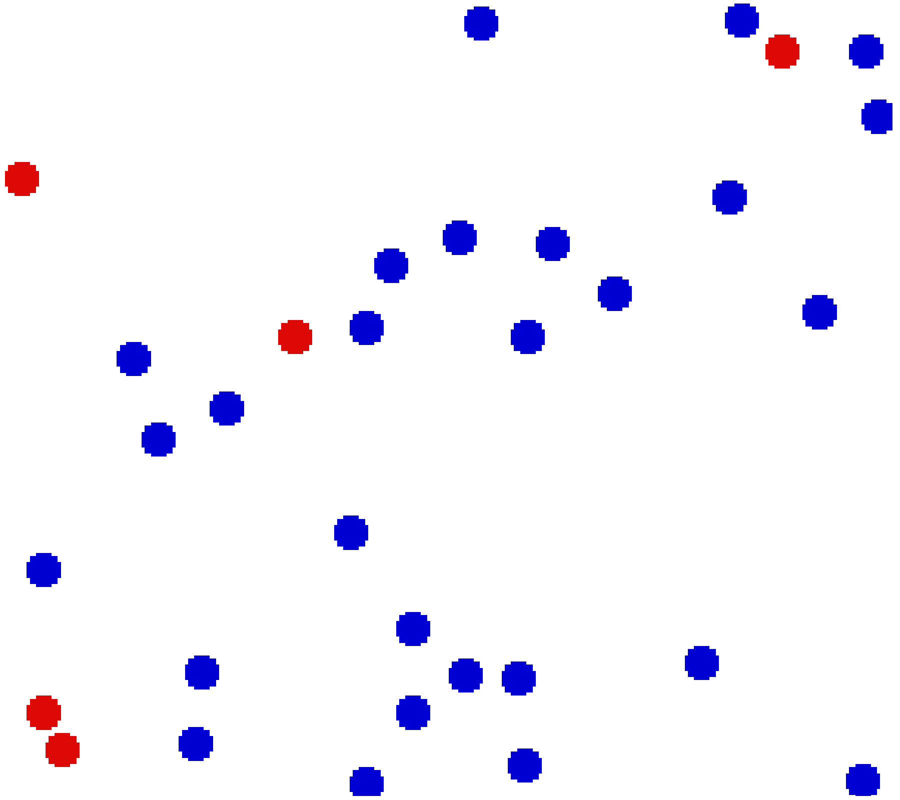
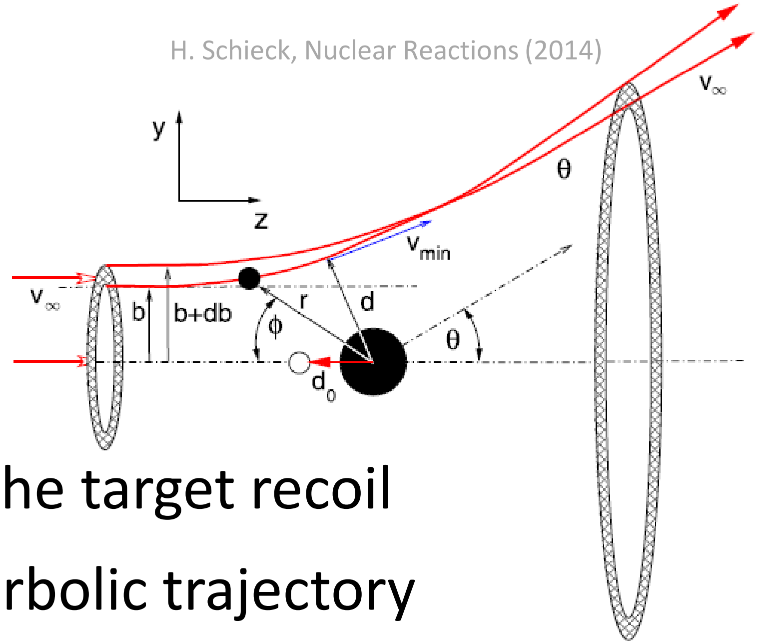
Lecture 14: Scattering

- Rutherford scattering
- Nuclear elastic scattering
- Nuclear inelastic scattering
- Quantum description
- The optical model



Rutherford (a.k.a. elastic Coulomb) scattering

H. Schieck, Nuclear Reactions (2014)



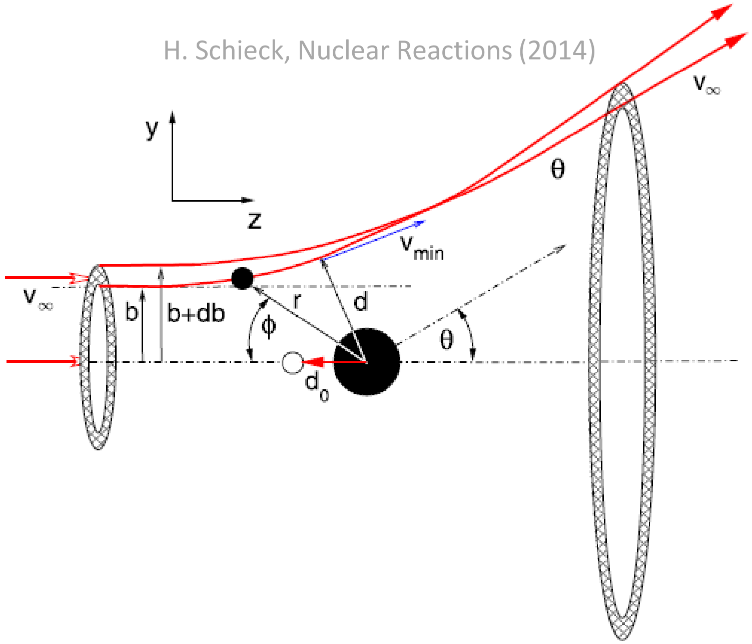
- Consider an interaction where only the Coulomb force matters
  - The force between projectile and target is $F_c = \frac{Z_p Z_t}{r^2}\alpha\hbar c$, and so the potential energy is $V_c = \frac{Z_p Z_t}{r}\alpha\hbar c$
- Furthermore, take the case for which $A_t \gg A_p$, so we can ignore the target recoil
- The $\propto \frac{1}{r}$ repulsive potential means the projectile will follow a hyperbolic trajectory
- For a projectile a distance b above the target projectile-target center line and incoming with initial kinetic energy $T = \frac{1}{2}m_p v_i^2$, we can solve for the scattering angle with a trick:
  - Note that, at closest approach d, energy conservation dictates: $\frac{1}{2}m_p v_i^2 = \frac{1}{2}m_p v_{min}^2 + \frac{Z_p Z_t}{d}\alpha\hbar c$
  - A.k.a. $v_i^2 = v_{min}^2 + \frac{2Z_p Z_t}{m_p d}\alpha\hbar c$ ....or, $\left(\frac{v_{min}}{v_i}\right)^2 = 1 - \frac{2Z_p Z_t}{v_i^2 m_p d}\alpha\hbar c$
  - From angular momentum conservation, $m_p v_i b = m_p v_{min} d$

