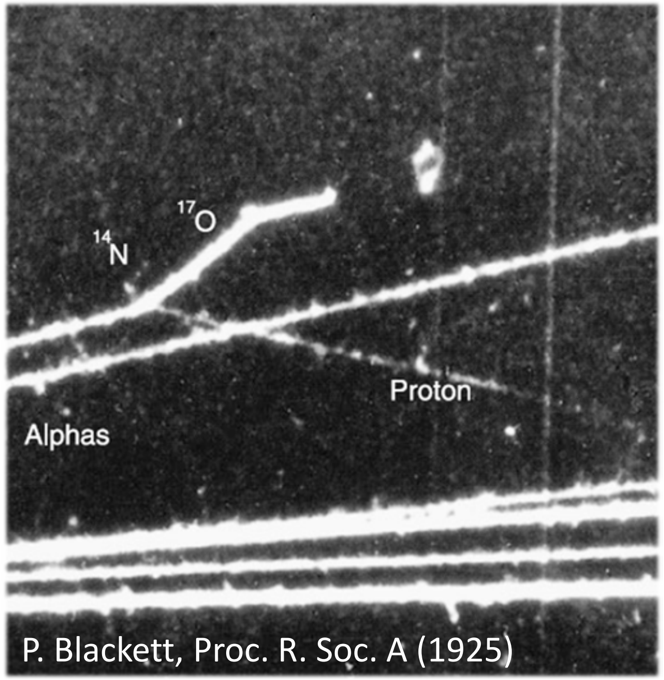
Lecture 14: Reactions Overview

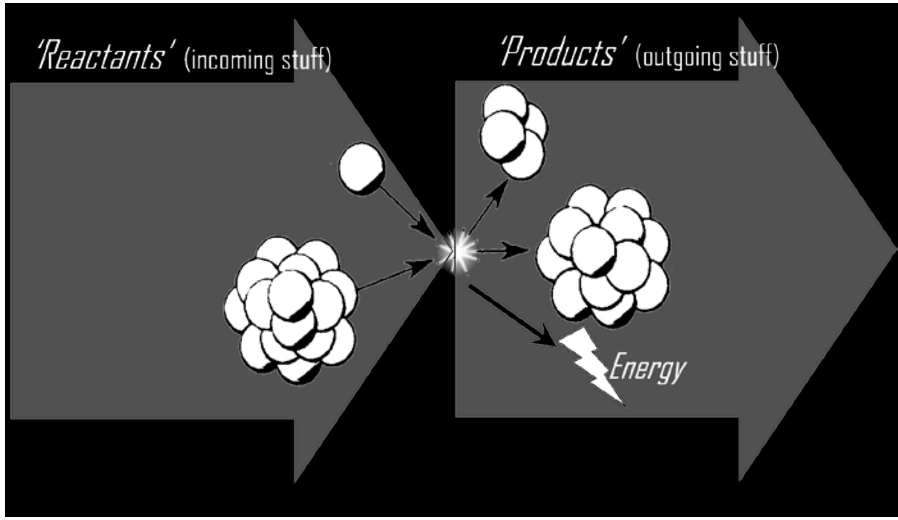
- Terminology and types
- Energetics & kinematics
- Cross sections

Defining a reaction

- A nuclear reaction consists of the interaction of two or more nuclei or nucleons that results in some final product
- The initial stuff is known as the reactants [projectile and target, in the lab] and the final stuff is known as the products [recoil and ejectile, in the lab]
- Several sets of products are often possible for a pair of reactants colliding at a given energy, including simple scattering
  - The ways of "decaying" from the nucleus briefly formed by the reaction are known as channels
- The modern notation for a reaction is always to put the lighter of the reactants and products on the inside of a pair of brackets, like A(b,c)D, where $M_A > M_b$ and $M_D > M_c$
- Nuclear reactions are governed by the strong force and so they conserve baryon number, nuclear charge, energy, linear momentum, angular momentum, and parity

1st nuclear fusion reaction observed in the lab





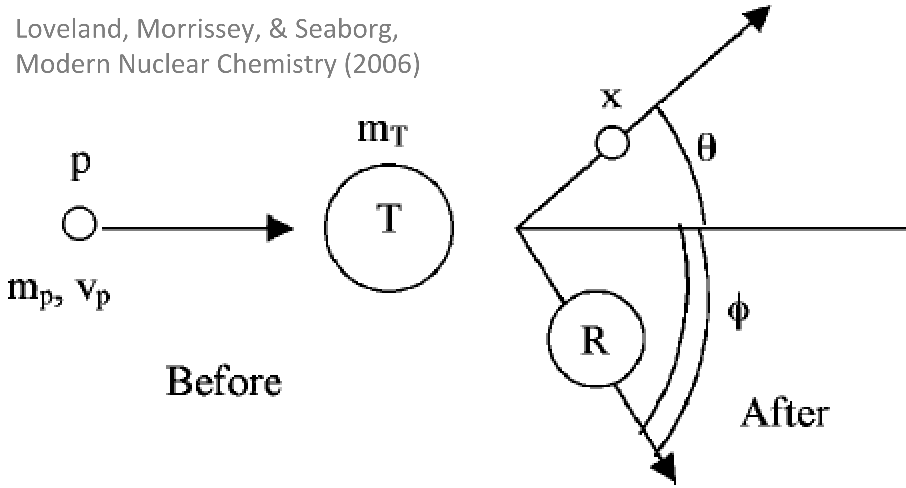
Reaction types

- Reactions are categorized by the participants and by the mechanism
- Participants:
    - Particle comes in and only a γ comes out: *radiative capture*
    - γ comes in and one or a few particles comes out: *photodisintegration*
    - A multi-nucleon beam transfers a nucleon (or nucleons) to the target: *transfer*
    - A projectile exits as an ejectile, taking one or more nucleons with it: *knockout*
    - The projectile and target aren't transmuted and exit in their ground states: *elastic scattering*
    - The projectile and target aren't transmuted and one (or both) is excited: *inelastic scattering*
- Mechanism:
    - Few nucleons take part in the reaction (e.g. transfer): *direct*
    - The projectile and target briefly fuse, sharing energy amongst all nucleons: *compound*
    - The projectile and target briefly fuse, forming a loosely bound state: *resonant*

Energetics

- Conservation of energy requires
  $m_{proj}c^2 + KE_{proj} + m_{targ}c^2 = m_{rec}c^2 + KE_{rec} + m_{ejec}c^2 + KE_{ejec}$,
  where m_i could refer to the mass of a nucleus in an excited state
- The Q-value is equal to any excess kinetic energy resulting from the reaction,
  $$Q = ME_{proj} + ME_{targ} - ME_{rec} - ME_{ejec} = KE_{rec} + KE_{ejec} - KE_{proj}$$
- So, for a reaction to be energetically possible, $Q + KE_{proj} > 0$
- Conservation of momentum in x and y dictates:
  - x: $m_p v_p = m_e v_e \cos(\theta) + m_r v_r \cos(\phi)$
  - y: $0 = -m_e v_e \sin(\theta) + m_r v_r \sin(\phi)$

