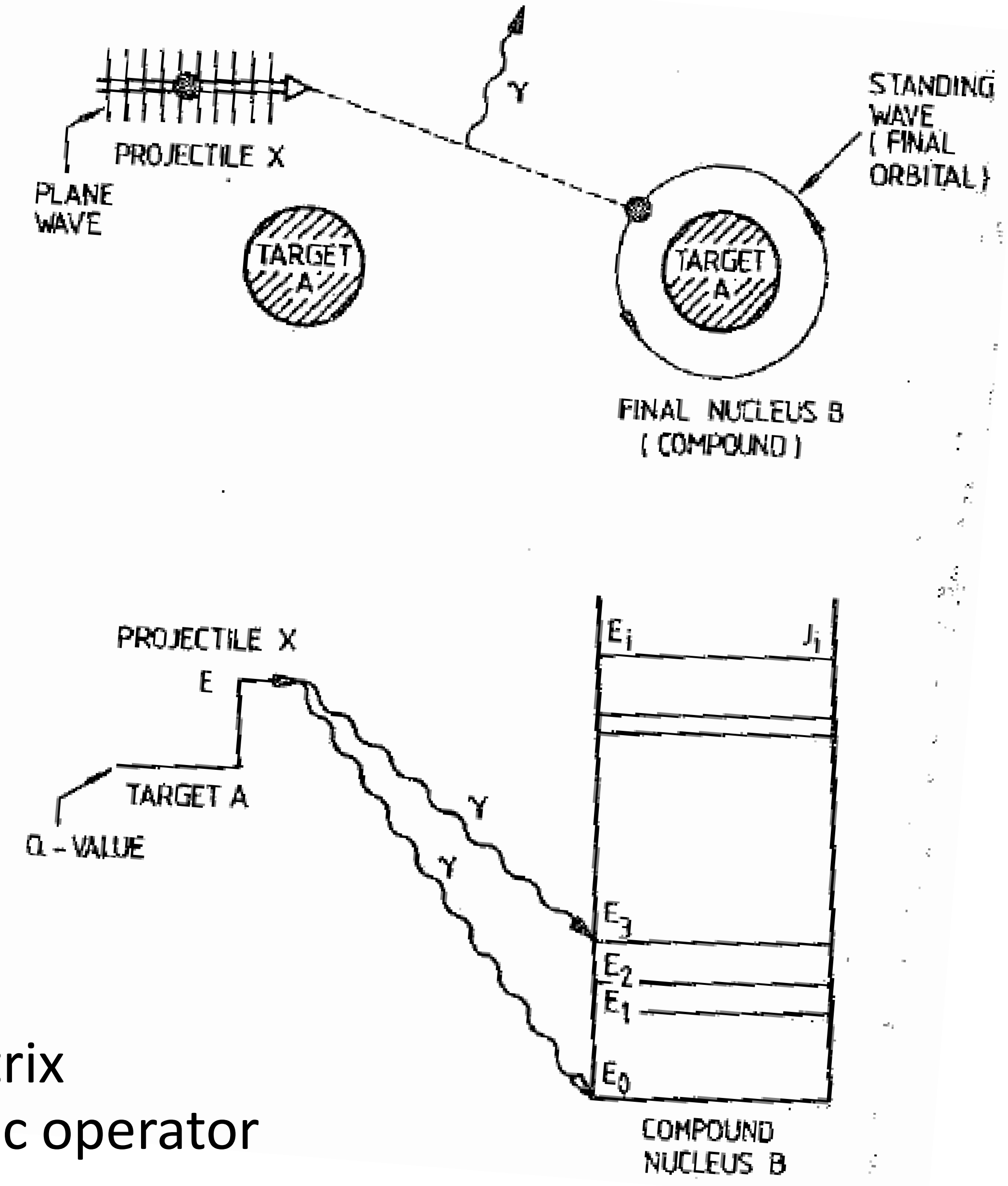
Lecture 16: Resonant Reactions

- Non-resonant reaction
- Resonant reaction picture
- Breit-Wigner formula

Non-resonant reactions

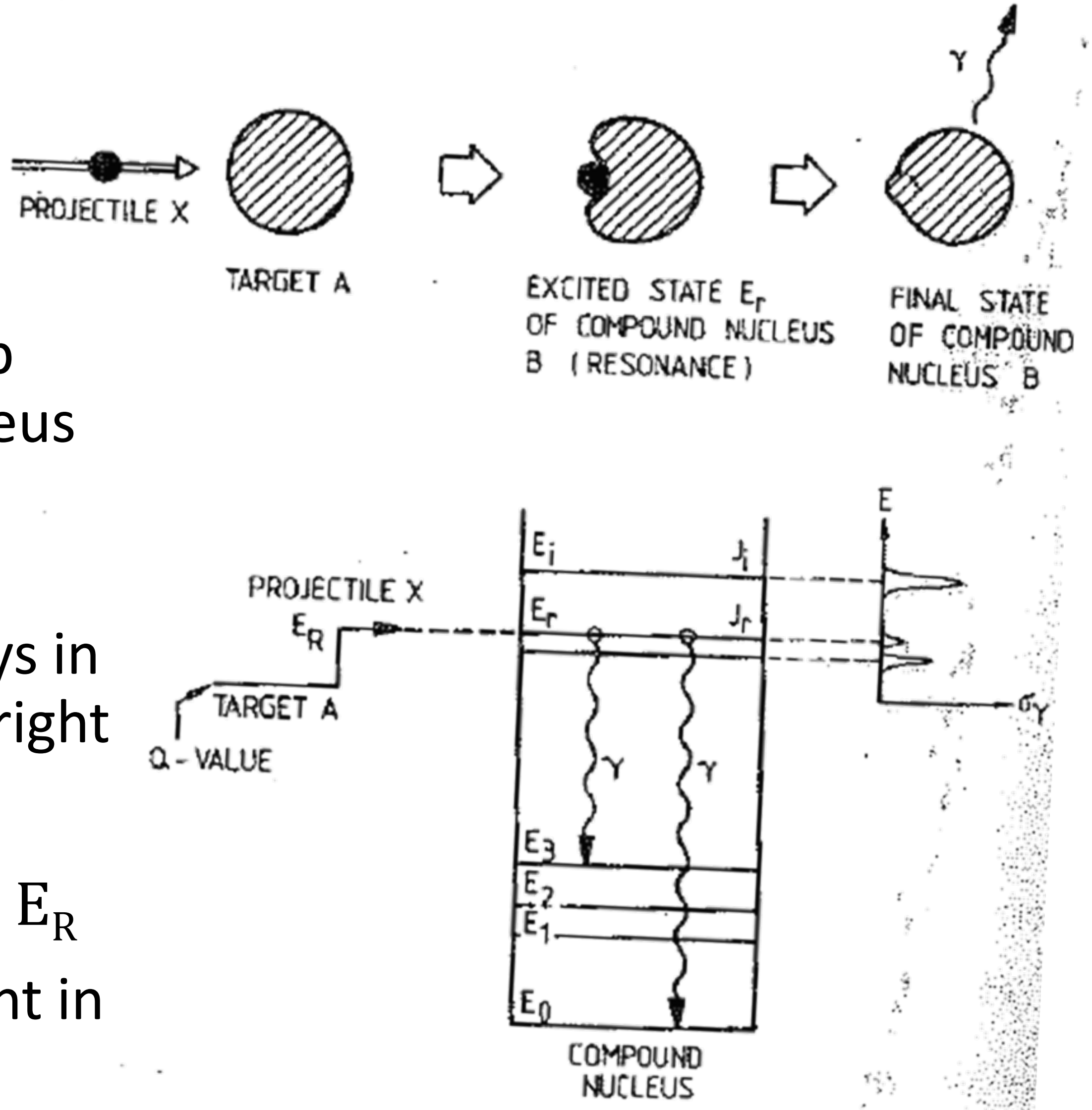
- Before we discuss resonant reactions, let's first consider a non-resonant reaction
- The non-resonant reaction is the process we've discussed so far, e.g. when we considered low-energy collisions with and without Coulomb effects included
- An example is the direct capture reaction shown in the figure on the right
  - The interaction of the plane-wave of the projectile with the potential of the target results in a standing-wave in the compound nucleus that is characterized by angular momentum l
  - The transition between the initial and final states is accomplished directly via photon emission, so the matrix element connecting these states is the electromagnetic operator $\sigma_{A(a,\gamma)B} \propto \left|< B\middle|H_\gamma\middle|A + x >\right|^2$, where semi-classical estimates can be obtained as we did previously