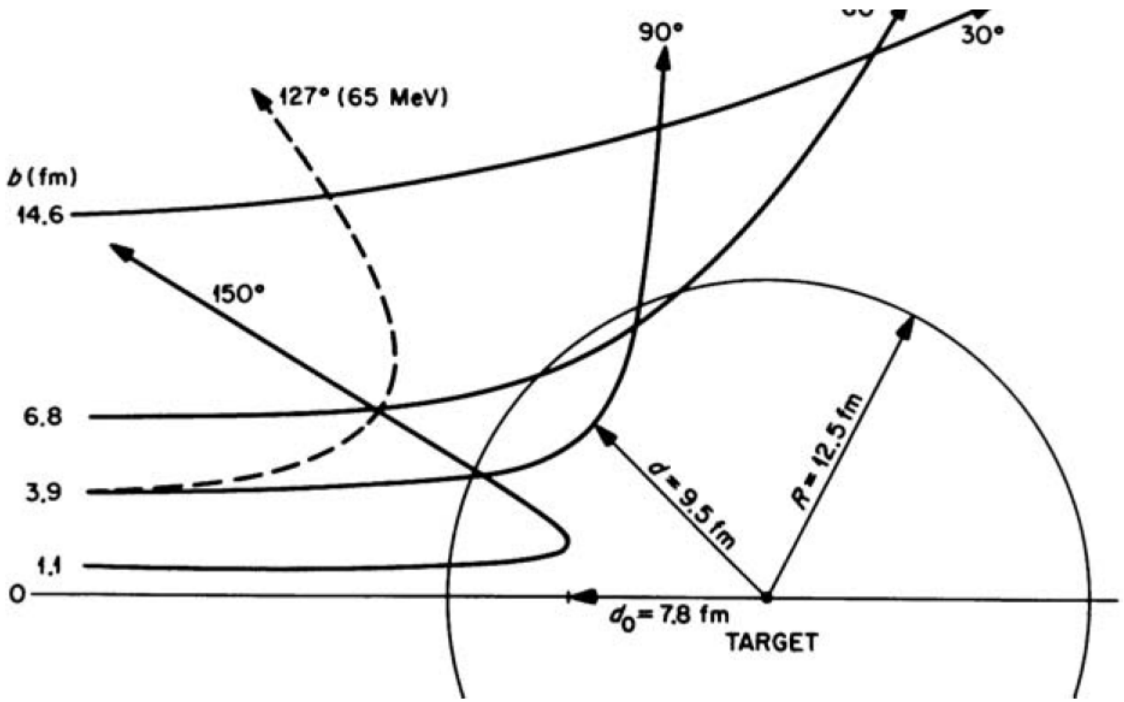
... so $b^2 = d^2\left(\frac{v_{min}}{v_i}\right)^2 = d^2\left(1 - \frac{2Z_p Z_t}{v_i^2 m_p d}\alpha\hbar c\right)$

Rutherford (a.k.a. elastic Coulomb) scattering

- $b^2 = d^2 \left(\frac{v_{min}}{v_i}\right)^2 = d^2 \left(1 - \frac{2Z_pZ_t}{v_i^2 m_p d}\alpha\hbar c\right)$
- You can either integrate the force to get the change in momentum (as in Schieck) or just accept the fact that $\propto \frac{1}{r}$ repulsive potentials result in hyperbolic trajectories, and note that for such cases: $\cot\left(\frac{\pi-\theta}{4}\right) = \frac{d}{b}$
- So, $\cot\left(\frac{\theta}{2}\right) = \frac{2b}{\frac{2Z_pZ_t}{v_i^2 m_p}\alpha\hbar c}$

H. Schieck, Nuclear Reactions (2014)

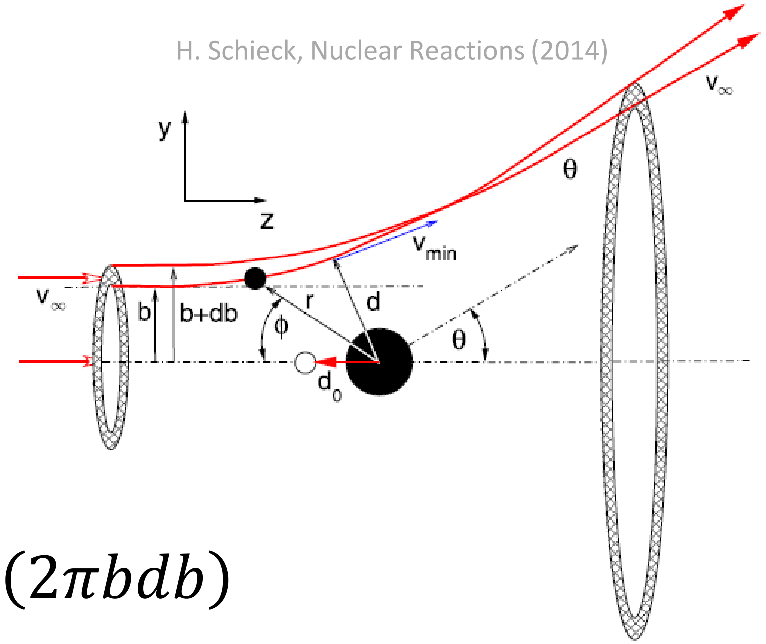




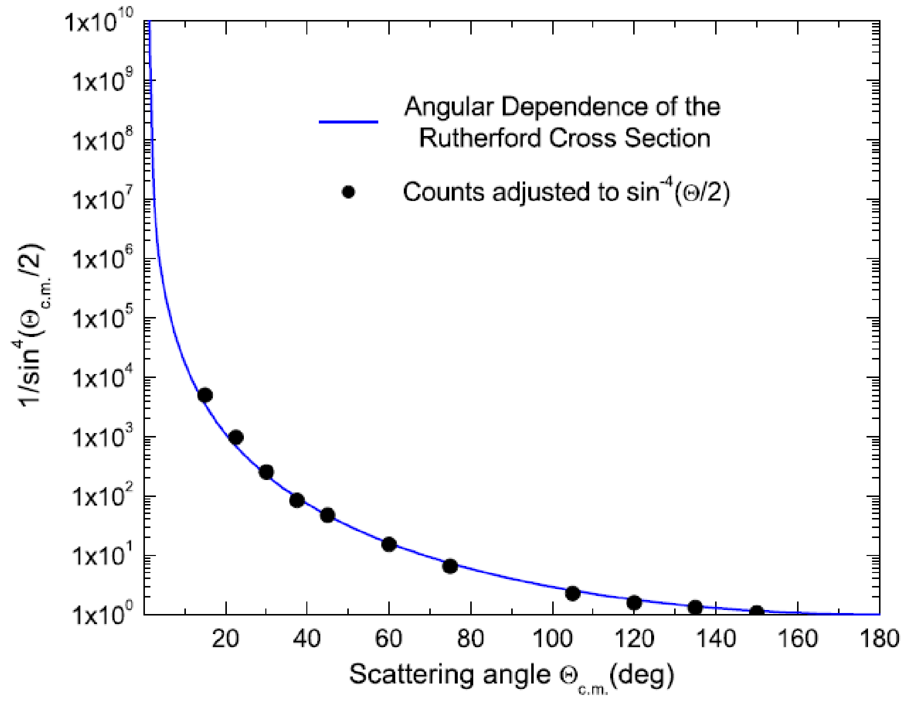
G.R. Satchler, Introduction to Nuclear Reactions (1990)