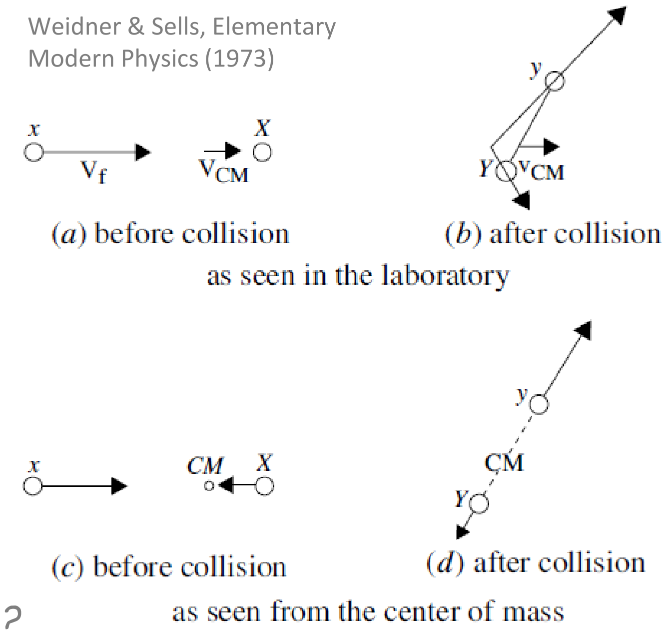
Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)



- Noting $p = \sqrt{2mKE}$ and doing some algebra,
  $$m_p KE_p - 2\sqrt{(m_p KE_p m_e KE_e)}\cos(\theta) + m_e KE_e = m_r KE_r$$
- Noting that $Q = KE_r + KE_e - KE_p$, we get the "Q equation"
  $Q = KE_e\left(1 + \frac{m_e}{m_r}\right) - KE_p\left(1 - \frac{m_p}{m_r}\right) - \frac{2}{m_r}\sqrt{m_p m_e KE_p KE_e}\cos(\theta)$ which tells us Q can be determined by measuring the angle and energy of the ejectile alone

Energy in the center of mass (CM) frame

Weidner & Sells, Elementary Modern Physics (1973)



- For mathematical convenience *(and to keep the outsiders out!)* often the center-of-mass (CM) frame is employed
- For the energetic conversion between the two, consider the kinetic energy in the CM frame: $KE_{cm} = KE_{lab}\frac{A_t}{A_p+A_t}$
- So, the center of mass energy is always lower than the laboratory energy, but how much lower depends on which reactant is the beam and which is the target

What does this imply about the choice of forward vs inverse kinematics measurements?

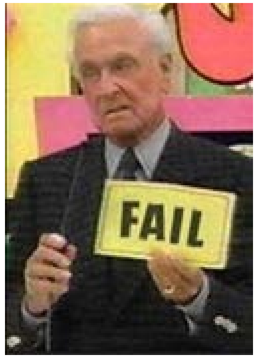
For an accelerator with a limited minimum voltage (i.e. all of them), inverse kinematics will reach a lower center-of-mass energy.

- As an aside, consider an alternative way to specify the laboratory beam energy: MeV/u
- Let's look at an example reaction and use Energy/nucleon for the forward & inverse kinematics:
  - ^{96}Zr(α,n) at $E_{cm} = 7.68MeV$
  - For an α beam on a ^{96}Zr target, $E_{\alpha,lab} = \frac{96+4}{96}7.68MeV = 8MeV = \left(\frac{8}{4}\right) = 2MeV/u$
  - For a ^{96}Zr beam on an α target, $E_{^{96}Zr,lab} = \frac{96+4}{4}7.68MeV = 192MeV = \left(\frac{192}{96}\right) = 2MeV/u$
  - Hence, MeV/u is often a more useful way to speak about reaction energies

Reaction threshold

- If you want to initiate a particular reaction in the lab by impinging a beam on a target and the reaction is endothermic, what should your beam energy be?

The Q-value? ***No!***



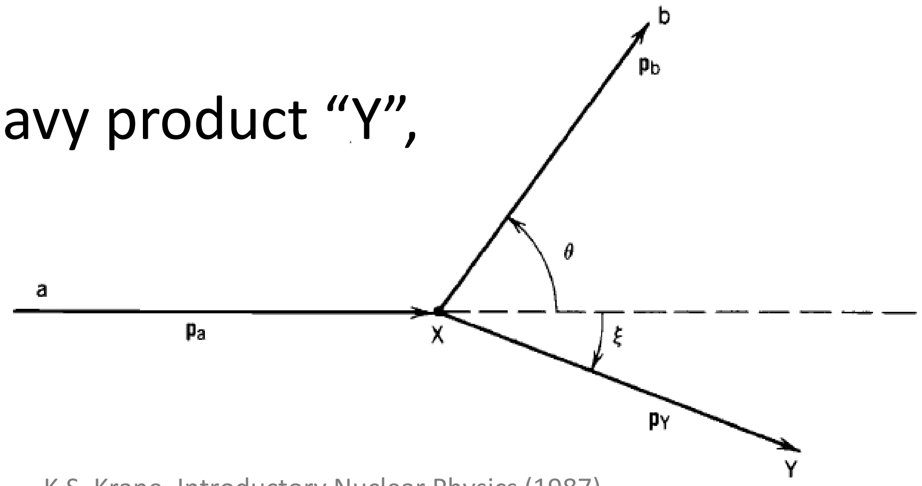
- You need to pay the extra energy that is carried away by the center-of-mass of the system
- Recalling our energy conversion, $KE_{cm} = KE_{lab}\frac{A_t}{A_p+A_t}$,

  the lab energy must be $KE_p \geq (-Q)\frac{A_t+A_p}{A_t}$

Kinematics

- Using the Q-equation, $Q = KE_e\left(1+\frac{m_e}{m_r}\right) - KE_p\left(1-\frac{m_p}{m_r}\right) - \frac{2}{m_r}\sqrt{m_p m_e KE_p KE_e}\cos(\theta)$, and conservation of momentum, we can determine the relationship between our reaction products outgoing energy and angle
- Calling the projectile "a", target "X", light product "b", and heavy product "Y", some gymnastics results in

$$\sqrt{KE_b} = \frac{\sqrt{m_a m_b KE_a}\cos(\theta) \pm \sqrt{m_a m_b KE_a \cos^2(\theta) + (m_Y + m_b)[m_Y Q + (m_Y - m_a)KE_a]}}{m_Y + m_b}$$

