (other than confirming the evaporation picture)
- Comparisons with HF calculations can be used to experimentally constrain level densities!
- To remove most of the normalization worries, what's actually done is:

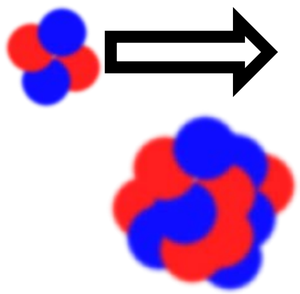
$$\rho_{expt} = \rho_{theory} \frac{d\sigma/_{dE_b}|_{expt}}{d\sigma/_{dE_b}|_{theory}}$$

A.P.D.Ramiriez et al. Phys.Rev.C (2015)

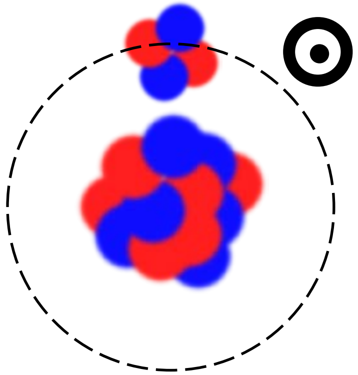


Ejectile **angular** distribution

- If we take the independence hypothesis to heart, then we would expect the compound nucleus to have lost all information about how it was formed
- As such, in the center-of-mass frame we expect an isotropic emission of ejectiles
  (in the lab there will be a bias toward forward angles from momentum conservation)
  … however, that picture is a bit too naïve



- Consider the classical picture of a projectile bringing in some angular momentum $\vec{l}$
- In order to rid the angular momentum from the system,
  the best scenario would be for the ejectile to be emitted perpendicular to $\vec{l}$
- By considering all trajectories corresponding to $\left|\vec{l}\right|$, we realize that ejectiles
  will be preferentially emitted at forward and backward angles,
  since these angles are perpendicular to all $\vec{l}$ for a given $\left|\vec{l}\right|$,
  and symmetric about $\theta_{cm} = 90°$



- As we might anticipate from our qualitative picture,
  the anisotropy is only appreciable for heavy projectiles and/or large incident energies
  (*larger incident energies will face more competition with forward-peaked direct processes)

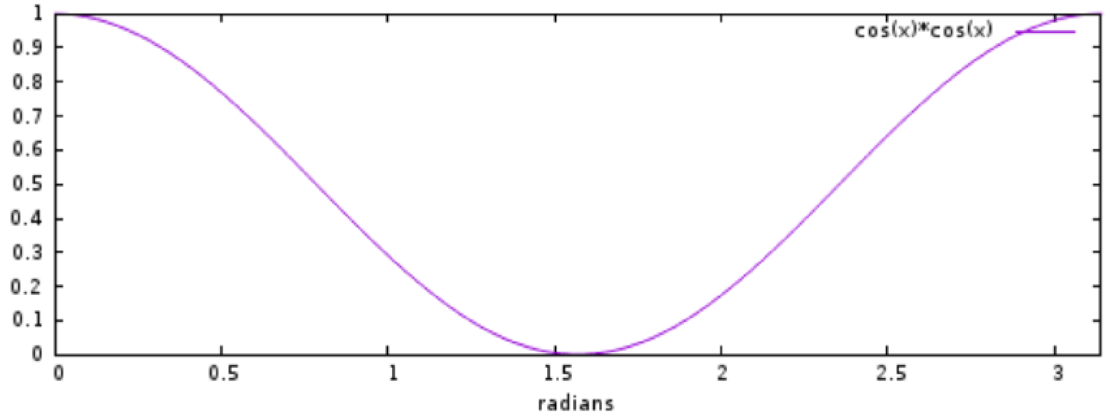
Ejectile angular distribution

- For a slightly more quantitative analysis, consider the fact that the scattered wave goes like: $\psi_{sc} \propto \sum_{l=0}^{\infty} P_l(\cos(\theta))$
- To satisfy the symmetry about $90°$, only even-l enter in the sum
- Since we expect high l to be suppressed by the centrifugal barrier, to first order the anisotropy will be similar to the $l = 2$ Legendre polynomial: $\propto \cos^2(\theta)$
- Each l is also weighted (among other factors) by the final density of levels with J that can be accessed by angular momentum transfer l
- Recall that the level density is the state density weighted by the spin-distribution