Rutherford (a.k.a. elastic Coulomb) scattering

H. Schieck, Nuclear Reactions (2014)



- Now that we know the scattering angle θ corresponding to an impact parameter b, we can solve for an experimentally useful property: *the angular distribution*
- Let's consider the intensity of particles arriving within a ring in with the impact parameter range $b + db$
  - $dI = \left(\frac{Intensity}{Unit\ Area}\right)(ring\ area) = F_0(\pi(b+db)^2 - \pi b^2) \approx F_0(2\pi b db)$
- By swapping in our previous result for $\theta(b)$, we get the intensity of particles scattered through a solid angle $d\Omega$ at angle θ
  - $dI = \frac{\pi}{4} I_0 \left(\frac{2Z_1Z_2\alpha\hbar c}{m_p v_i^2}\right)^2 \frac{\cos(\theta/2)}{\sin^3(\theta/2)} d\theta$
- The number scattered through $d\Omega = 2\pi \sin(\theta) d\theta$ sr at θ is:
  - $\frac{d\sigma}{d\Omega} = \frac{dI}{F_0}\frac{1}{d\Omega} = \left(\frac{Z_1Z_2\alpha\hbar c}{4KE_{cm}}\right)^2 \frac{1}{\sin^4(\theta_{cm}/2)}$

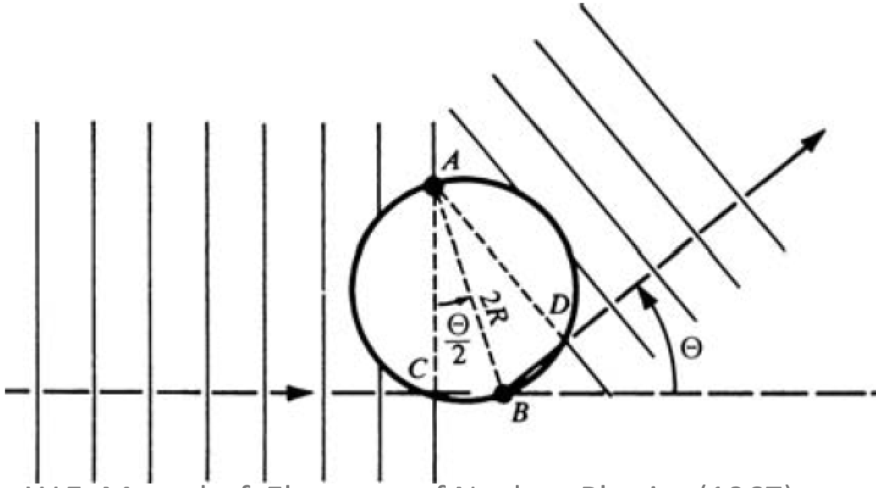


Geiger & Marsden, Phil. Mag. (1913), as reproduced by H. Schieck, Nuclear Reactions (2014)

Rutherford scattering creates a background for all charged particle experiments, but is minimal at backward angles

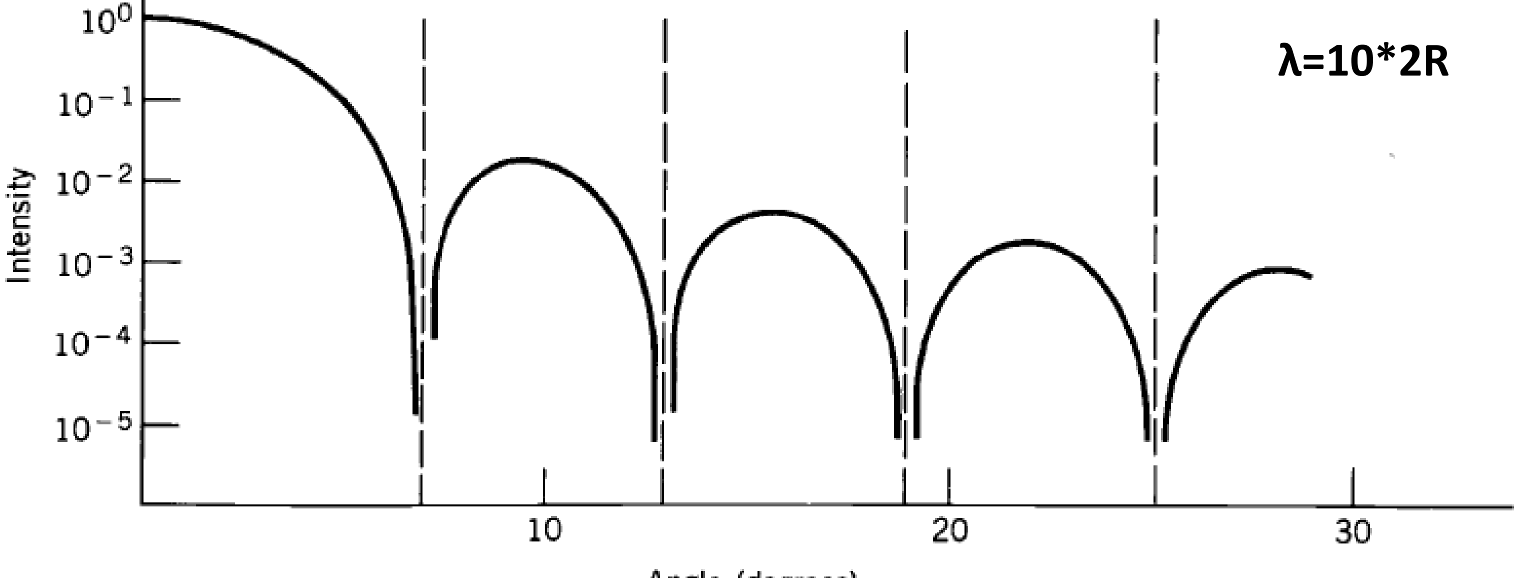
Elastic Nuclear Scattering

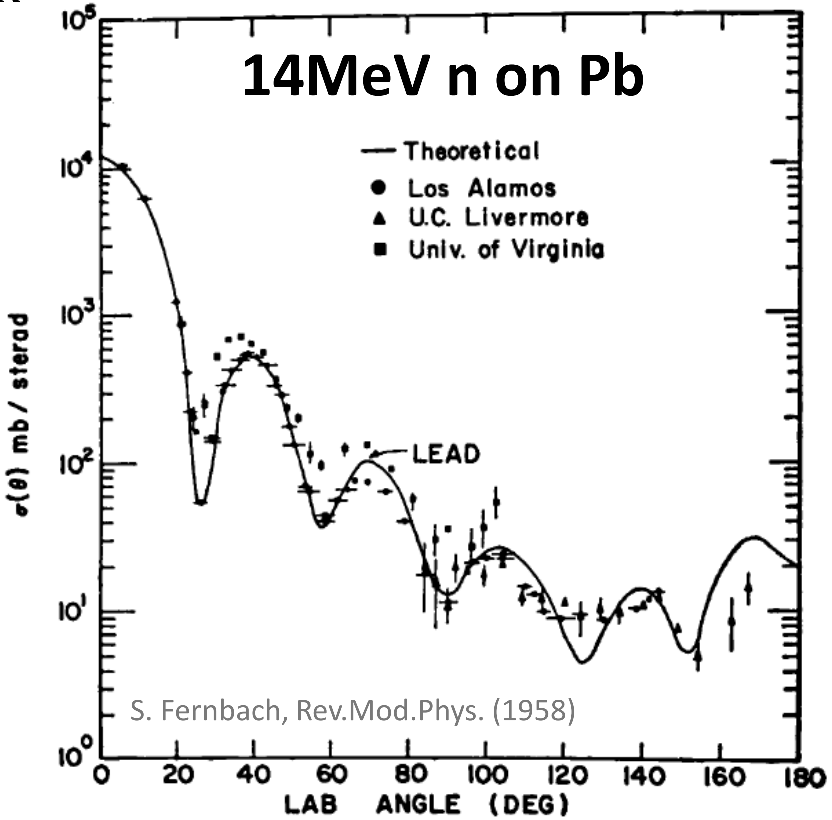
- Considering a projectile as a plane wave and a target nucleus a opaque disk, a creative person realizes this situation looks like diffraction of light off of an opaque disk
- The opticians among us recall that diffraction on a sharp edge results in a diffraction pattern with the first minimum at $\theta \approx \frac{\lambda}{2R}$ and succeeding minima at roughly equal spacing, with a decreasing maxima [like the $\mathrm{sinc}(\theta)$ function]