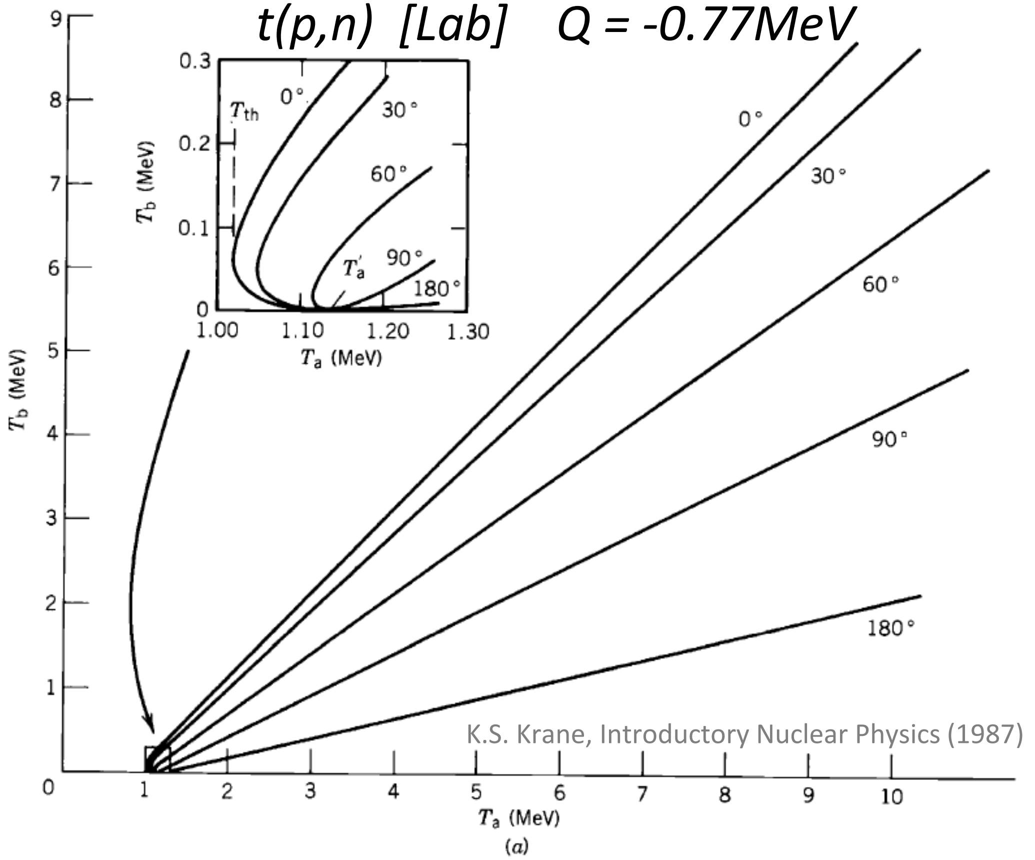


K.S. Krane, Introductory Nuclear Physics (1987)

- That looks pretty awful … and it is!
- Luckily, some noble souls have already worked-out the math for us and developed calculators used to figure out the laboratory and center-of-mass kinematics for two-body reactions of interest:
  - A common option is Catkin, from W. N. Catford: kinematics with an Excel spreadsheet
  - Another is the LISE++ kinematics calculator, from O. Tarasov & D. Bazin: kinematics with a Windows/Apple app
  - My recent favorite is SkiSickness, from S.K.L. Sjue: kinematics with an online tool

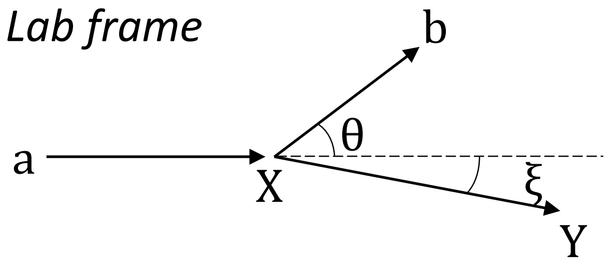
Kinematics, common features

- For a given projectile energy, usually only one ejectile energy corresponds to a given angle
- Ejectiles are emitted at 0° for projectiles at the threshold energy
- For $Q < 0$ reactions, there is a double-valued behavior for projectile energies near the threshold



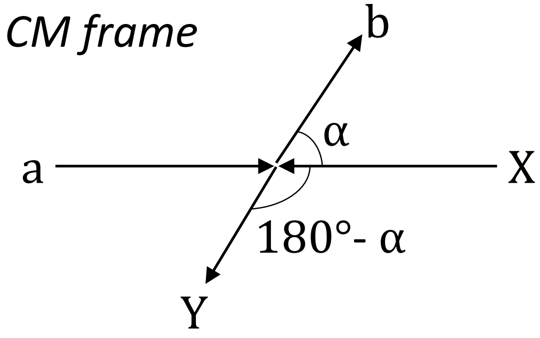
- The double-valued ejectile-angle relationship exists between the threshold energy and an upper limit: $(-Q)\frac{m_X+m_a}{m_X} \leq T_a \leq (-Q)\frac{m_Y}{m_Y-m_a}$
- If $A_{recoil} \gg A_{projectile}$, the double-value region is obviously inconsequential
- The angular range which can have the double-valued behavior is limited
  - At the upper limit, the double-valued behavior happens between $\theta \leq 90°$
  - Near the threshold, the double-valued behavior can only happen near $\theta \approx 0°$
  - In general, the maximum angle for double-valued behavior θ_{max} is $\cos^2(\theta_m) = -\frac{(m_Y+m_b)[m_Y Q+(m_Y-m_a)T_a]}{m_a m_b KE_a}$

Center of Mass Ejectile Properties



- If low-mass particle b is observed at an angle θ with energy $E_{b,L}$
- The center-of-mass energy of b is

$$E_{b,CM} = (E_{a,L} + Q)\left(\frac{m_X m_Y}{(m_a+m_X)(m_b+m_Y)}\right)\left(1 + \frac{m_a Q}{m_X(E_{a,L}+Q)}\right)$$



- The center-of-mass angle of b is

$$\alpha = \sin^{-1}\left(\sqrt{\frac{E_{b,L}}{(E_{a,L}+Q)\left(\frac{m_X m_Y}{(m_a+m_X)(m_b+m_Y)}\right)\left(1+\frac{m_a Q}{m_X(E_{a,L}+Q)}\right)}}\sin(\theta)\right)$$

- The moral of the story is: use a kinematics calculator!