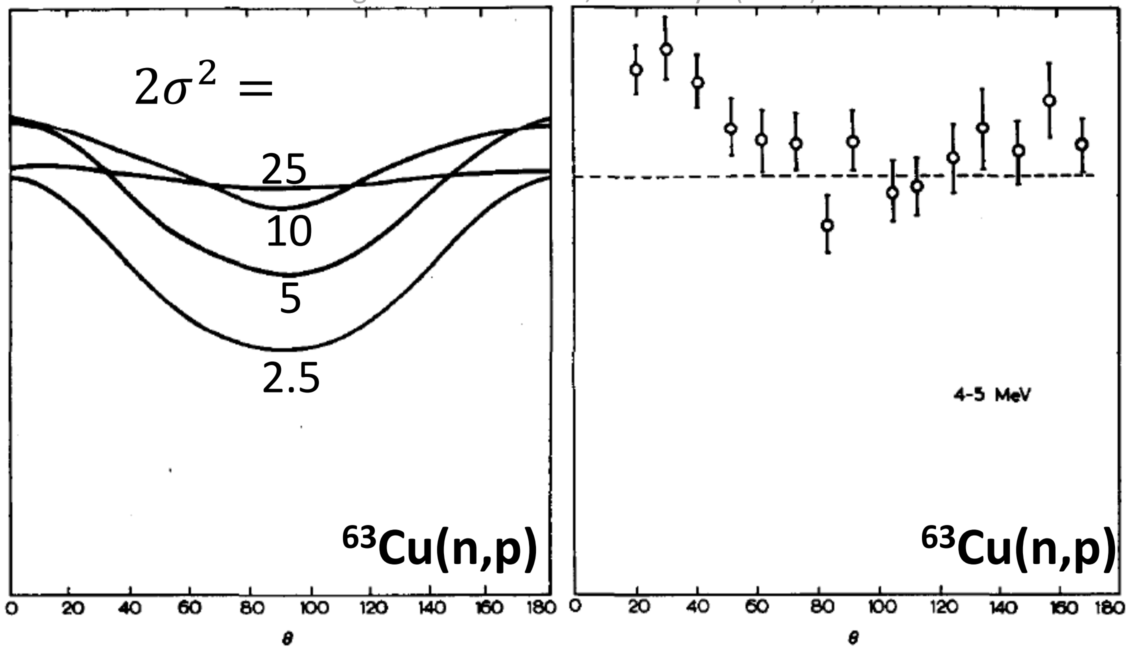
$$\rho(E^*, J) \approx \rho(E^*)\frac{2J+1}{2\sigma^2}\exp\left(\frac{-\left(J+\frac{1}{2}\right)^2}{2\sigma^2}\right),$$

where σ is the spin-cutoff parameter

- Smaller σ will result in a narrower spin distribution, and therefore a more anisotropic angular distribution