W.E. Meyerhof, Elements of Nuclear Physics (1967)

K.S. Krane, Introductory Nuclear Physics (1987)



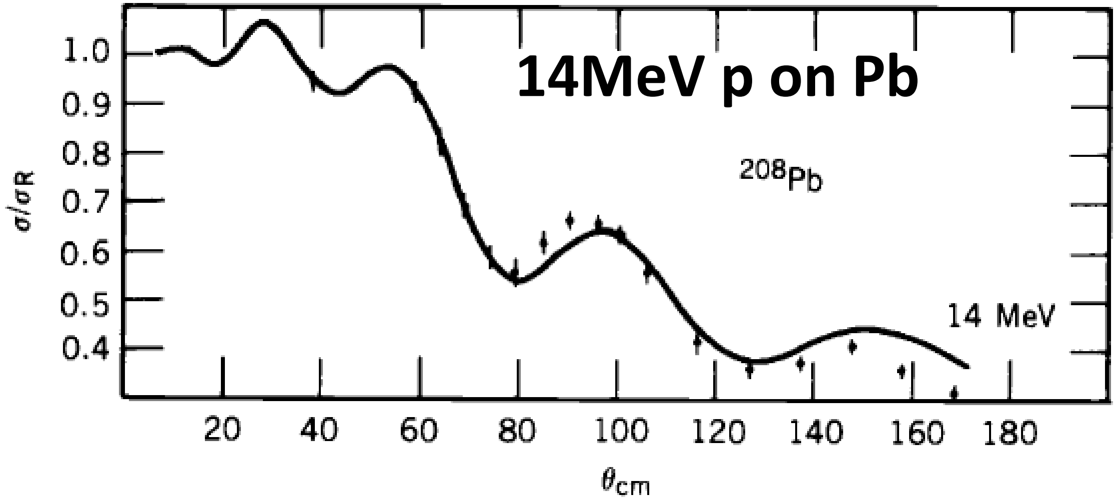


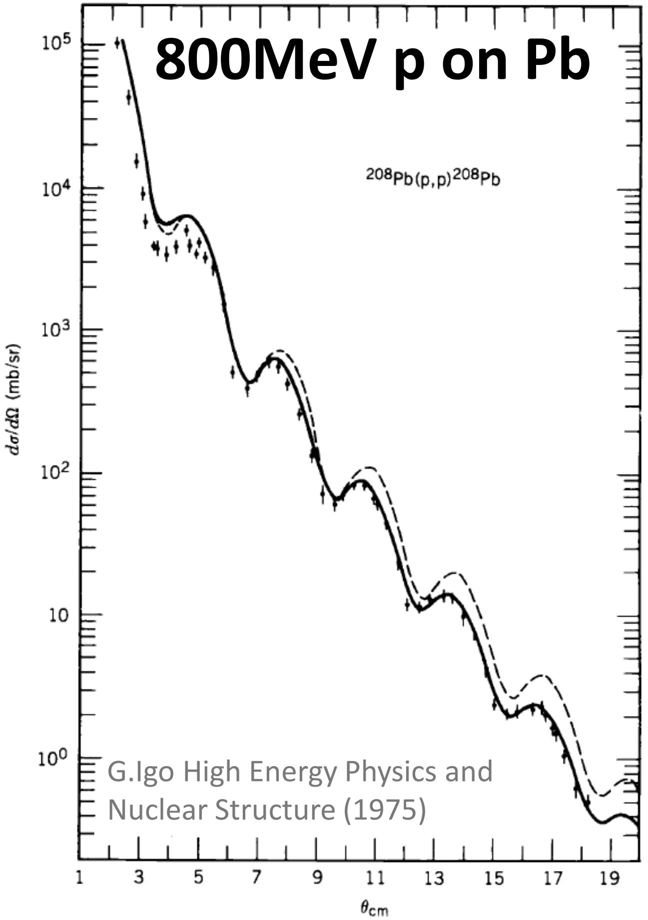
S. Fernbach, Rev.Mod.Phys. (1958)

Coulomb + Elastic Nuclear

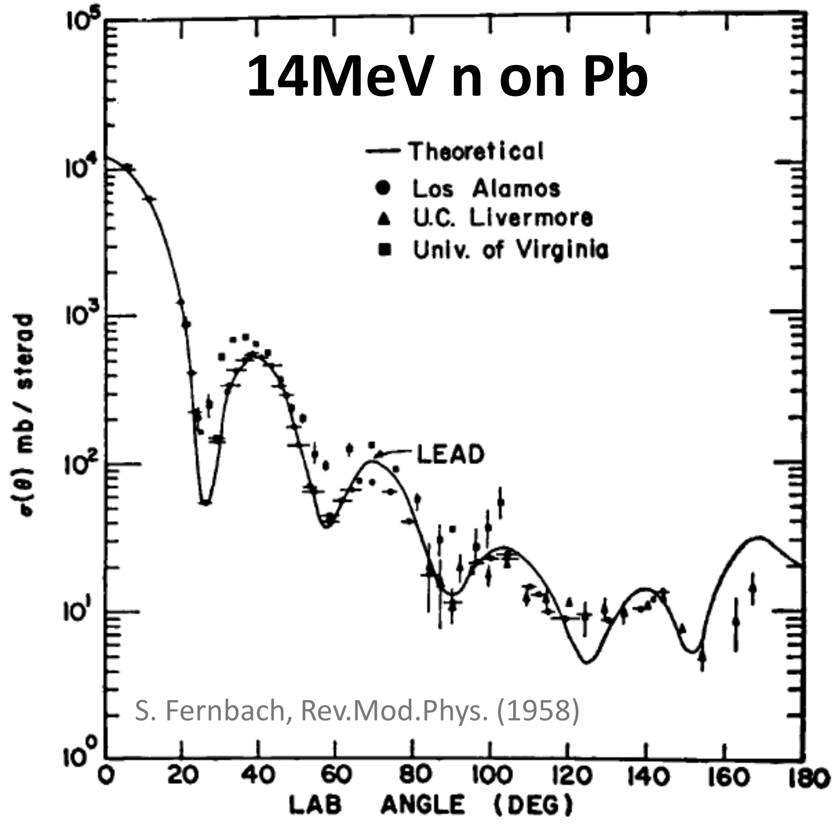
- Coulomb scattering dominates for charged particles at low angles at low energies
- …but at high energies nuclear scattering effects can be seen even at low angles