Reaction likelihood: The cross section σ

- To quantify the probability of a nuclear interaction occurring, we use the cross section
- The number of reactions that occur N when we impinge an ion beam of intensity I (in units of particles/second) for some time t on a target with areal density n_T (in units of atoms/cm^2) is scaled by a sort of reaction probability σ: $N = Itn_T\sigma$
- For the units to work: # = (s^{-1})*(s)*(atoms/cm^2), [σ] = cm^2, so we call this area the cross section.
- You can think of it as the overlap between the wave-like projectile and target, though it can be very different from the classical value from a physical overlap
- Practically, a reaction product detection efficiency ε needs to be included: $\sigma = \frac{N_{detected}}{Itn_T\varepsilon}$
- And the cross section and detection efficiency need not be uniformly distributed, so we should consider the cross section for particles detected at some angle θ by a detector with some solid-angle $\Omega = \frac{Detector\ Area}{(Target-Detector\ Distance)^2}$:

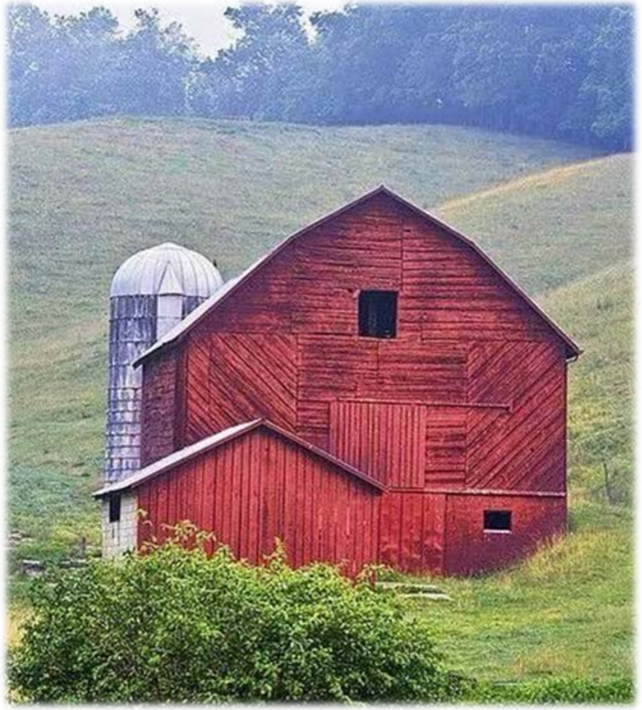
$$\sigma(\theta) = \frac{N_d}{I_b t_{meas} n_T \varepsilon_{det}(\theta)\Omega_{det}(\theta)}$$

Fun fact: Beam intensities are usually measured by the current those ions create on a beam-stopping device. To convert to I_{pps}, you need to use the fact that 1 unit of electric charge is 1.602×10^{-19}C and 1A=1C/s.

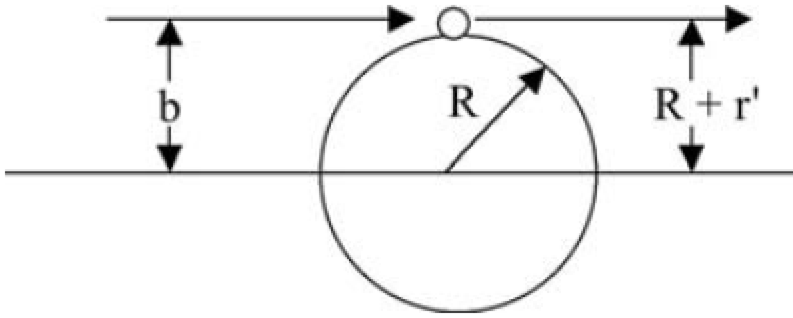
Cross section units

- To estimate the scale of cross sections we're going to deal with, consider the area an incoming projectile would see for a typical nucleus
- Take $A = 100$, since it's near the middle of the nuclear chart
  - $R = 1.2A^{1/3} fm \approx 5.5 fm$
  - $\pi R^2 \approx 100 fm^2$
  - Boy, that's huge! As big as a barn even. So let's call it a barn: $1b \equiv 100 fm^2 = 10^{-24} cm^2$
- For convenience, to save you from unit conversions, the most commonly encountered magnitudes in low energy nuclear physics are:
  - $1b = 10^{-24} cm^2$
  - $1mb = 10^{-27} cm^2$
  - $1\mu b = 10^{-30} cm^2$
  - $1nb = 10^{-33} cm^2$

- *Some of the largest cross sections we see are for neutron capture, e.g. $n_{thermal}+^{125}Xe \sim 10^6 b$*
- *$\sigma \sim 100mb$ can be measured with $\sim 10^4 pps$ on a target of $\sim 10^{19} atoms/cm^2$, like with re-accelerated radioactive on beams (like at the NSCL)*
- *$\sigma \sim 1$ nb is usually the limit for stable ion beam facilities for measuring something within a reasonable time (like at the EAL)*
- *Element discovery nowadays involves production cross sections on the $\sigma \sim 1$ pb scale*



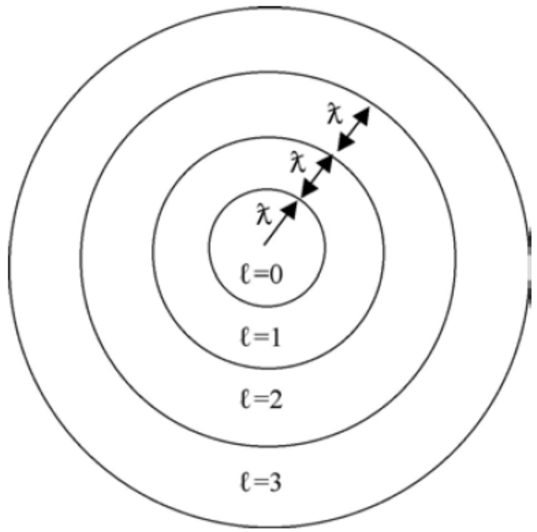
Cross sections from a semi-classical view



Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

- Consider a grazing collision between a projectile with radius r' and a target with radius R, where the distance between them is the impact parameter $b = R + r$'
- The relative orbital angular momentum is $\left|\vec{l}\right| = l\hbar = |\vec{r} \times \vec{p}| = pb$
- Since our projectile is a wave after all, $p = \frac{h}{λ} = 2\pi\hbar/λ$ …. and so $l\hbar = 2\pi b\frac{\hbar}{λ}$ …i.e. $b = \frac{l}{2\pi}λ$
- Collisions at a given impact-parameter slice $b \pm db$ correspond to a given l
- The geometric cross section for one slice is:
  - $\sigma_l = \sigma_{b\pm db} = \sigma_{b+db} - \sigma_{b-db} = \pi(b+db)^2 - \pi(b-db)^2$
    $= \pi(l+1)^2\left(\frac{λ}{2\pi}\right)^2 - \pi l^2\left(\frac{λ}{2\pi}\right)^2 = \pi\left(\frac{λ}{2\pi}\right)^2(2l+1)$
- So a total cross section is a sum over l: $\sigma_{total} = \sum_l \pi\left(\frac{λ}{2\pi}\right)^2(2l+1)$
- Including quantum mechanics, the probability of tunneling through a barrier corresponding to l is the transmission coefficient T_l,

so $\sigma_{total} = \pi\left(\frac{λ}{2\pi}\right)^2\sum_l (2l+1)\ T_l$ where $0 \leq T_l \leq 1$