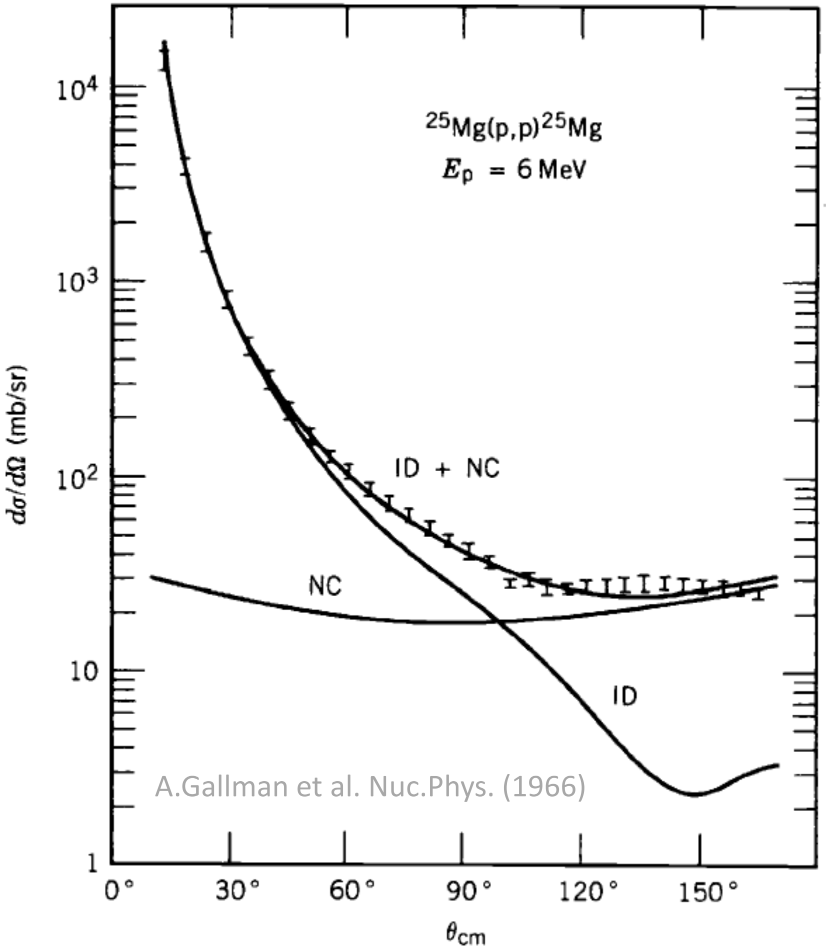


Douglas & Macdonald, Nuc.Phys. (1959)



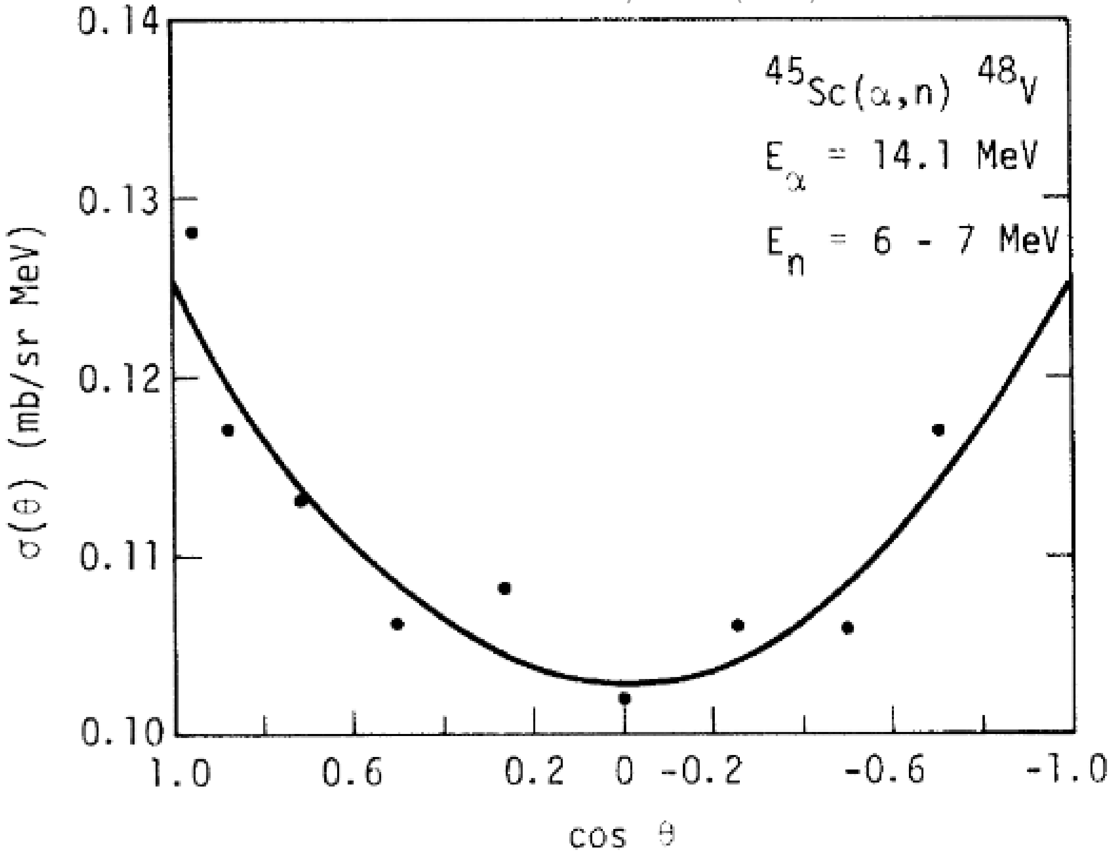
Ejectile angular distribution, experimental considerations