Eck & Thompson, Nuc.Phys.A (1975)



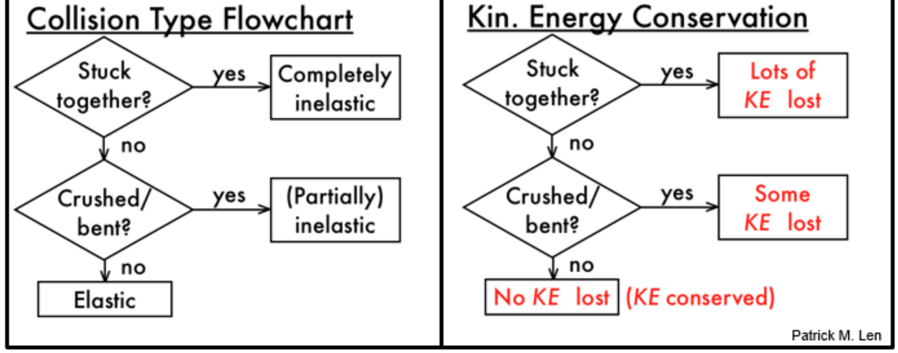


G.Igo High Energy Physics and Nuclear Structure (1975)



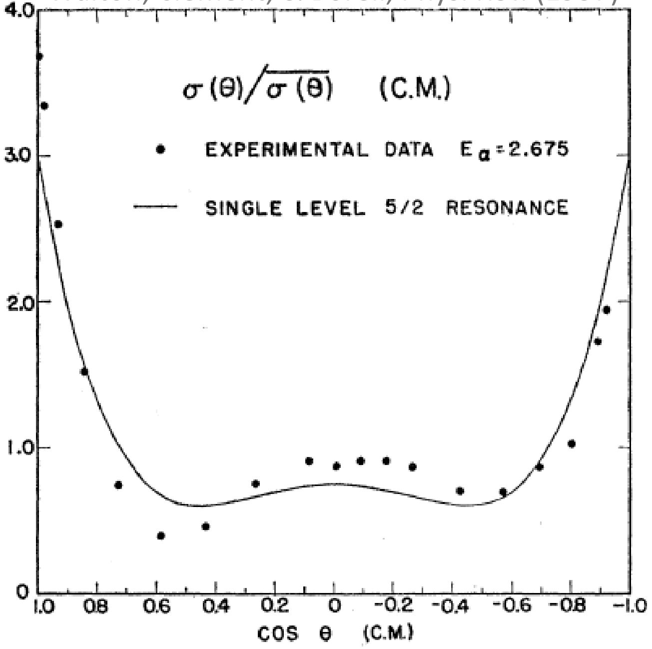
S. Fernbach, Rev.Mod.Phys. (1958)