Rolfs & Rodney, Cauldrons in the Cosmos (1988)

Resonant reaction

- Now, like the good capitalists we are, we're going to add a middle-man
- If it so happens that the sum of the mass excesses of our reactants and their center-of-mass energy lines-up with an excited state in the reactant's compound nucleus $\{(^{A_{compound}}Z_{compound} = {}^{A_{target}+A_{projectile}}(Z_{product}+Z_{projectile})\}$, then capture can proceed into that state
- The excited state of the compound nucleus then decays in a second step, e.g. via γ-emission in the figure on the right
- This process requires $E_{cm} = E_B^* - Q_{A+x \rightarrow B+y}$
  - This energy is referred to as the resonance energy, $\mathrm{E_R}$
- This process, as we'll see, causes a strong enhancement in the cross section near the center-of-mass energy that fulfills the condition above



Rolfs & Rodney, Cauldrons in the Cosmos (1988)

Why a resonance?

- For a resonant reaction,
  the basic phenomenon is that the incoming plane wave is scattering on a potential well
- You could solve for the wavefunction consistently inside and outside the well,
  applying appropriate matching conditions for the function and its derivative,
  and you would find that there are characteristic wavenumbers for which an integer number of wavelengths occur for the part of the wavefunction inside the well
- For those cases you would see the ratio of the amplitude of the wavefunction outside of the well to inside of the well is maximized
- You could see this, but you'll have to go to Iliadis's book, Nuclear Physics of Stars, to see that
- Instead, recall our result from the Scattering lecture for the scattering wavefunction:

$$\sigma_{total} = \sigma_{elastic} + \sigma_{reaction} = \sum_{l=0}^{\infty} 2\pi \left(\frac{\lambda}{2\pi}\right)^2 (2l+1)(1 - \mathrm{Real}(\eta_l(E)))$$

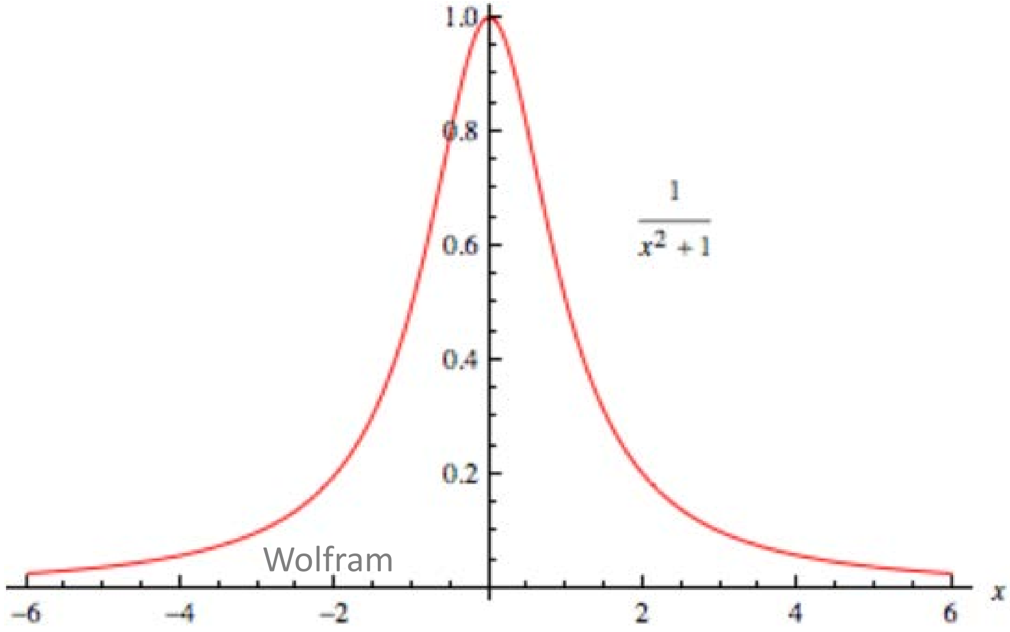
- Since $|\eta_l| < 1$, we can see the total cross section will have a maximum when $\mathrm{Real}(\eta_l) = -1$
- This corresponds to the scenario where $\delta_l(E) = \frac{\pi}{2}$, where $\eta_l = e^{2i\delta_l(E)}$

Why a resonance?

- To get the energy dependence, we need to expand $\delta_l(E = E_R) = \frac{\pi}{2}$ in terms of energy
- Ultimately we're concerned with the scattering cross section, which you'll recall goes as $\sigma_{sc} = \sum_{l=0}^{\infty} 4\pi \left(\frac{\lambda}{2\pi}\right)^2 (2l+1)\sin^2(\delta_l(E))$
- Someone far more clever than me *(Kenneth Krane, or whoever he copied for his book)*, realized that the best approach is to expand $\cot(\delta_l(E))$ about E_R so that you get something that converges
- Scientists tell us that this expansion looks like $\cot\big(\delta_l(E)\big) = \cot\big(\delta_l(E_R)\big) + (E - E_R)\left(\frac{\partial \cot(\delta_l(E))}{\partial E}\right)|_{E=E_R} + \frac{1}{2}(E - E_R)^2\left(\frac{\partial^2 \cot(\delta_l(E))}{\partial^2 E}\right)|_{E=E_R} + \cdots$
- If you listen to Krane, or spend too much time on Wolfram Alpha verifying his claims, you will see that $\frac{\partial \cot(\delta_l(E))}{\partial E}|_{E_R} = \left(\frac{\partial \delta_l(E)}{\partial E}\right)|_{E_R}\left(\frac{-1}{\sin^2(\delta_l(E_R))}\right)$
- Since $\delta_l(E_R) = \frac{\pi}{2}$, $\sin^2(\delta_l(E_R)) = 1$
- As such, $\frac{\partial \cot(\delta_l(E))}{\partial E} \approx -\left(\frac{\partial \delta_l(E)}{\partial E}\right)$
- It turns out, in taking the derivative of $\left(\frac{-1}{\sin^2(\delta_l(E))}\right)$ and evaluating the result at E_R, the second-order term will go to zero

Why a resonance?