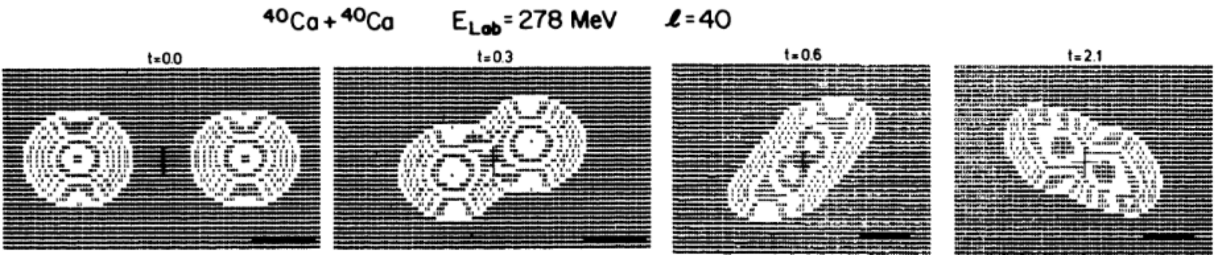
Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

Cross sections from a semi-classical view: *High-E collision*

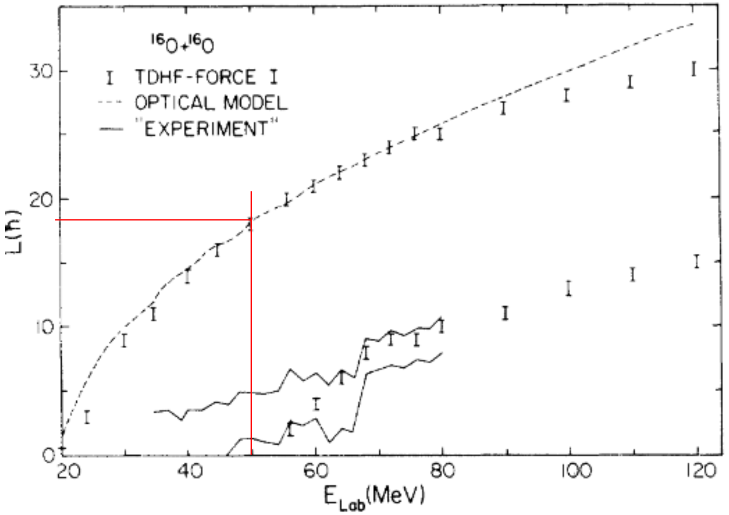
- $\sigma_{total} = \pi \left(\frac{\lambda}{2\pi}\right)^2 \sum_l \ (2l+1)\ T_l$ where $0 \leq T_l \leq 1$
- High-energy limiting case for T_l:

  For relatively high energy (e.g. a ~100MeV heavy-ion beam), $T_l = \begin{matrix} 1 \ \ for\ l < l_{max} \\ 0 \ \ for\ l > l_{max} \end{matrix}$ This is the "sharp cutoff limit"
- l_{max} is estimated by assuming barely any beam-target overlap: $b = R$, so $l_{max} = \frac{2\pi R}{\lambda}$,

  …or can be estimated using fancy calculations (e.g. Bonche, Grammaticos, & Koonin, PRC 1978)
- Example: 50MeV ^{16}O on an ^{16}O target
  - $l_{max} = \frac{2\pi(1.2)(16)^{1/3} fm}{h} \sqrt{2(16 * \frac{931.5 MeV}{c^2})50 MeV} = \frac{(3.02 fm)(1221 MeV)}{197 MeV fm} \approx 19$



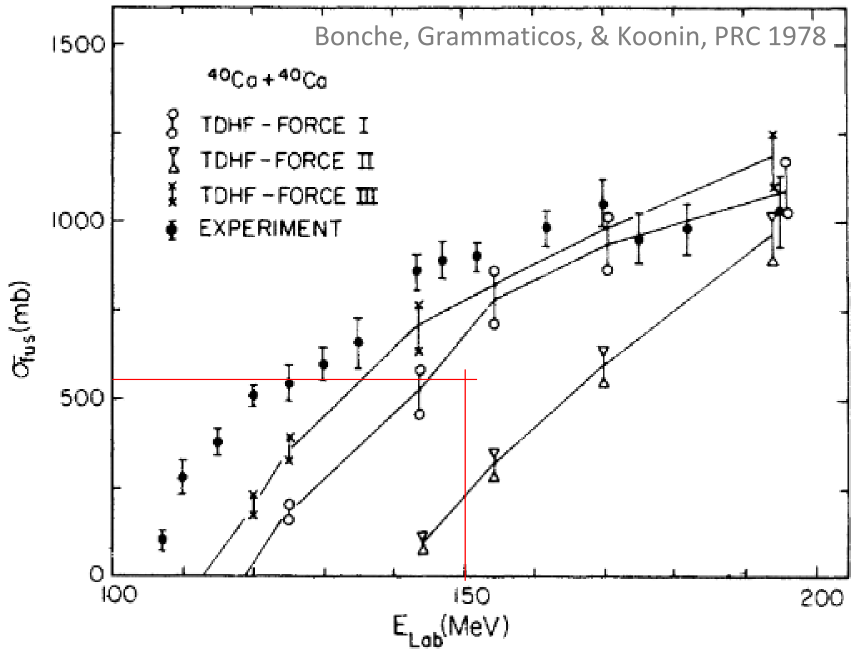
Bonche, Grammaticos, & Koonin, PRC 1978



Cross sections from a semi-classical view: *High-E collision*

- $\sigma_{total} = \pi \left(\frac{\lambda}{2\pi}\right)^2 \sum_l (2l+1)\ T_l$ where $0 \leq T_l \leq 1$
- High-energy limiting case for T_l:

  For relatively high energy (e.g. a ~100MeV heavy-ion beam), $T_l = \begin{matrix} 1 \ \ for\ l < l_{max} \\ 0 \ \ for\ l > l_{max} \end{matrix}$ This is the "sharp cutoff limit"
- Example: 150MeV ^{40}Ca on a ^{40}Ca target: $l_{max} \approx 70$
- $\sigma_{total} = \pi \left(\frac{\lambda}{2\pi}\right)^2 \sum_{l=0}^{70}(2l+1) = \pi \left(\frac{\hbar c}{\sqrt{2(40*931.5MeV)150MeV}}\right)^2 5041 \approx 55fm^2 = 550mb$



Cross sections from a semi-classical view: *Low-E collision*

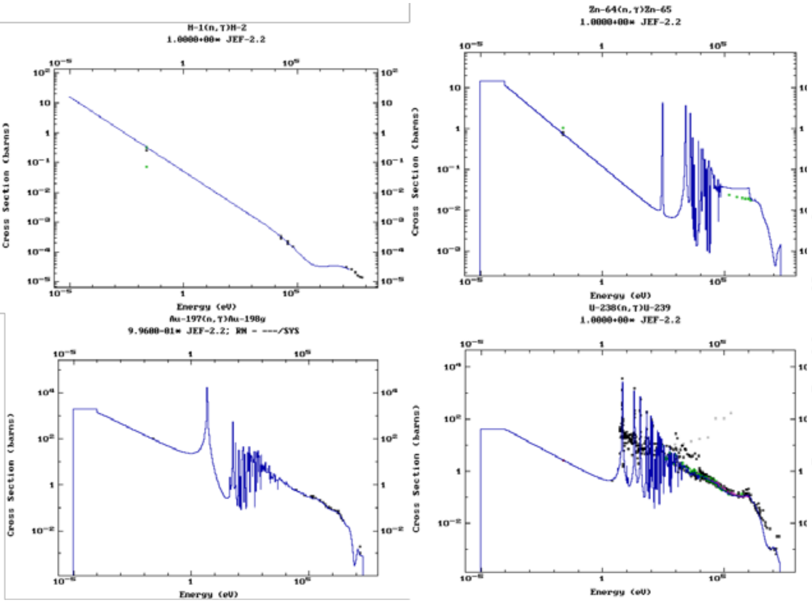
- $\sigma_{total} = \pi\left(\frac{\lambda}{2\pi}\right)^2 \sum_l\ (2l+1)\ T_l$ where $0 \le T_l \le 1$
- Low-energy limiting case for T_l:

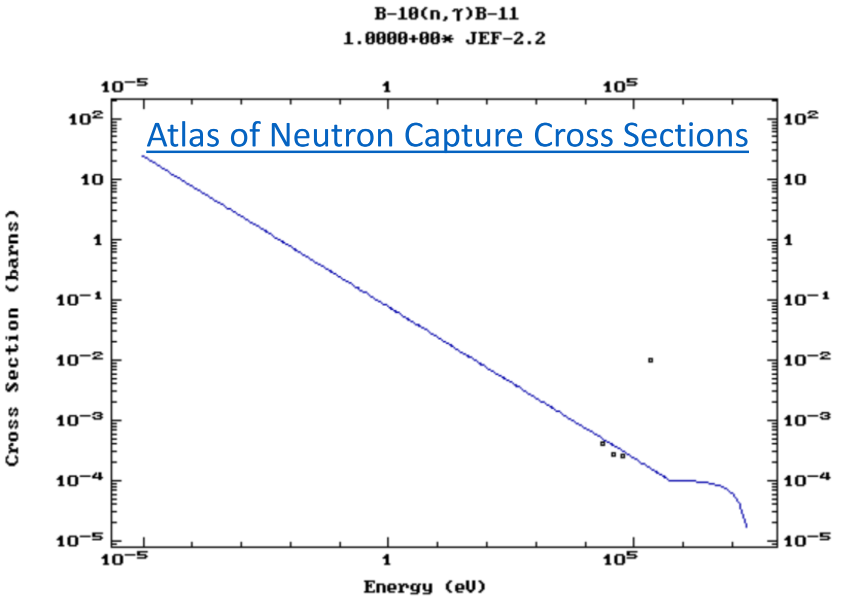
  For relatively low energy (e.g. a <keV beam), $T_l \propto \begin{matrix} \sqrt{KE_p}\ \ for\ l = 0 \\ 0\ \ for\ l > 0 \end{matrix}$

The basic reasoning for T~SQRT√(KE)~v is that lower velocities mean the neutron will spend more time in proximity to the target.
For l>0, there isn't enough energy to tunnel through the centrifugal barrier.

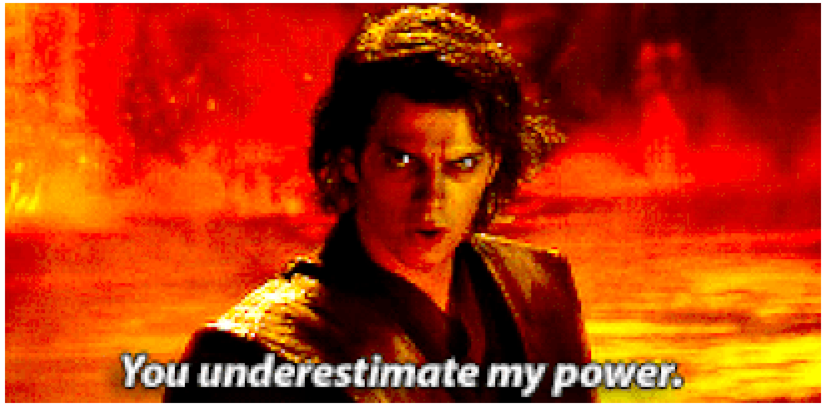
- Consider a neutron at low energy,

$$\sigma_{total} \propto \pi\left(\frac{\lambda}{2\pi}\right)^2 \sqrt{KE_n} \propto \pi\left(\frac{h}{2\pi\sqrt{2m_n KE_n}}\right)^2 \sqrt{KE_n} \propto \pi\left(\frac{h^2}{4\pi^2(m_n^2 v_n^2)}\right)\sqrt{\frac{1}{2}m_n v_n} \propto \frac{1}{v_n}$$

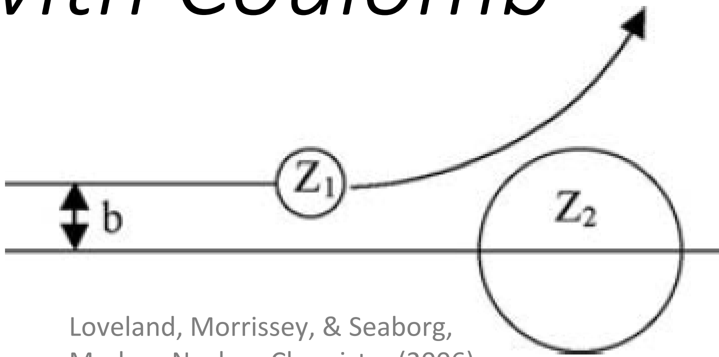




[Atlas of Neutron Capture Cross Sections]