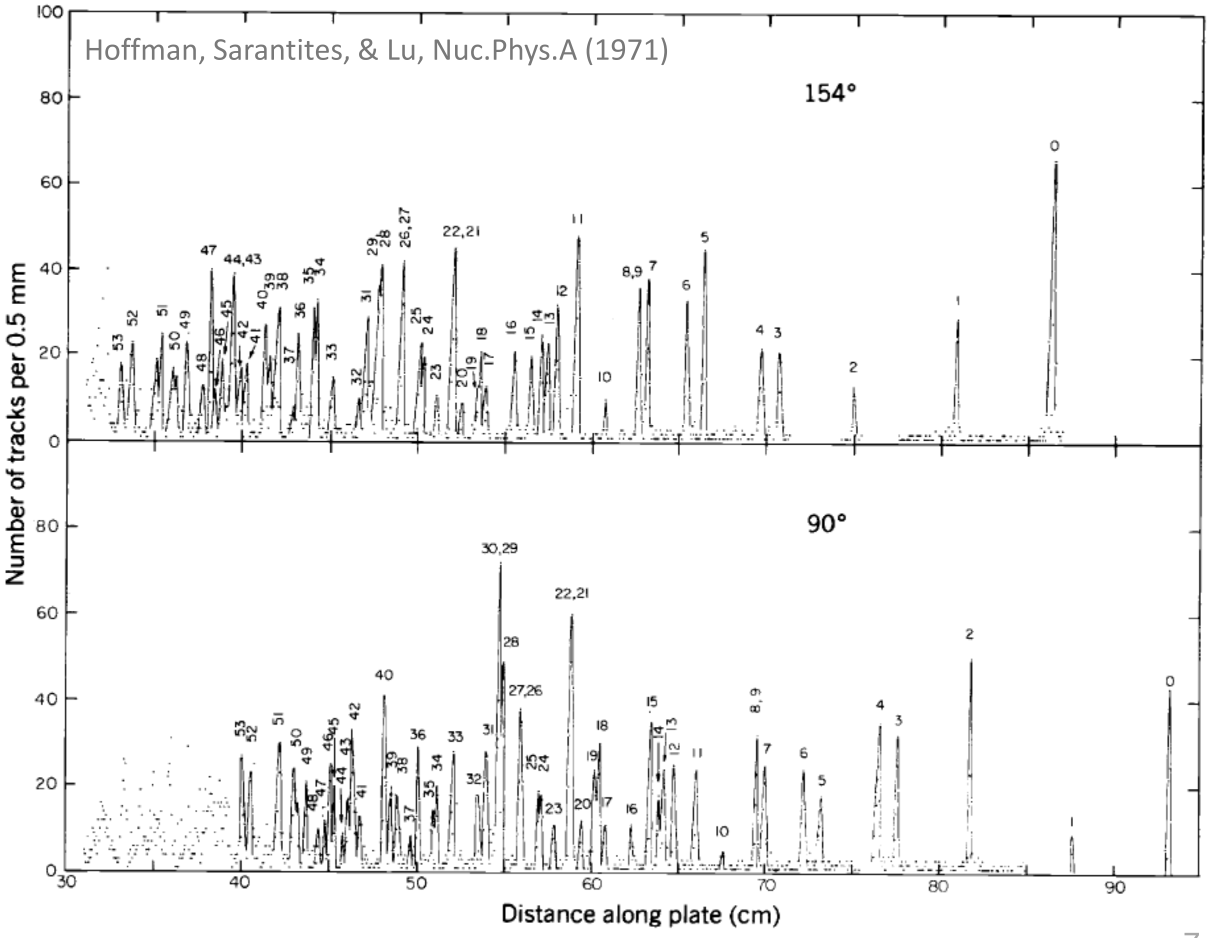
Inelastic Scattering



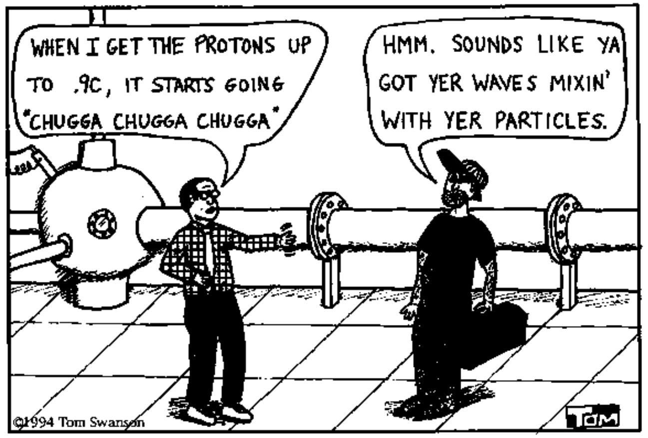
- When the target absorbs energy from the projectile and becomes excited, inelastic scattering has occurred
- At a fixed angle, charged particles observed at energies below the energy expected for elastic scattering are signatures of excited states that were populated
- The relative intensity of each peak is related to the wavefunction overlap between the initial state (beam + target) and final state (recoil + ejectile)
- Angular distributions constrain state spins and parities
  - E.g. for $^{13}C(\alpha,n)^{16}O$, the ground-state J^{π} are known for ^{13}C, ^{16}O, α, and n.
  - At low energy, only low ℓ are relevant, and here ℓ=1 is the lowest ℓ that conserves spin …so the neutron angular distribution gives ℓ-neutron, which is determined by J of the ^{17}O state

Walton, Clement, & Boreli, Phys. Rev. (1957)





Scattering in terms of Quantum Mechanics



QUANTUM MECHANIC

- We've avoided the cold, harsh world for too long! *Of course, that's one of the perks of grad school.*

  To get more useful information, we have to bust out some QM
- Consider the beam as an incident plane wave: $\psi_{inc} = Ae^{ikz}$
- For a central potential, we can make the switch to spherical-polar coordinates,
  $\psi_{inc} = Ae^{ikz} = A\sum_{l=0}^{\infty} i^l(2l+1)j_l(kr)P_l(\cos(\theta))$, where j_l are spherical Bessel functions
- For $kr \gg 1$, $j_l(kr) \approx \frac{\sin\left(kr - \frac{l\pi}{2}\right)}{kr} = i\,\frac{e^{-i(kr-l\pi/2)} - e^{i(kr-l\pi/2)}}{2kr}$
- So, $\psi_{inc} \approx \frac{A}{2kr}\sum_{l=0}^{\infty} i^{l+1}(2l+1)\left(e^{-i(kr-l\pi/2)} - e^{i(kr-l\pi/2)}\right)P_l(\cos(\theta))$
- The math enthusiasts among us will notice that our plane wave is now described in terms of an incoming spherical wave (e^{-ikr}) and an outgoing spherical wave (e^{ikr})
- Scattering off of a central potential, known to its friends as the target nucleus, of course will only impact the outgoing spherical wave

Scattering in terms of Quantum Mechanics

- Scattering will modify $\psi_{inc} = \frac{A}{2kr}\sum_{l=0}^{\infty} i^{l+1}(2l+1)\left(e^{-i\left(kr-\frac{l\pi}{2}\right)} - e^{i\left(kr-\frac{l\pi}{2}\right)}\right)P_l(\cos(\theta))$
  to $\psi = \frac{A}{2kr}\sum_{l=0}^{\infty} i^{l+1}(2l+1)\left(e^{-i(kr-l\pi/2)} - \boldsymbol{\eta_l} e^{i(kr-l\pi/2)}\right)P_l(\cos(\theta))$
- $\boldsymbol{\eta_l}$ is a complex coefficient that describes the impact to outgoing wavefunction for a particular l (a.k.a. a particular "partial wave") which can describe a change in
  the angular distribution (a.k.a. a change in "phase")
  and/or a change in
  amplitude (e.g. due to absorption or emission of an ejectile different from the projectile)
- Subtracting the upper equation from the lower one results in the scattered wave:
  $\psi_{sc} = \frac{A}{2kr}\sum_{l=0}^{\infty} i^{l+1}(2l+1)(1-\eta_l)\left(e^{i\left(kr-\frac{l\pi}{2}\right)}\right)P_l(\cos(\theta))$
  which is equal to $\psi_{sc} = \frac{A}{2k}\frac{e^{ikr}}{r}\sum_{l=0}^{\infty} i(2l+1)(1-\eta_l)P_l(\cos(\theta))$
- If this all seems like formalism for it's own sake, have some patience!
  We can now use the scattered and incident wave functions to determine cross sections

Scattering cross sections from quantum mechanics

- When you were younger and more full of life, you learned in your QM class that a particle current density for a particle described by a wavefunction is $j = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial\psi}{\partial x} - \psi \frac{\partial\psi^*}{\partial x}\right)$
- Moving to 3D and considering our scattered wave, $j_{sc} = \frac{\hbar}{2mi}\left(\psi_{sc}^* \frac{\partial\psi_{sc}}{\partial r} - \psi_{sc} \frac{\partial\psi_{sc}^*}{\partial r}\right)$
- From our previous work, $j_{sc} = |A|^2 \frac{\hbar}{4mkr^2} \left|\sum_{l=0}^{\infty} i(2l+1)(1-\eta_l)P_l(\cos(\theta))\right|^2$
- For the incident wave $j_{inc} = |A|^2 \frac{\hbar k}{m}$
- Logically, the probability particles will be scattered at some angle θ (through an area $r^2 d\Omega$) is $\frac{d\sigma}{d\Omega}(\theta) = \frac{j_{sc}}{j_{inc}} r^2 = \frac{1}{4k^2} \left|\sum_{l=0}^{\infty} i(2l+1)(1-\eta_l)P_l(\cos(\theta))\right|^2$
- For the total cross section, we integrate over $\sin(\theta)\, d\theta d\varphi$, where it turns-out the integral over the Legendre polynomial product is $\frac{4\pi}{2l+1}$
- So, the total scattered cross section is $\sigma_{sc} = \sum_{l=0}^{\infty} \pi \left(\frac{\lambda}{2\pi}\right)^2 (2l+1)|1-\eta_l|^2$

η_l and the scattering phase shift, δ_l

- $\frac{d\sigma}{d\Omega}(\theta) = \frac{1}{4k^2}\left|\sum_{l=0}^{\infty} i(2l+1)(1-\eta_l)P_l(\cos(\theta))\right|^2$, $\sigma_{sc} = \sum_{l=0}^{\infty} \pi \left(\frac{\lambda}{2\pi}\right)^2 (2l+1)|1-\eta_l|^2$
- When only elastic scattering can occur, the outgoing amplitude is maintained and so $|\eta_l|$=1
- For that case, people like to write $\eta_l = e^{2i\delta_l}$, where δ_l is the phase shift of partial wave l
- Then, $|1-\eta_l|^2 = 4\sin^2(\delta_l)$, so $\sigma_{sc} = \sum_{l=0}^{\infty} 4\pi \left(\frac{\lambda}{2\pi}\right)^2 (2l+1)\sin^2(\delta_l)$
- … so how do we determine η_l or δ_l?
  - Theorists:
    1) Posit some potential for the scattering interaction
    2) Solve the Schrödinger equation for $kr \to \infty$
    3) Profit
  - Experimentalists:
    1) Measure scattering and differential cross sections
    2) Match the calculations to the data
    3) Profit