- But wait, there's more! Noting that $\cot\big(\delta_l(E_R)\big) = 0$, our hard-won effort for the expansion is $\cot\big(\delta_l(E)\big) = -(E - E_R)\left(\frac{\partial \delta_l(E)}{\partial E}\right)|_{E=E_R}$
- For reasons that will become clear in a minute, we define the width $\Gamma \equiv 2\left(\frac{\partial \delta_l(E)}{\partial E}\right)^{-1}|_{E=E_R}$ which is just some number that scales how rapidly the cross section falls-off near the resonance
- So, $\cot\big(\delta_l(E)\big) = -\frac{(E-E_R)}{\Gamma/2}$
- The trigonometry wizards among us can show that, therefore, $\sin\big(\delta_l(E)\big) = \frac{\Gamma/2}{\sqrt{(E-E_R)^2+\Gamma^2/4}}$
- At long last, we finally have something we can use! Returning to the scattering cross section, and considering a single angular momentum transfer $\sigma_{sc,l}(E) = \pi\left(\frac{\lambda}{2\pi}\right)^2(2l+1)\frac{\Gamma^2}{(E-E_R)^2+\Gamma^2/4}$



- This is the Lorentzian distribution, which you've probably seen for resonances, e.g., in the damped harmonic oscillator

Breit-Wigner formula

- When we found the cross section for scattering a plane wave of a given λ transferring some l, $\sigma_{sc,l}(E) = \pi \left(\frac{\lambda}{2\pi}\right)^2 (2l+1) \frac{\Gamma^2}{(E-E_R)^2+\Gamma^2/4}$, we were a bit hasty and left out necessary details for reactions that aren't elastic scattering
- Namely, we ignored the fact that the projectile entering the reaction is not necessarily the same particle as the ejectile leaving the reaction
- Since the width for forming the compound nucleus isn't the same as decaying from it, Γ^2 in the numerator goes instead, for *X(a,b)Y*, to $\Gamma_{aX}\Gamma_{bY}$
    - Γ_{aX} is the rate (recall $\Gamma = h/\tau$) at which *a+X* forms the compound nucleus, …which is the same as the rate at which the compound nucleus decays via channel *a+X*
    - Γ_{bY} is the rate at which the compound nucleus decays via channel *b+Y*
- However, Γ^2 in the denominator is a sort of weighting factor that corresponds to the total decay rate of the compound nucleus, and therefore $\Gamma = \sum \Gamma_i$
- To make life more complicated, each of these widths are energy dependent, $\Gamma(E)$
- Finally, $2l+1$ degeneracy was valid for a spinless particle. For an excited state with spin J being populated by a projectile with spin J_a impinging on a nucleus with ground-state spin J_X, $2l+1$ becomes the factor $g = \frac{2J+1}{(2J_a+1)(2J_X+1)}$

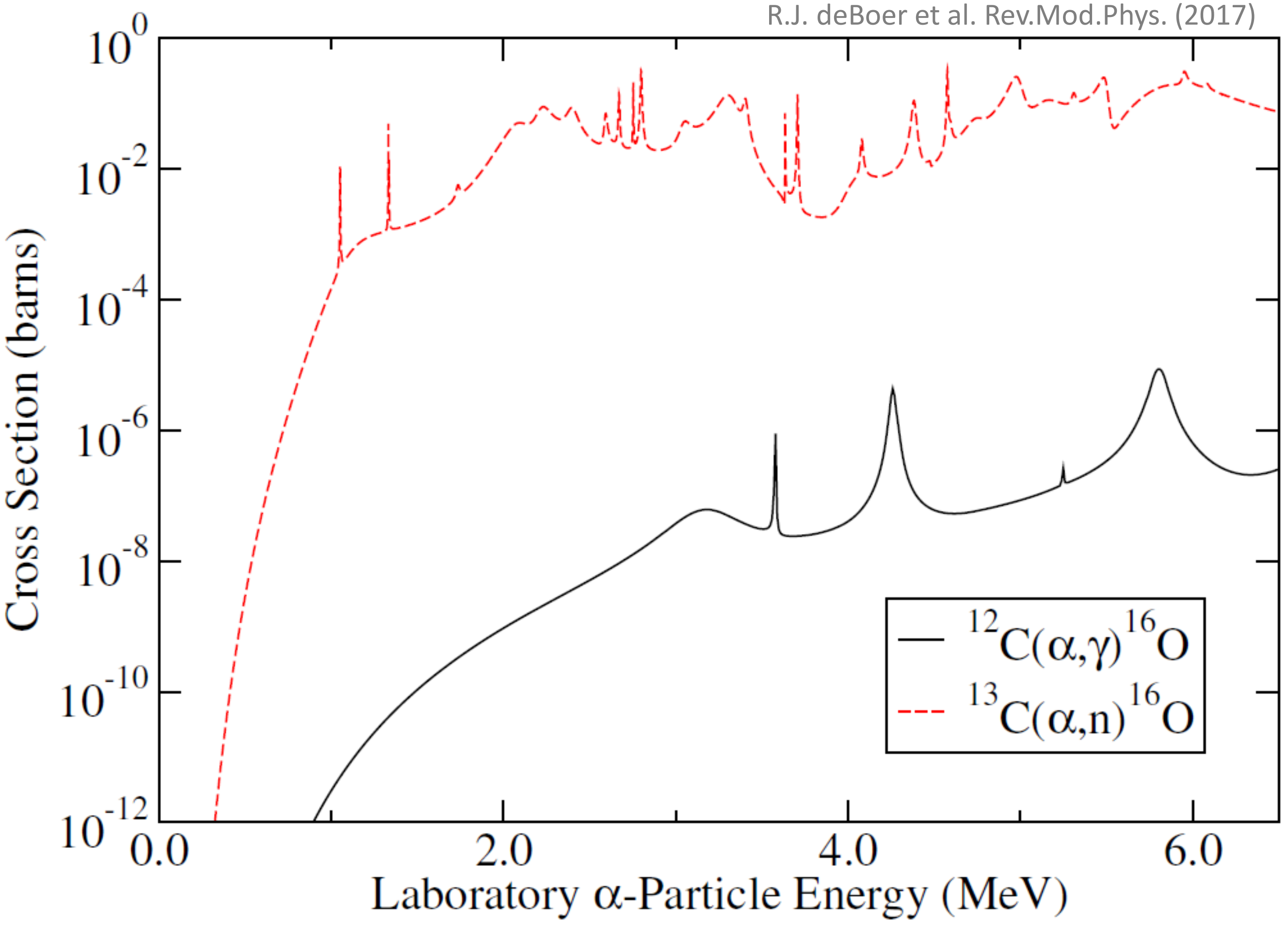
Breit-Wigner formula

- Our ho-hum Lorentzian, now becomes the bright and shiny Breit-Wigner formula,

$$\sigma_{BW,X(a,b)Y}(E) = \pi\left(\frac{\lambda}{\pi}\right)^2 \frac{2J+1}{(2J_a+1)(2J_X+1)} \frac{\Gamma_{aX}(E)\Gamma_{bY}(E)}{(E-E_R)^2+(\Gamma(E))^2/4}$$

- Each resonance adds a sharp spike onto the non-resonant cross section
- This has some major implications:
  1. If we just want to make a reaction happen, it's best to pick an energy on a resonance
  2. If we don't want a reaction to happen (e.g. background), we had better avoid the resonance energy
  3. If we're considering an environment with an energy distribution (e.g. a star), the resonant rate is mostly what matters
  4. Since σ_{BW} has such a strong energy dependence, we can use it to measure energy-loss and therefore target thicknes