Cross sections from a semi-classical view: *with Coulomb*



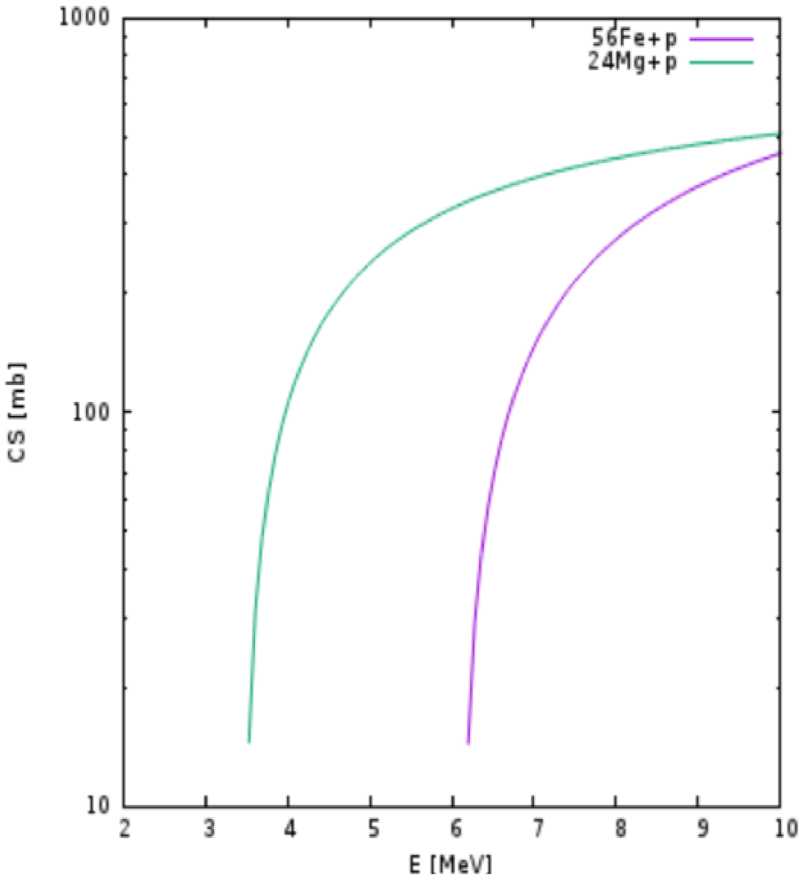
Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

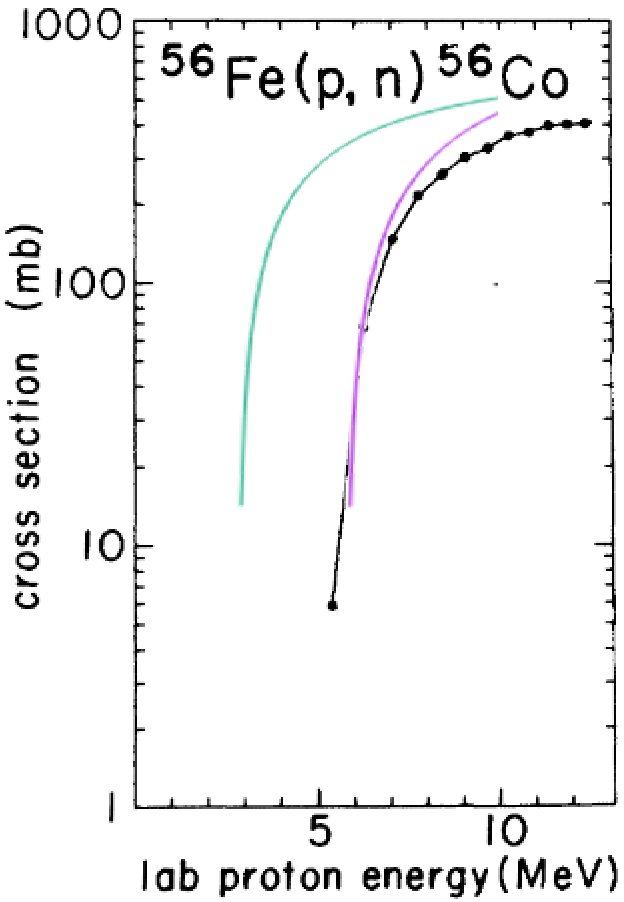
- Let's take a step back and now include Coulomb effects
- For a charged projectile, there is a Coulomb barrier between the projectile and the target:
  $V_C = \frac{Z_p Z_t e^2}{R} = \frac{Z_p Z_t}{r_0\left(A_p^{1/3}+A_t^{1/3}\right)}\frac{e^2}{\hbar c}\hbar c \approx 1.44\frac{Z_p Z_t}{r_0\left(A_p^{1/3}+A_t^{1/3}\right)} MeV$
- At closest approach ($r = R$), the initial center of mass energy ε is reduced by,
  $\varepsilon' = \varepsilon - V_C$, so the momentum is $p = \sqrt{2\mu\,(\varepsilon - V_C)} = \sqrt{2\mu\varepsilon}\sqrt{1-\frac{V_C}{\varepsilon}}$
- The orbital angular momentum is then $\left|\vec{l}\right| = l\hbar = |\vec{r}\times\vec{p}| = R\sqrt{2\mu\varepsilon}\sqrt{1-\frac{V_C}{\varepsilon}}$,
  $\sigma_{tot} = \pi(l_{max}+1)^2\left(\frac{\lambda}{2\pi}\right)^2 \approx \pi l_{max}^2\left(\frac{\lambda}{2\pi}\right)^2 = \pi R^2\left(\frac{2\mu\varepsilon}{\hbar^2}\right)\left(1-\frac{V_C}{\varepsilon}\right)\left(\frac{\lambda}{2\pi}\right)^2 = \pi R^2\left(1-\frac{V_C}{\varepsilon}\right)$