- The statistical nuclear reaction mechanism will compete with direct reactions, so backward angles are where one looks for information about the compound nucleus
- Similarly, lower bombarding energies and channels that would require multiple nucleon-transfer for a direct reaction are more promising for statistical nuclear reaction signatures



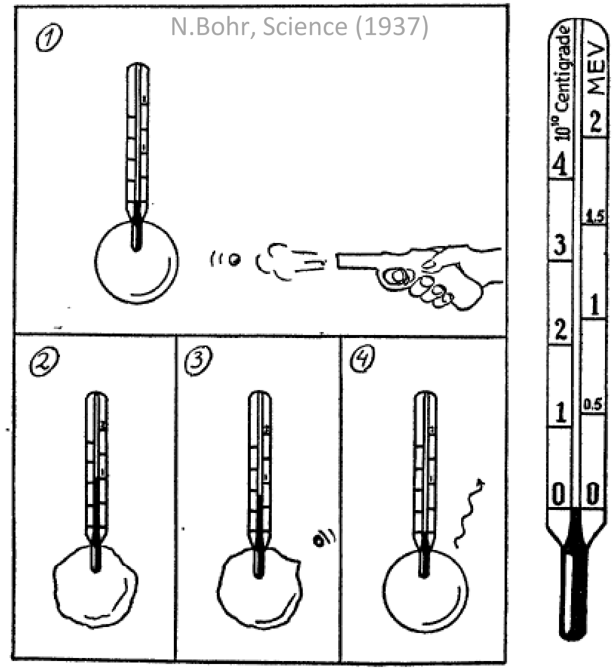
A.Gallman et al. Nuc.Phys. (1966)

S.M. Grimes et al. Phys.Rev.C (1974)