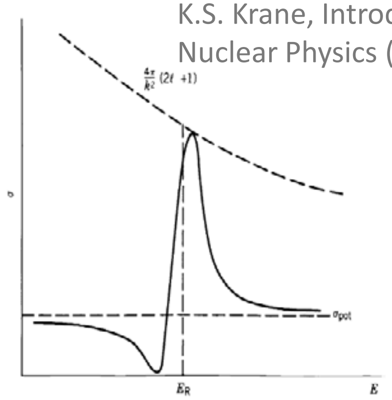
R.J. deBoer et al. Rev.Mod.Phys. (2017)

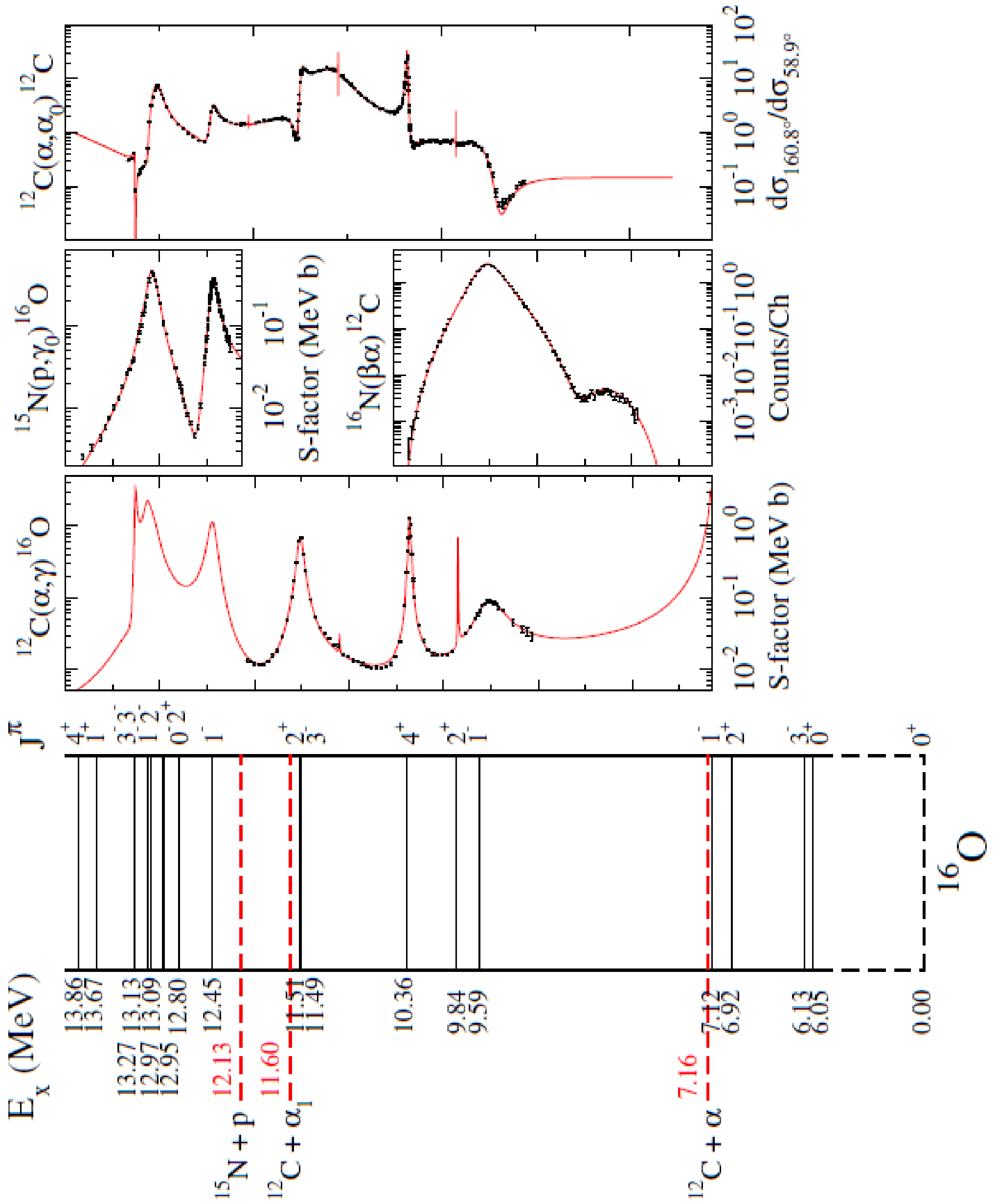


Resonance features

- Before we get too fat and sassy and start ignoring the direct reaction contribution, we need to remember that scattering directly off of the potential happens too
- This is the nuclear elastic scattering from the Scattering lecture, which fancy folks like to call potential scattering *or* shape elastic scattering
- The waves from shape elastic scattering can interfere with scattering off of the resonance, producing a neat shape
- When we remove the trivial component of the cross section (which we'll cover in a moment), we can see these interference effects in our data.
- Resonances can also interfere with other resonances, if one of them is broad enough

K.S. Krane, Introductory Nuclear Physics (1987)

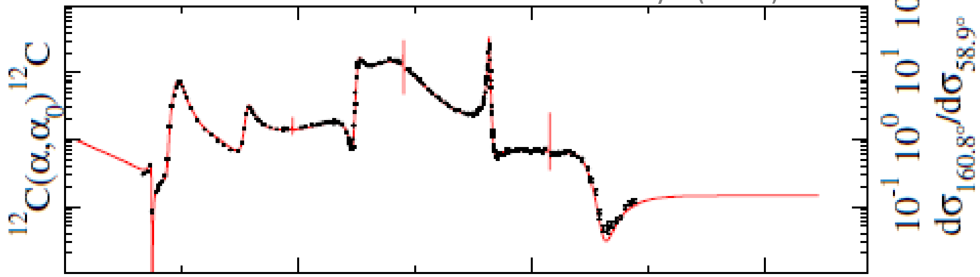




R.J. deBoer et al. Rev.Mod.Phys. (2017)

Resonance features

R.J. deBoer et al. Rev.Mod.Phys. (2017)



- How can we understand this feature near the resonance?
- Because wave mechanics is in play here,
  we can't just add the outgoing wavefunctions for the different types of scattering
- Instead, we need to combine them as $\eta_l(E) = e^{2i\left(\delta_{l,res}(E)+\delta_{l,shape}(E)\right)}$
- The scattering cross section becomes:
  $$\sigma_{scatt,X(a,b)Y}(E) = \pi\left(\frac{\lambda}{\pi}\right)^2 g\left|e^{-2i\delta_{l,shape}} - 1 + \frac{i\Gamma}{(E-E_R)+i\Gamma/2}\right|^2$$
- The last term inside || becomes negligible for $(E - E_R) \gg \Gamma/2$, and we just have shape elastic scattering
- The last term dominates for $E \approx E_R$ and we just have the resonance scattering
- We also see that we expect a dip just below $E \approx E_R$, because the last term is significant, but negative

J^{π} considerations

- Resonant reactions are due to the strong interaction, so spin is conserved
- Therefore, for a spin J_1 particle impinging on a spin J_2 target, bringing in an orbital angular momentum l, can only populate excited states for a limited range of spins J
  - For example, nucleon capture on an even-A nucleus can only populate states with $\left|l - \frac{1}{2}\right| \leq J \leq \left|l + \frac{1}{2}\right|$
- Similarly, the parity is constrained by $\pi(J) = \pi_1 \pi_2 (-1)^l$
  - If $\pi_1 = \pi_2 = +1$, then $\pi(J) = (-1)^l$,
    i.e. the parity of the resonance is determined by the orbital angular momentum of the reaction channel
  - Such a resonant state is said to have "natural parity".
  - If $\pi(J) \neq (-1)^l$, then that resonant state has "unnatural parity"
    - These can occur because nuclear levels are seldom pure single-particle states, but are rather a superposition of many single-particle states