Reaction and Total cross sections

- Let's call everything other than elastic scattering, when considered together, the *reaction*
- For these other processes, some absorption does happen, so $|\eta_l| < 1$
- The total reaction cross section will be the difference between the incoming particle current density and the outgoing particle current density

$$|j_{in}| - |j_{out}| = $$
$$= |A|^2 \frac{\hbar}{4mkr^2}\left[\left|\sum_{l=0}^{\infty} i^{l+1}(2l+1)e^{i\left(\frac{l\pi}{2}\right)}P_l(\cos(\theta))\right|^2 - \left|\sum_{l=0}^{\infty} i^{l+1}(2l+1)e^{-i\left(\frac{l\pi}{2}\right)}P_l(\cos(\theta))\right|^2\right]$$

- Which reduces to $\sigma_{reaction} = \sum_{l=0}^{\infty} \pi\left(\frac{\lambda}{2\pi}\right)^2 (2l+1)(1-|\eta_l|^2)$
- The total cross section is therefore

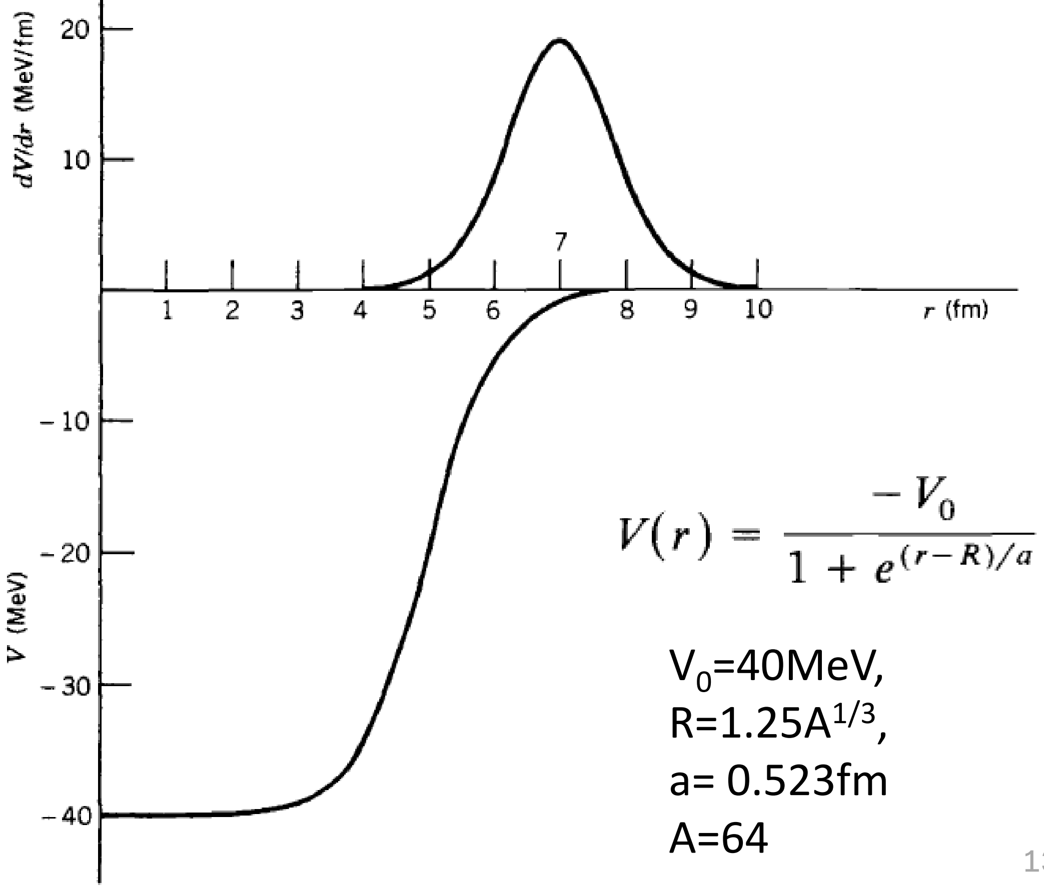
$$\sigma_{total} = \sigma_{elastic} + \sigma_{reaction} = \sum_{l=0}^{\infty} 2\pi\left(\frac{\lambda}{2\pi}\right)^2 (2l+1)(1-\mathrm{Real}(\eta_l))$$

A take-away from the above work is that pure elastic scattering can happen, but reactions with no elastic scattering component cant. **Elastic scattering always happens.**

The optical model

- As we just saw, the total and differential cross sections can be calculated by scattering a plane wave off of some nuclear potential
- Therefore, we can turn the problem on its head and instead use measured scattering data to figure out what potential we would need to have our solution to the Schrödinger equation provide the observed data
- The optical model is a way to do this, where the potential is something like $U(r) = V(r) + iW(r)$
- As we saw much earlier in class, the Woods-Saxon form is the best bet for $V(r)$
- Since absorption is mostly going to happen on the surface, typically $W(r) \propto dV/dr$
- Solving for the optical model parameters for one case means reaction cross sections can be solved for many more cases with the same projectile and similar A for the target

K.S. Krane, Introductory Nuclear Physics (1987)