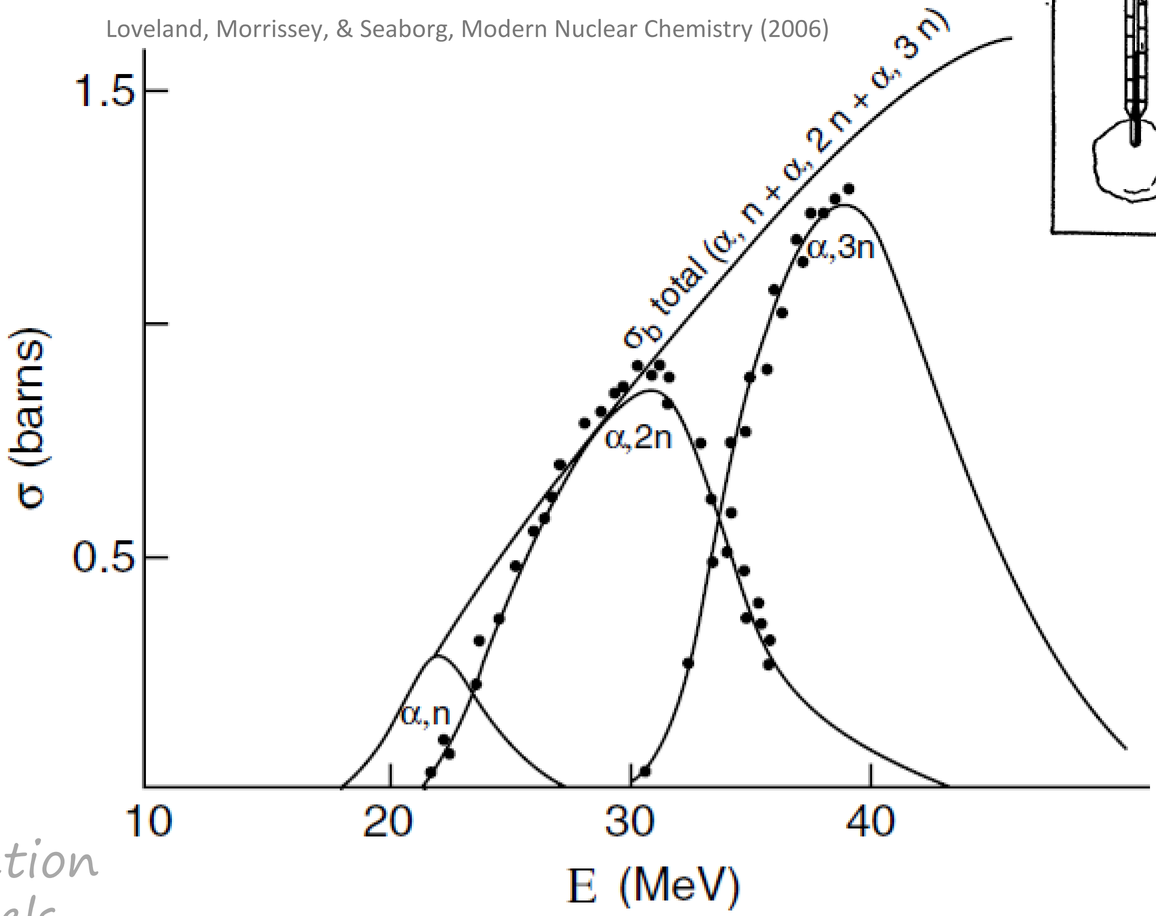


Inclusive cross section

- In the evaporation picture, wouldn't it be more effective to "cool off" by "boiling" off more than one nucleon?
- Indeed! That's exactly what happens, once it's energetically favorable
- This sort of energy behavior is characteristic for cross sections from evaporation processes
- For the case on the right, the sum over all neutron-emitting channels is written as (α, xn) and is called the "inclusive cross section"



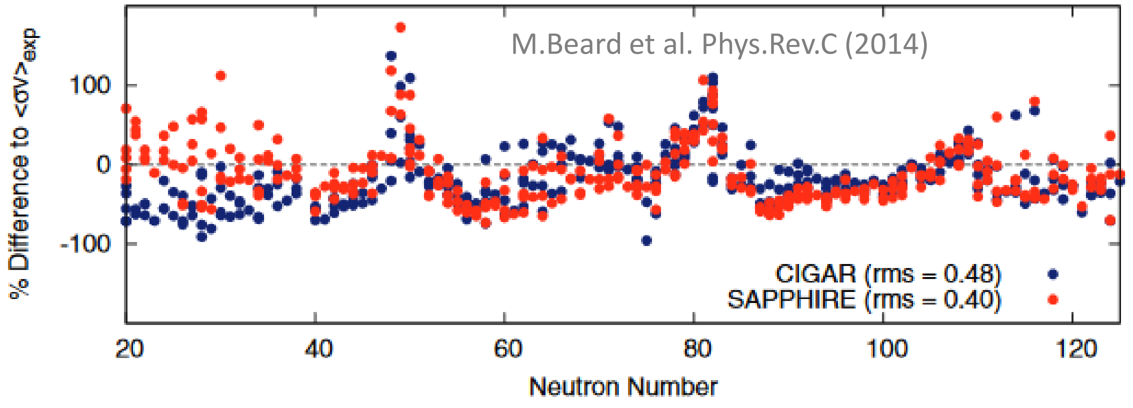
Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