Resonance widths

$$\sigma_{BW,X(a,b)Y}(E) \propto \frac{\Gamma_{aX}(E)\Gamma_{bY}(E)}{(E-E_R)^2+(\Gamma(E))^2/4}$$

- As we noted, the resonant cross section is mostly described by the probability of forming the compound nucleus, represented by $\Gamma_{aX}(E)$, multiplied by the probability of decaying from the compound nucleus by a particular channel, $\Gamma_{bY}(E)$, weighted by the total probability of decaying from the compound nucleus via any channel $\Gamma(E)$
- For charged particles, we can pretty easily imagine $\Gamma_{aX}(E)$ is related to the tunneling probability, which we made an analytic estimate of when we went over α decay
  - Recall, $P_{tunnel}(E) \approx e^{-2\pi\eta_{Sommerfeld}} = e^{-2\pi\left(Z_1Z_2e^2/\hbar v\right)} = e^{-2\pi(Z_1Z_2\alpha c/v)}$
- But, there's also a centrifugal barrier, so $P_l(E) \approx e^{-2l(l+1)\sqrt{\hbar^2/(2\mu Z_1Z_2e^2R)}}e^{-2\pi\eta}$

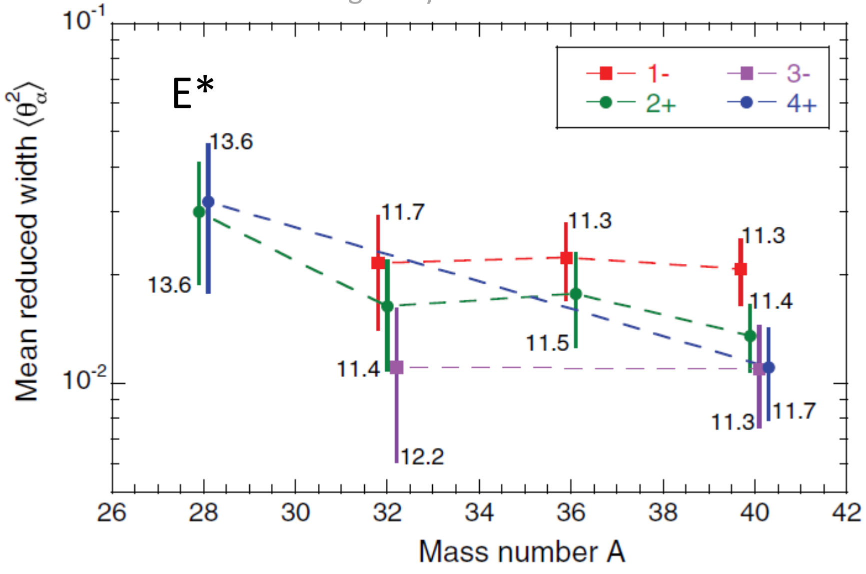
  should actually use the higher-order form presented in the α-decay lecture
- Since nuclear excited states don't necessarily correspond to a pure shell-model state that would be populated by the angular momentum transfer l, we need to take into account that probability as well
- Finally, $\Gamma_{aX}(E) = 2P_{l,aX}(E_R)\gamma_l^2$, where the last term describes the aforementioned overlap
  - If you have a shell-model result (or guess) for γ_l^2, then you can update $\Gamma_{aX}(E)$ by scaling it with $P_{l,aX}(E_R)$
- For photons, the decay probability scales as the γ-decay rate we have discussed in the past, $\Gamma_\gamma(E) = B_lE_\gamma^{2l+1}$, where B_l is the matrix element connecting the resonance and decay product

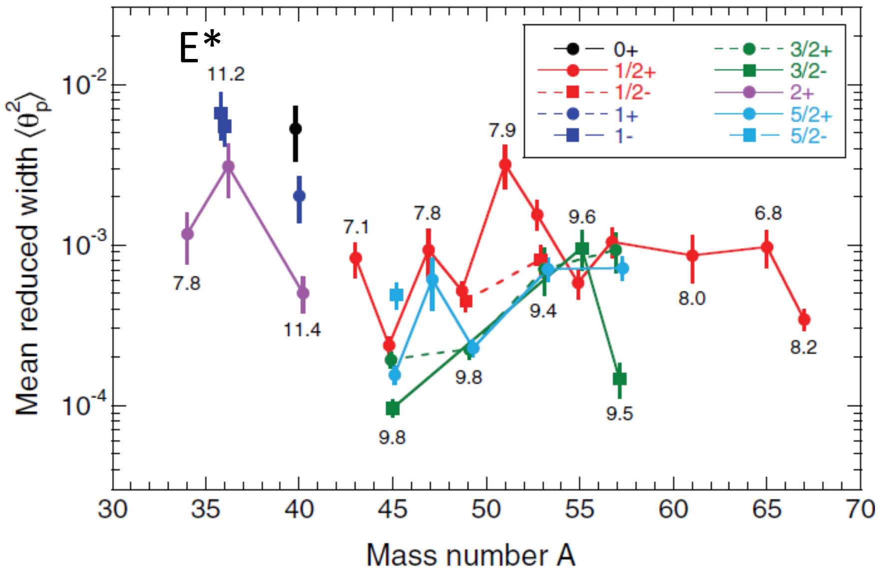
The Wigner Limit

- Particle widths have a theoretical upper-bound, known as the Wigner limit
- This is the case where we assume the single-particle limit, i.e. a single nucleon populating (or de-populating) a single-particle level from our shell-model picture
- Here, $\Gamma_{aX}(E) = 2\frac{3\hbar^2}{2\mu R^2}P_{l,aX}(E_R)\theta^2$ with $\theta^2 = 1$, μ is the reduced mass and R is the "channel radius", which is a \$10 word for the sum of the particle & nuclear radii.
- Actual widths are generally far below this
- A decent way to estimate the width of a state is to use systematics to estimate the average reduced width (or to use an optical potential, as we'll discuss in the Statistical Reactions lecture). Then, randomly select the width from a Porter-Thomas distribution defined by that average width, where $P(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-^{x^2}/_2\right)$ and $\Gamma = x^2\langle\Gamma_c\rangle$

(see e.g. DOI:10.1007/978-3-030-58082-7_5)

Pogrebnyak et al. PRC 2013





(more accurate) Resonance widths

- The widths on the previous slide are analytic, which is nice, but they are approximate
- To do things properly, one wants to solve the Schrodinger equation assuming some potential
- Ultimately (see Appendix A of Nuclear Physics of Stars) the end result is that the penetration factor is $P_l = R\left(\frac{k}{F_l^2+G_l^2}\right)_{r=R}$
  where F_l and G_l are the spherical Bessel and Neumann functions for neutrons and the regular and irregular Coulomb functions for charged-particles. R is the channel radius, and k is the wave number
- This needs to be calculated with a code. You could use Alexander Volya's nutcracker web-app.
- Interestingly, you'll find that there's enough uncertainty in the potentials and in the channel radius, that the analytic estimate is often accurate enough
- For example, compare results of $\omega\gamma$ at the Wigner limit, for ^{24}Mg(α,γ) from NuCracker, Adsley+ *PRC* 2020, & the analytic form of Merz & Meisel *MNRAS* 2018
  ($P_l(E) \approx e^{-2l(l+1)\sqrt{\hbar^2/(2\mu Z_1 Z_2 e^2 R)}}\exp\left(2\pi\frac{e^2}{\hbar c}Z_\alpha Z_{daughter}\sqrt{\frac{\mu c^2}{2Q_\alpha}}\left(1-\frac{4}{\pi}\sqrt{\frac{r_0\left(A_\alpha^{1/3}+A_{daughter}^{1/3}\right)}{\frac{Z_\alpha Z_{daughter}e^2}{Q_\alpha}}}\right)\right)$).
  Or ^{26}Si(α,p) between the Analytic estimate, NuCracker, & Almeraz-Calderon+ PRC 2013