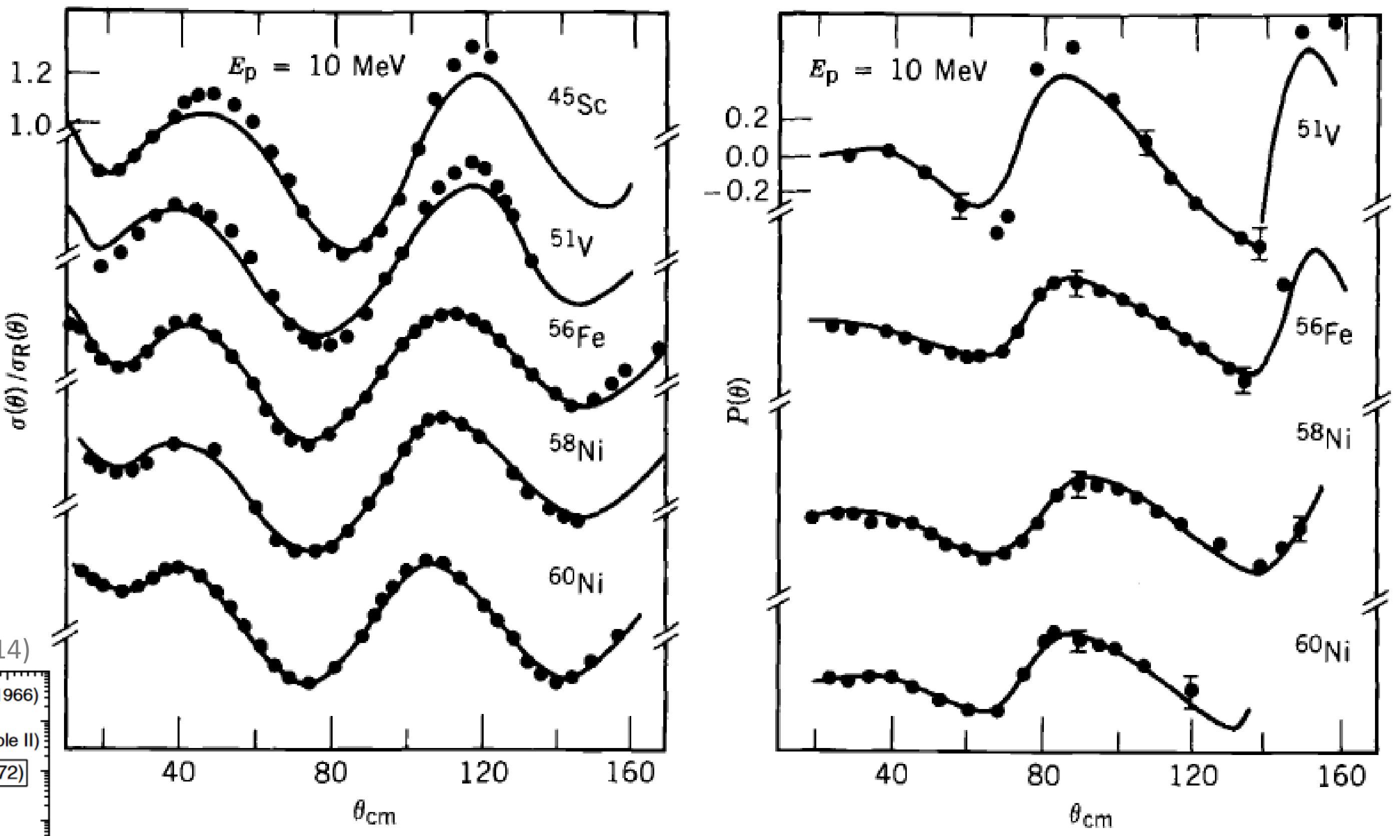


V_0=40MeV,
R=1.25$A^{1/3}$,
a= 0.523fm
A=64

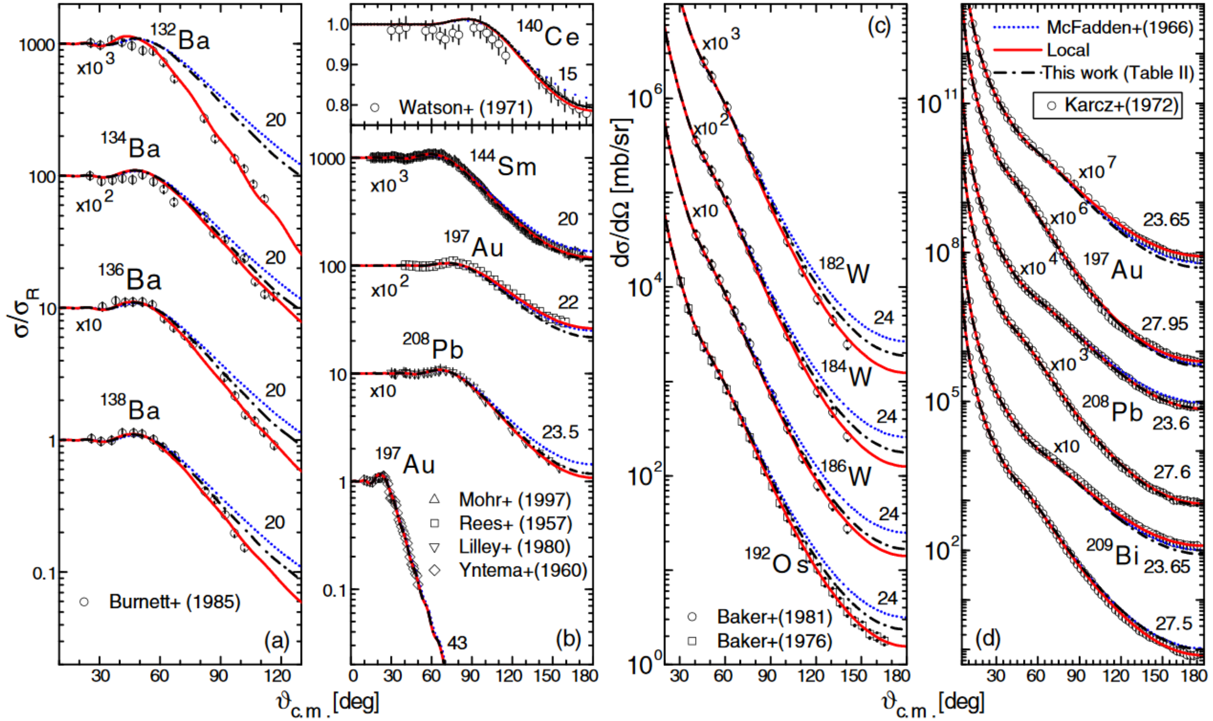
Optical model parameters

- If you need to calculate a cross section and you don't have the time or resources or inclination to measure the necessary scattering data, you're in luck!
- Several compilations exist of not only local fits, but global potentials obtained from systematics to these fits

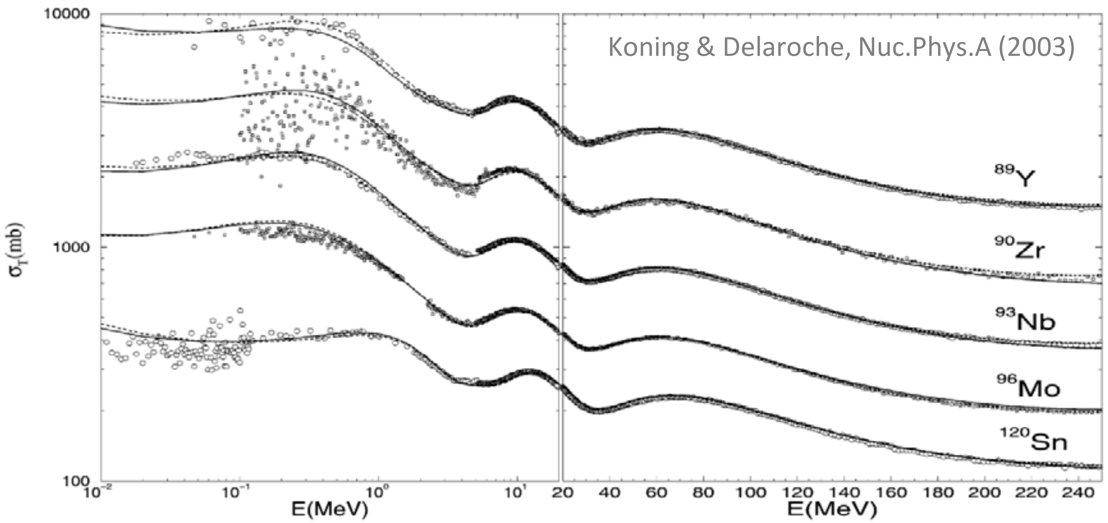
Becchetti & Greenlees, Phys.Rev. (1969)



Avrigeanu, Avrigeanu, & Manailescu, PRC (2014)



Koning & Delaroche, Nuc.Phys.A (2003)



Further Reading

- Chapter 10: Modern Nuclear Chemistry (Loveland, Morrissey, Seaborg)
- Chapter 11: Introductory Nuclear Physics (K.S. Krane)
- Chapter 17: Introduction to Special Relativity, Quantum Mechanics, and Nuclear Physics for Nuclear Engineers (A. Bielajew)
- Chapters 1 & 2: Nuclear Reactions (H. Schieck)
- J. Blatt & L.C. Biedenharn, Reviews of Modern Physics, 24, 258 (1952)
- Lecture Notes for Advanced Quantum Mechanics (Ben Simons)