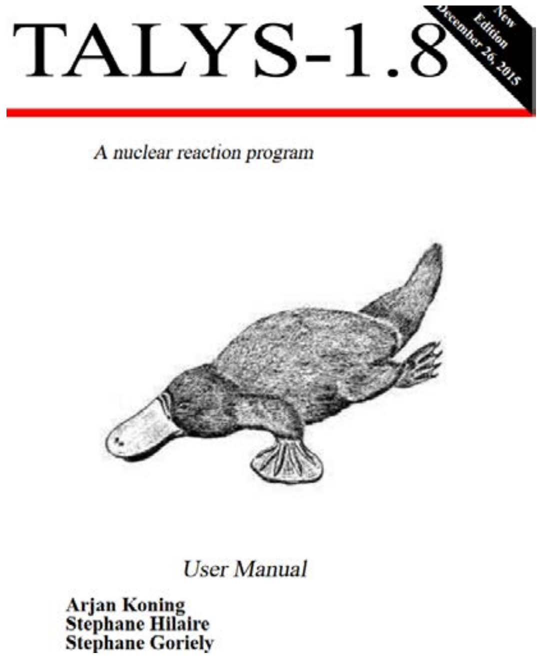


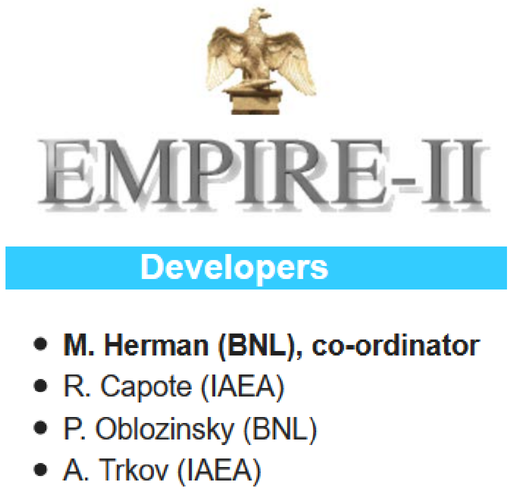
Fun fact to know & tell:
In the α-process of core-collapse supernovae, only the (α,xn) cross section is relevant, not the individual channels.
Which is to say that sometimes the inclusive cross section is the only thing that matters.

This HF business sounds like a lot of busy work, how should I actually do these calculations?

- Generally speaking, your best bet is to use open-source tools
- Many options are available, each with their own strengths, weaknesses, and assumptions.
- Unfortunately many under-the-hood assumptions lead to disagreements up to a factor of a few for what looks like the same inputs chosen by the user.
- At present, the most popular, best documented, easiest to use, and likely most tested HF code on the market is Talys, though another front-runner is EMPIRE
- If you use these, remember GI-GO (Garbage In, Garbage Out)