Eres[MeV]	Analytic [MeV]	Adsley [MeV]	nucracker [MeV]
0.198	2.2E-34	3.8E-34	2.1E-34
0.327	6.2E-26	1.4E-25	6.5E-26
0.684	8.2E-14	9.7E-14	6.8E-14
1.117	1.1E-13	1.7E-12	6.5E-13

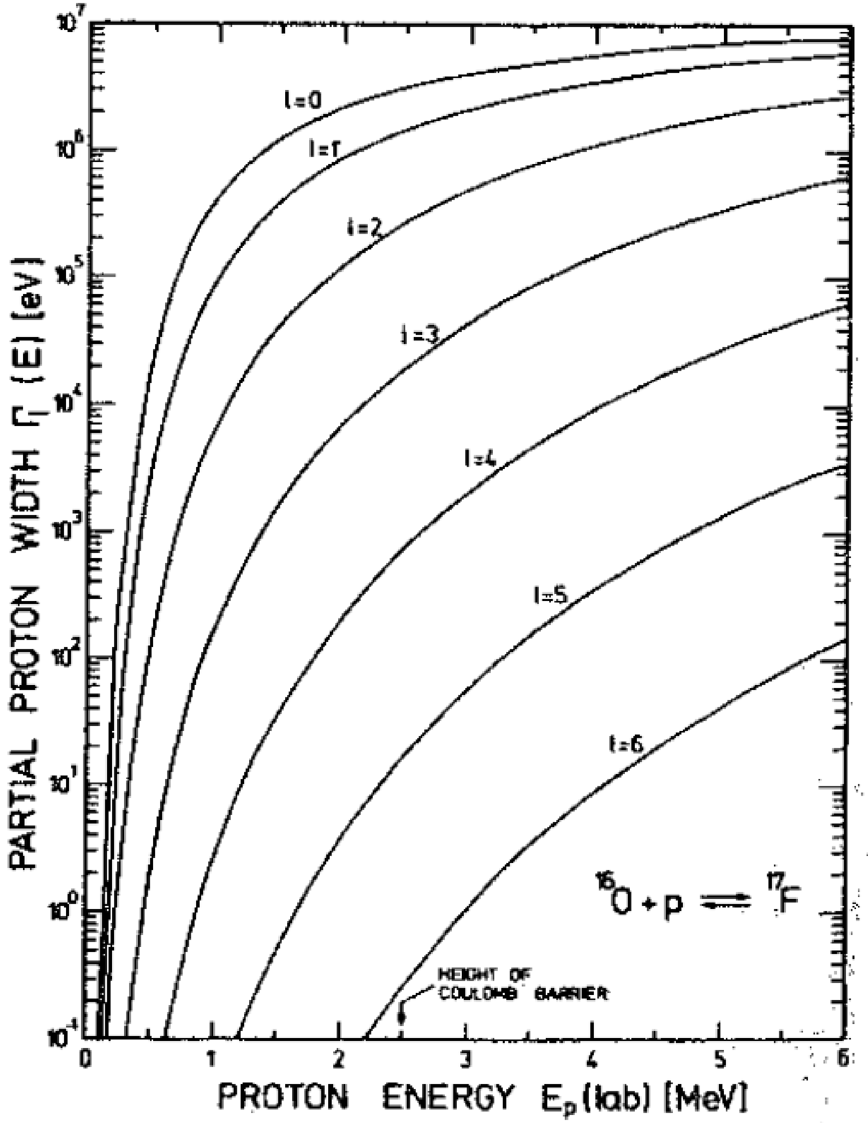
Eres[MeV]	Analytic [MeV]	Alm.-Cal. [MeV]	nucracker [MeV]
0.442	2.9E-23	1.4E-23	1.2E-22
0.932	6.7E-13	4.6E-14	5.2E-13
2.696	3.2E-03	4.8E-04	2.2E-03
2.696	5.2E-03	4.8E-04	1.4E-03

- If you have proper QM results for several energies and angular momenta, you can achieve a good extrapolation using an analytic fit: $P(E) = \exp\left({}^{-a}/_{\sqrt{E}} - bl(l+1)\right)$, fitting a and b. I've found this especially helps for very different l

Resonance widths

$$\sigma_{BW,X(a,b)Y}(E) \propto \frac{\Gamma_{aX}(E)\Gamma_{bY}(E)}{(E-E_R)^2+(\Gamma(E))^2/4}$$

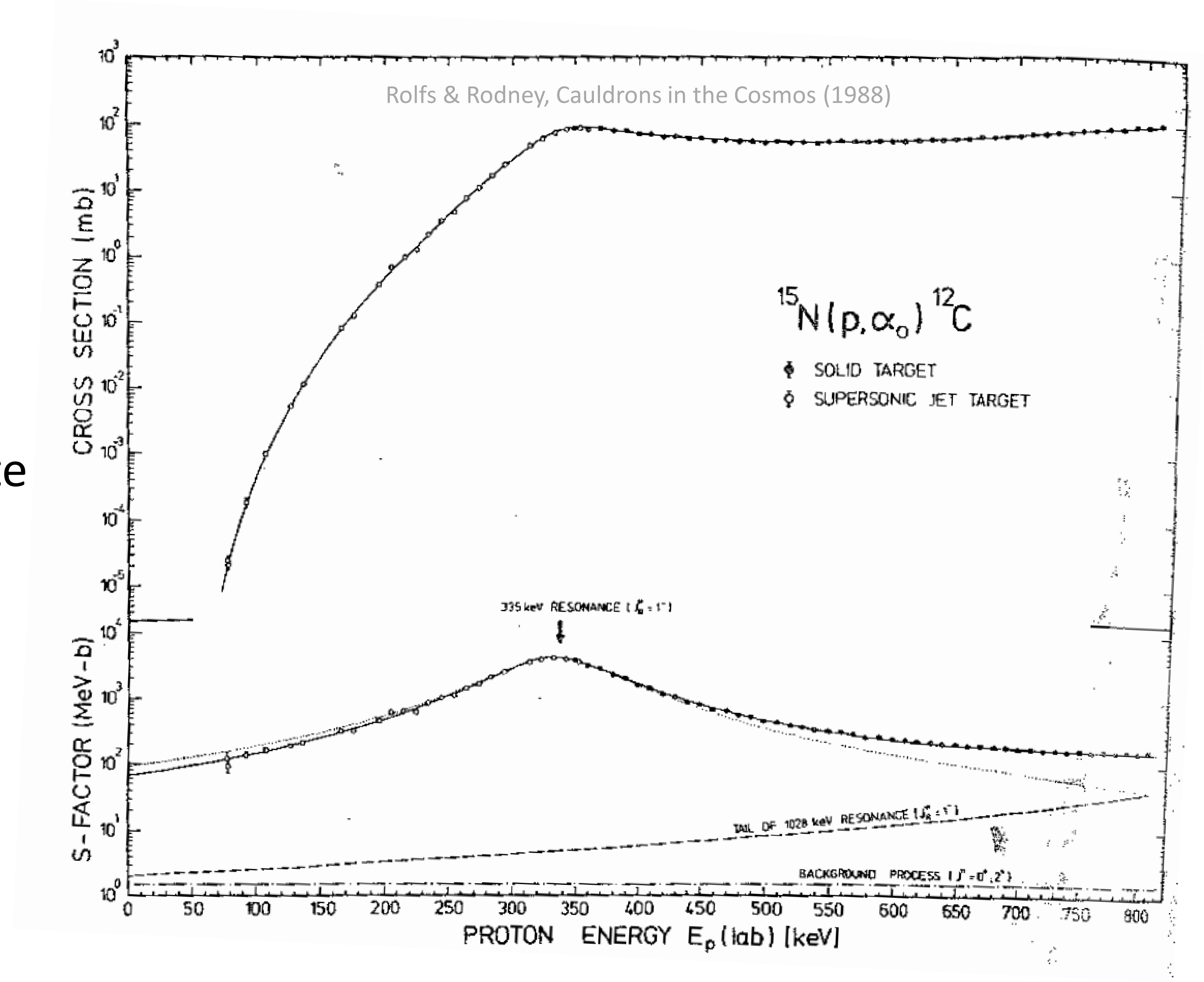
- Widths can have quite steep energy dependencies
- However, we have to pay attention to what E we're talking about in $\Gamma(E)$
- For a radiative capture, $E_{aX} = E_R$, $E_\gamma = Q + E_R$
- Otherwise, $E_{aX} = E_R$, $E_{bY} = S_b + E_R$ (where S_b is the b separation energy)
- Since $E_R \ll Q$ and $E_R \ll S_b$, it's clear that the incoming particle width has a strong dependence on E_R, but the outgoing width is relatively independent of E_R
- Note that there's still the energy dependence from λ^2



Rolfs & Rodney, Cauldrons in the Cosmos (1988)

Broad resonances

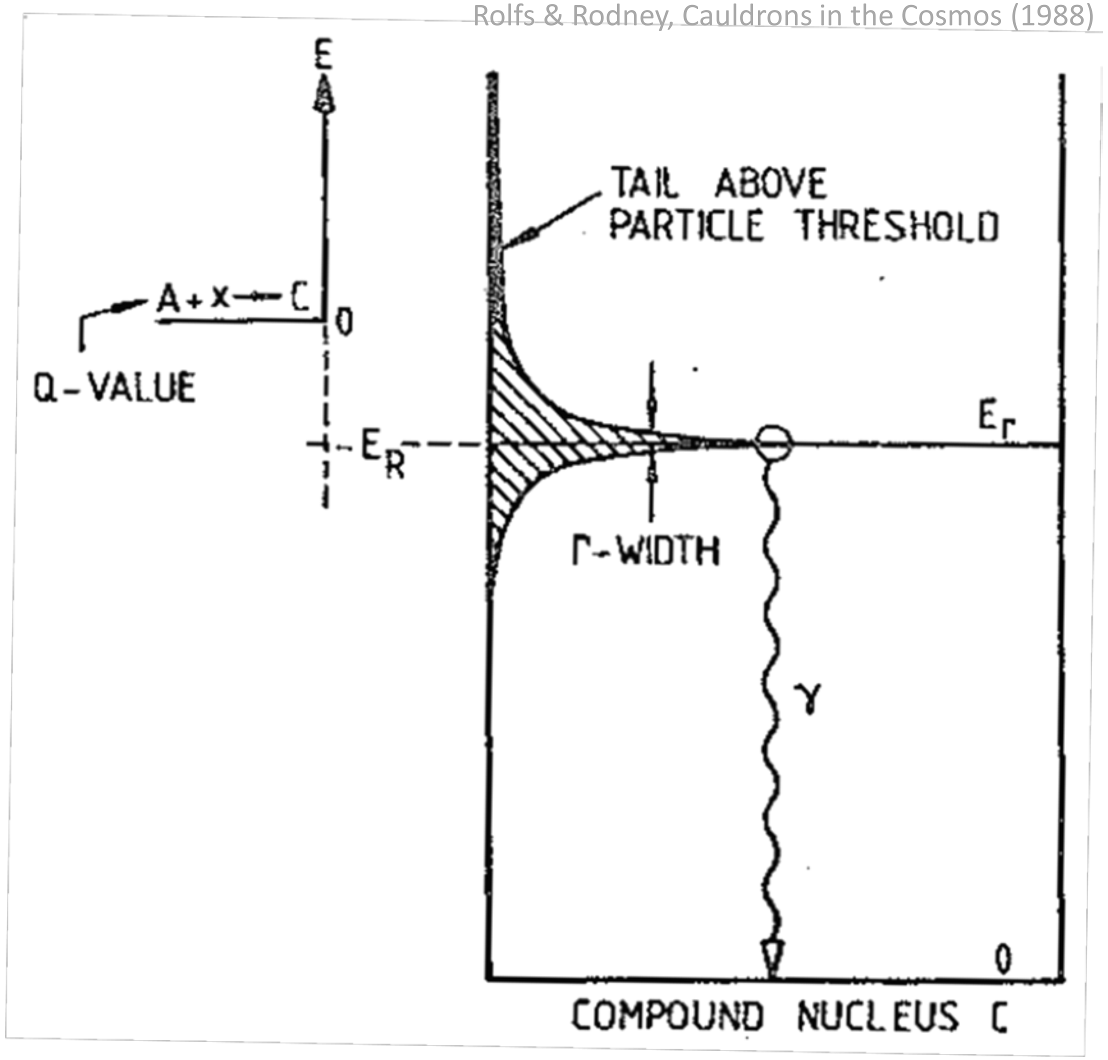
- When $\Gamma/E_R > 10\%$, this resonance is "broad"
- We'll need to take these into account in a different way when we calculate the astrophysical reaction rate in a future lecture