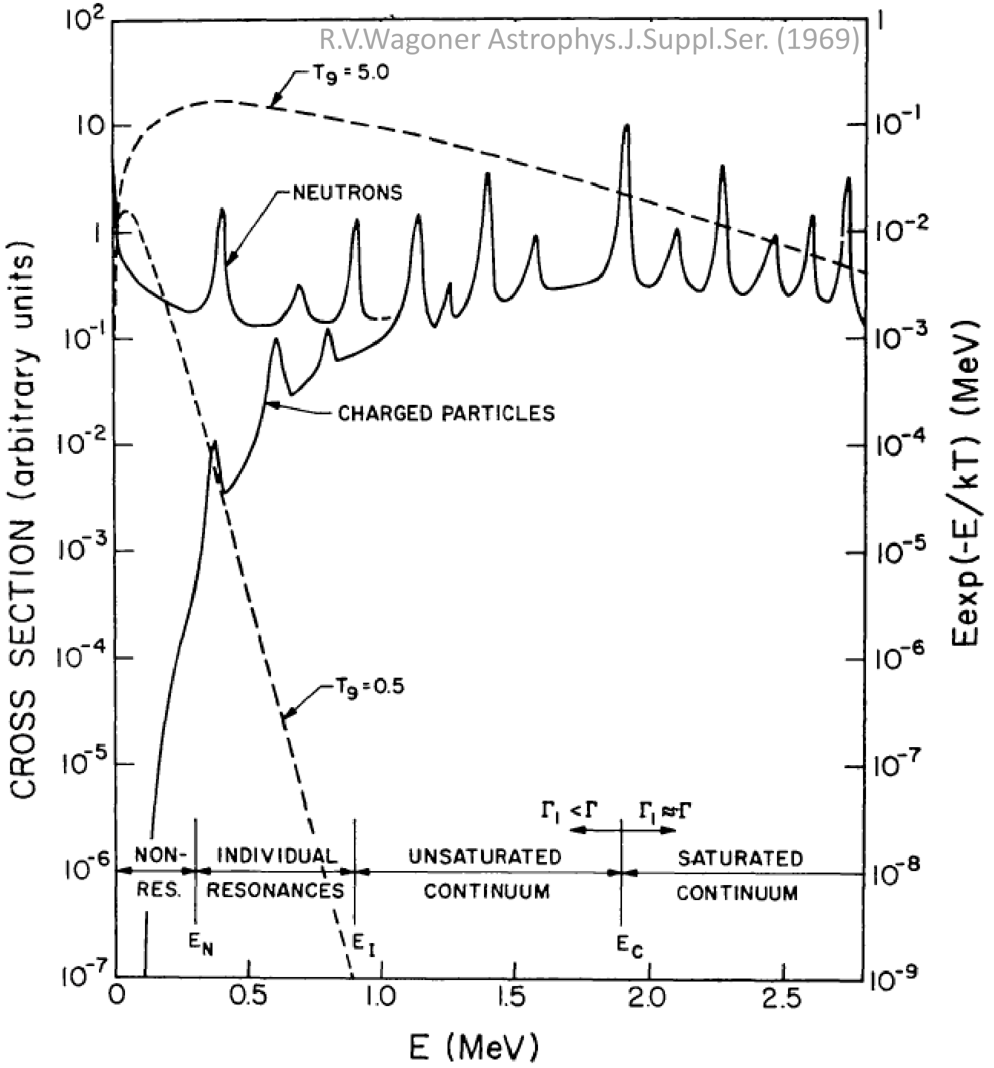


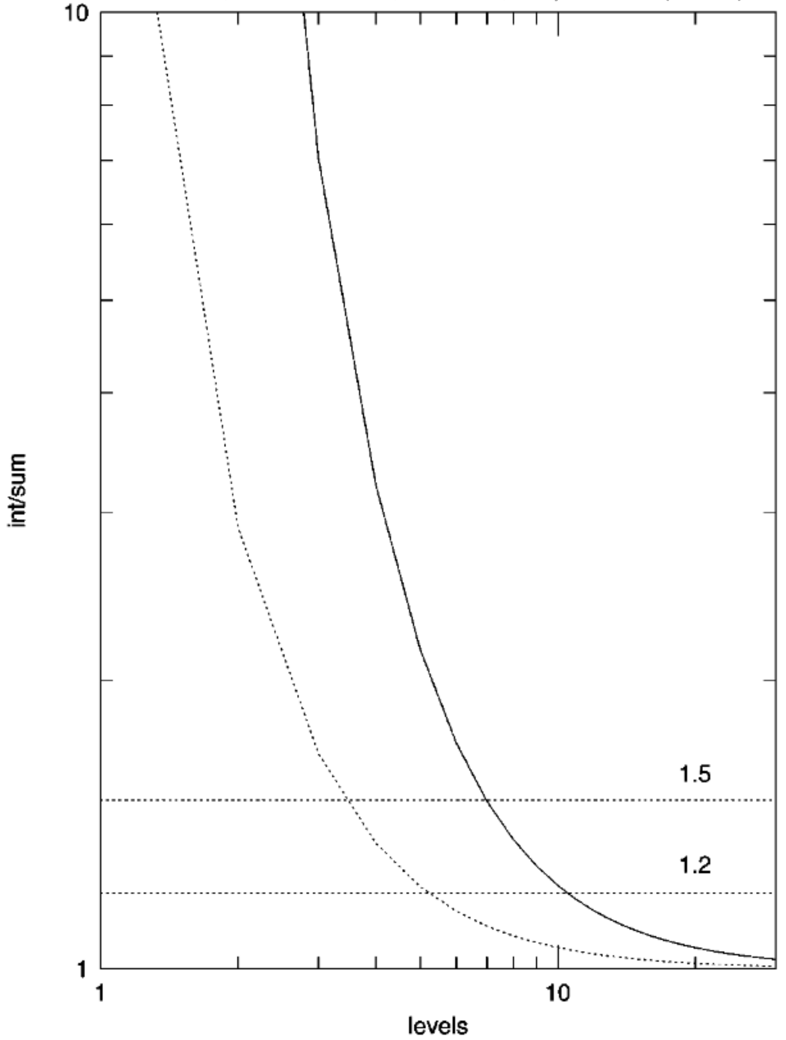




Applicability of the statistical approach

- You may be wondering exactly when we can use the Hauser-Feshbach approximation, as opposed to the sum over Briet-Wigner resonances
- Luckily, our forerunners have beat us to it and determined $\gtrsim$10 levels/MeV is sufficient