Subthreshold resonances

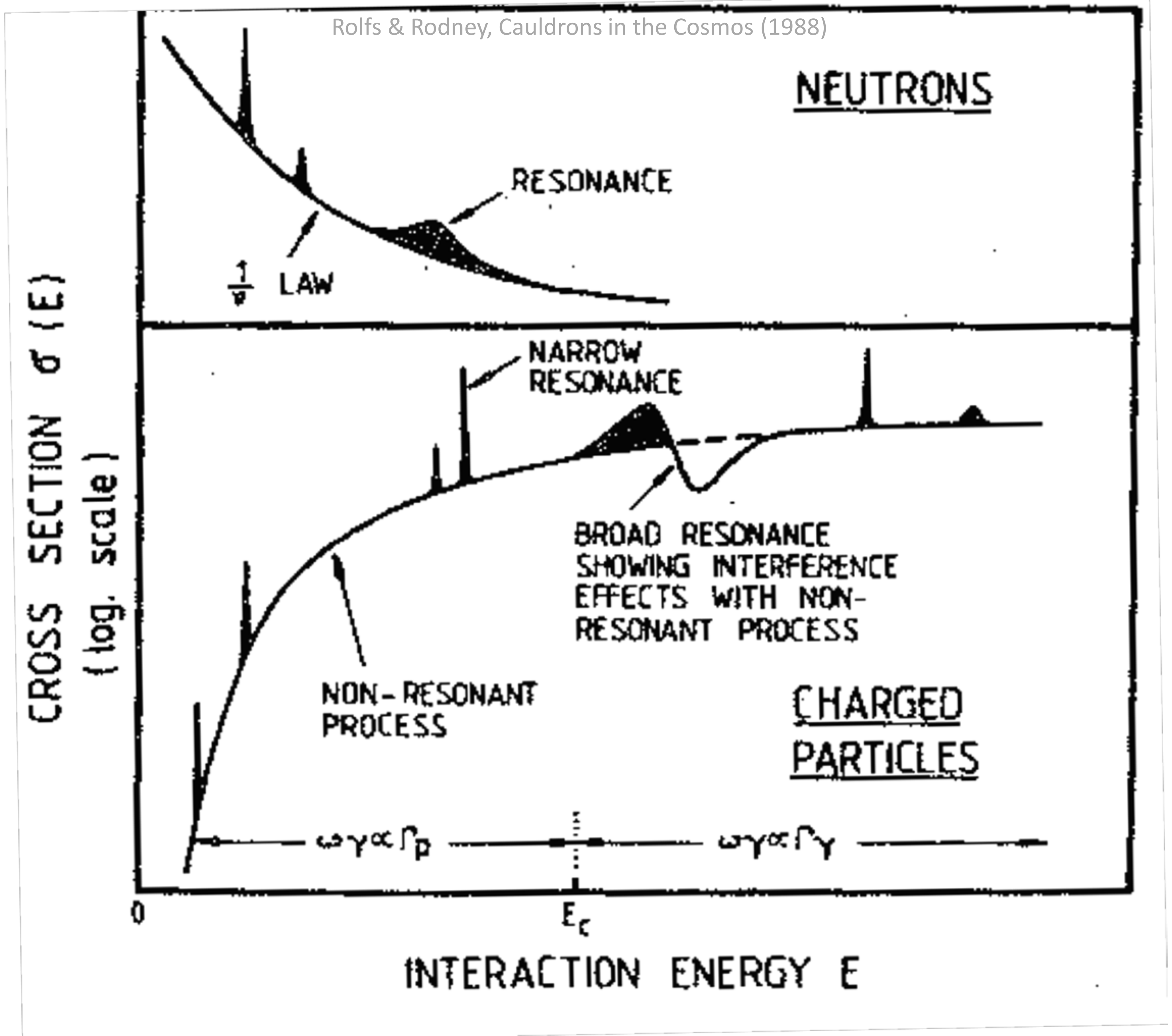
- Just because an excited state is below a reaction threshold doesn't mean it can't contribute as a resonance to the total reaction rate
- As long as the low-energy tail overlaps the threshold, the resonance can occur
- This is called a "sub-threshold" resonance
  (because another name would be pretty bizarre)

Rolfs & Rodney, Cauldrons in the Cosmos (1988)



Cross section features at low-ish energy ($E \sim V_{Coul}$)

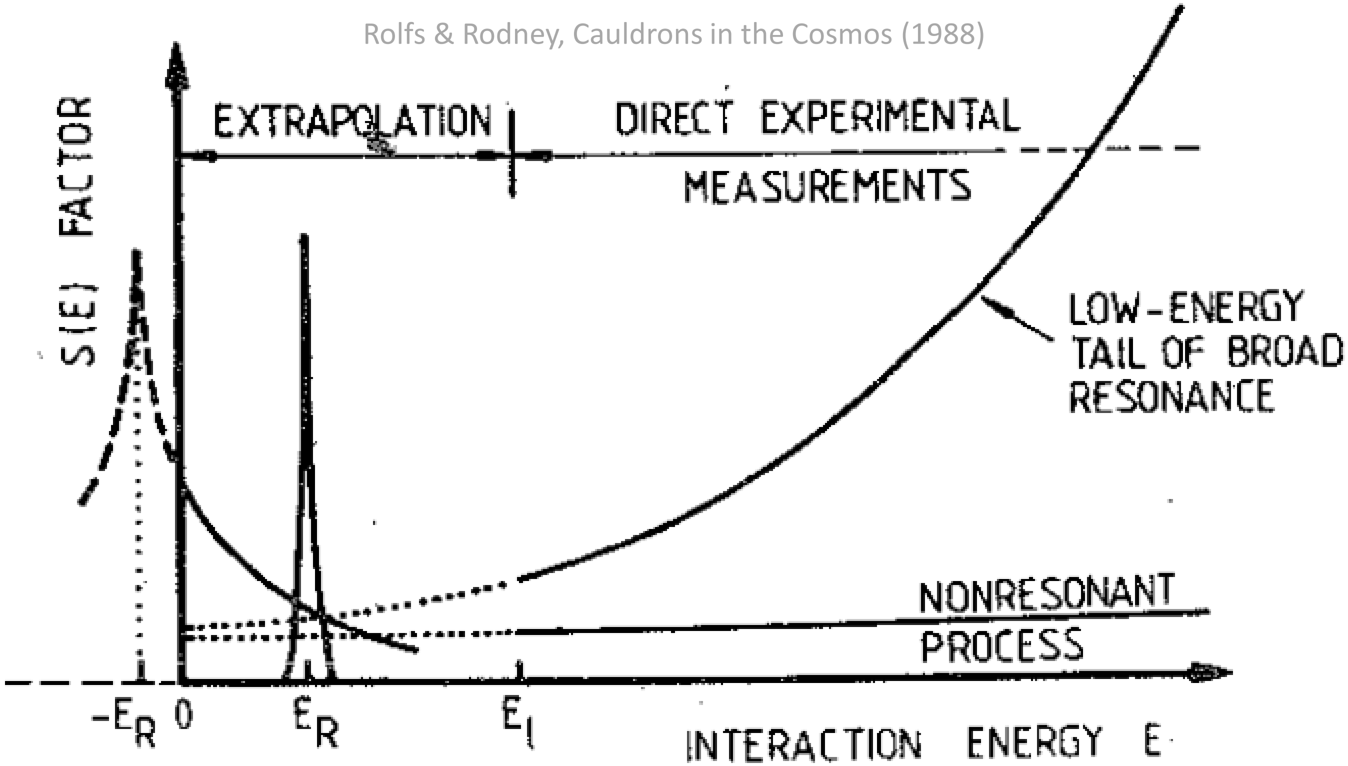
Rolfs & Rodney, Cauldrons in the Cosmos (1988)



The S-factor

- Often it's useful to remove the trivial energy dependence from the cross section, in particular for charged-particle reaction rates
- We'll discuss this more when we cover nuclear astrophysics, but suffice it to say for now that for a charged particle reaction rate, $\sigma(E) = \frac{1}{E} e^{-2\pi\eta} S(E)$, where $S(E)$ is the S-factor that contains all of the interesting physics

Rolfs & Rodney, Cauldrons in the Cosmos (1988)



Further Reading

- Chapter 11: Introductory Nuclear Physics (K.S. Krane)
- Chapter 2: Nuclear Physics of Stars (C. Iliadis)
- Chapter 4: Cauldrons in the Cosmos (C. Rolfs & W. Rodney)
- Chapter 17: Introduction to Special Relativity, Quantum Mechanics, and Nuclear Physics for Nuclear Engineers (A. Bielajew)