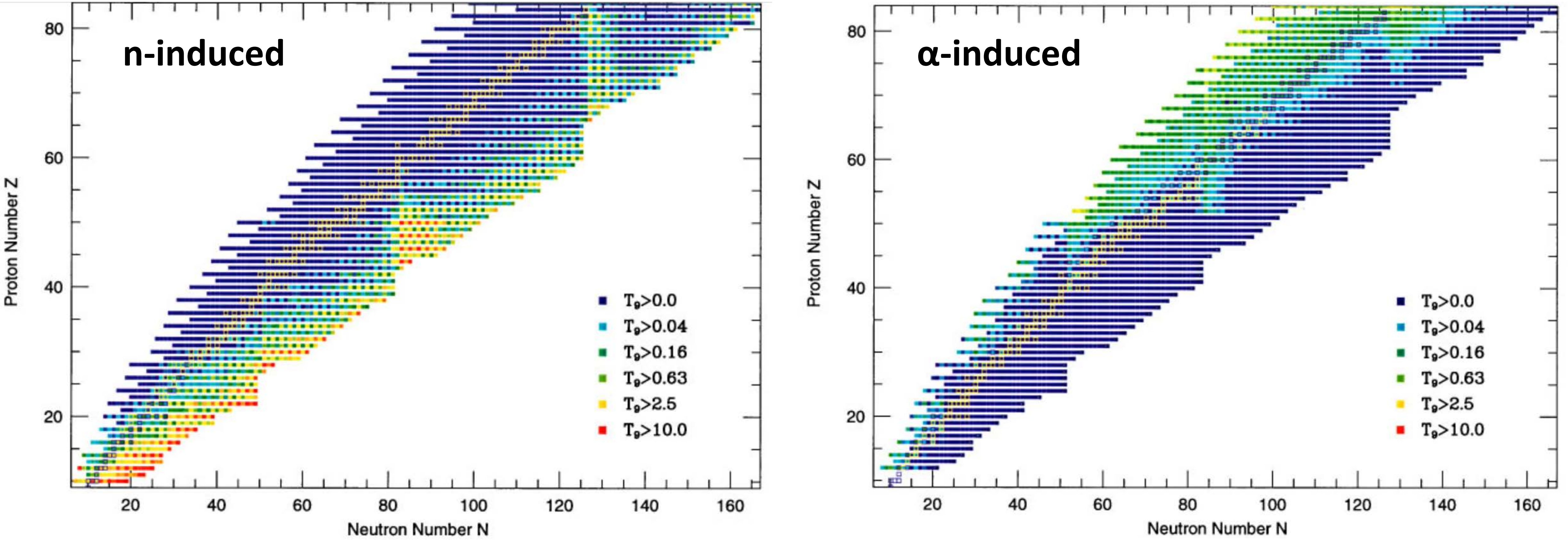




Applicability of the statistical approach

- For a given range of reaction energies, e.g. established by a high-temperature explosive environment, and level-density model, one can map-out when HF-derived reaction rates are applicable

Rauscher, Thielemann, & Kratz, Phys.Rev.C (1997)



Further Reading

- N. Bohr, “Transumations of Atomic Nuclei”, Science (1937)
- Chapter 10: Modern Nuclear Chemistry (Loveland, Morrissey, Seaborg)
- Chapter 11: Introductory Nuclear Physics (K.S. Krane)
- IAEA RIPL2 Handbook
- Chapters 3 &4: Talys User Manual
- LLNL Report UCRL-TR-201718 (F. Dietrich)
- Chapter 17: Introduction to Special Relativity, Quantum Mechanics, and Nuclear Physics for Nuclear Engineers (A. Bielajew)