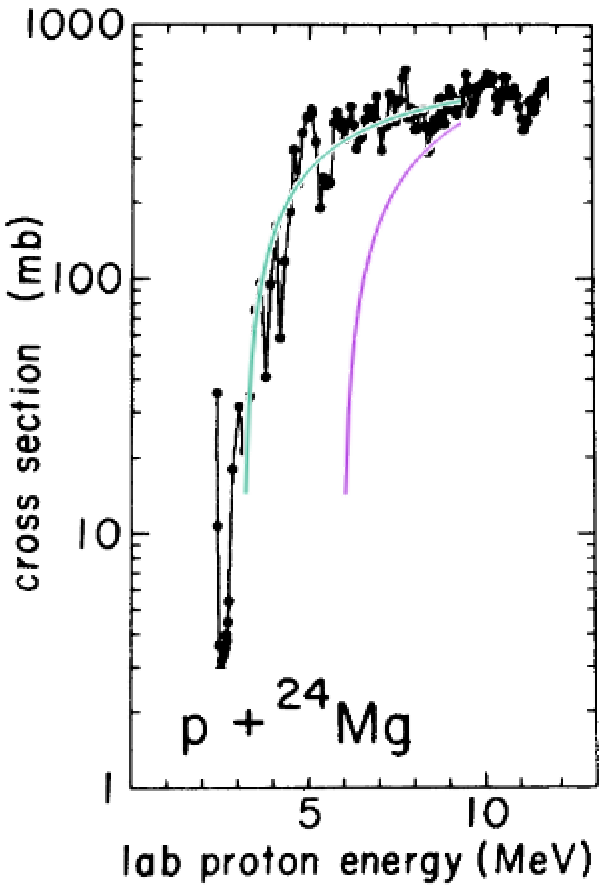
Cross sections from a semi-classical view: *with Coulomb*

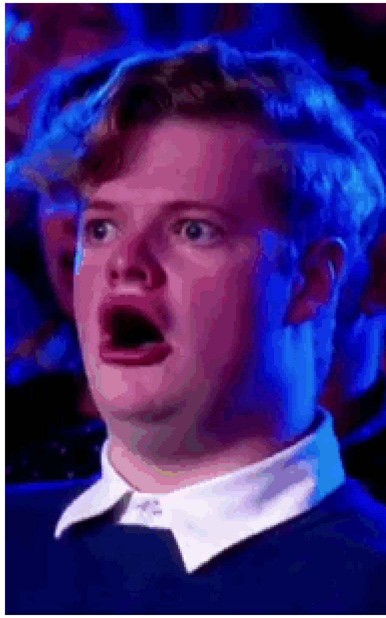
- The orbital angular momentum is then $\left|\vec{l}\right| = l\hbar = |\vec{r} \times \vec{p}| = R\sqrt{2\mu\varepsilon}\sqrt{1 - \frac{V_C}{\varepsilon}}$,

$$\sigma_{tot} = \pi(l_{max} + 1)^2\left(\frac{\lambda}{2\pi}\right)^2 \approx \pi l_{max}^2\left(\frac{\lambda}{2\pi}\right)^2 = \pi R^2\left(\frac{2\mu\varepsilon}{\hbar^2}\right)\left(1 - \frac{V_C}{\varepsilon}\right)\left(\frac{\lambda}{2\pi}\right)^2 = \pi R^2\left(1 - \frac{V_C}{\varepsilon}\right)$$



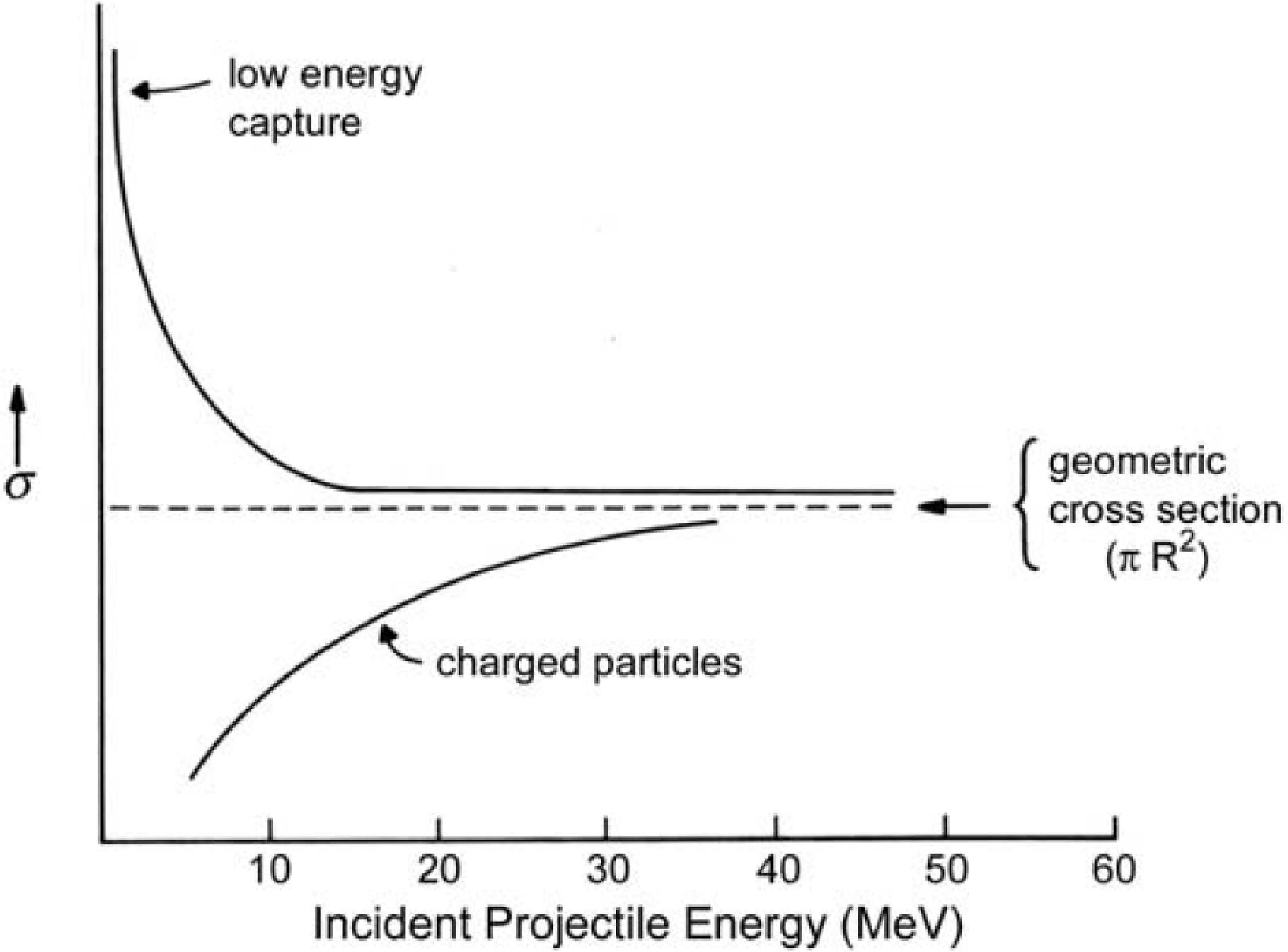






P. Dyer et al. PRC 1981

Cross sections from a semi-classical view: *Near Threshold*

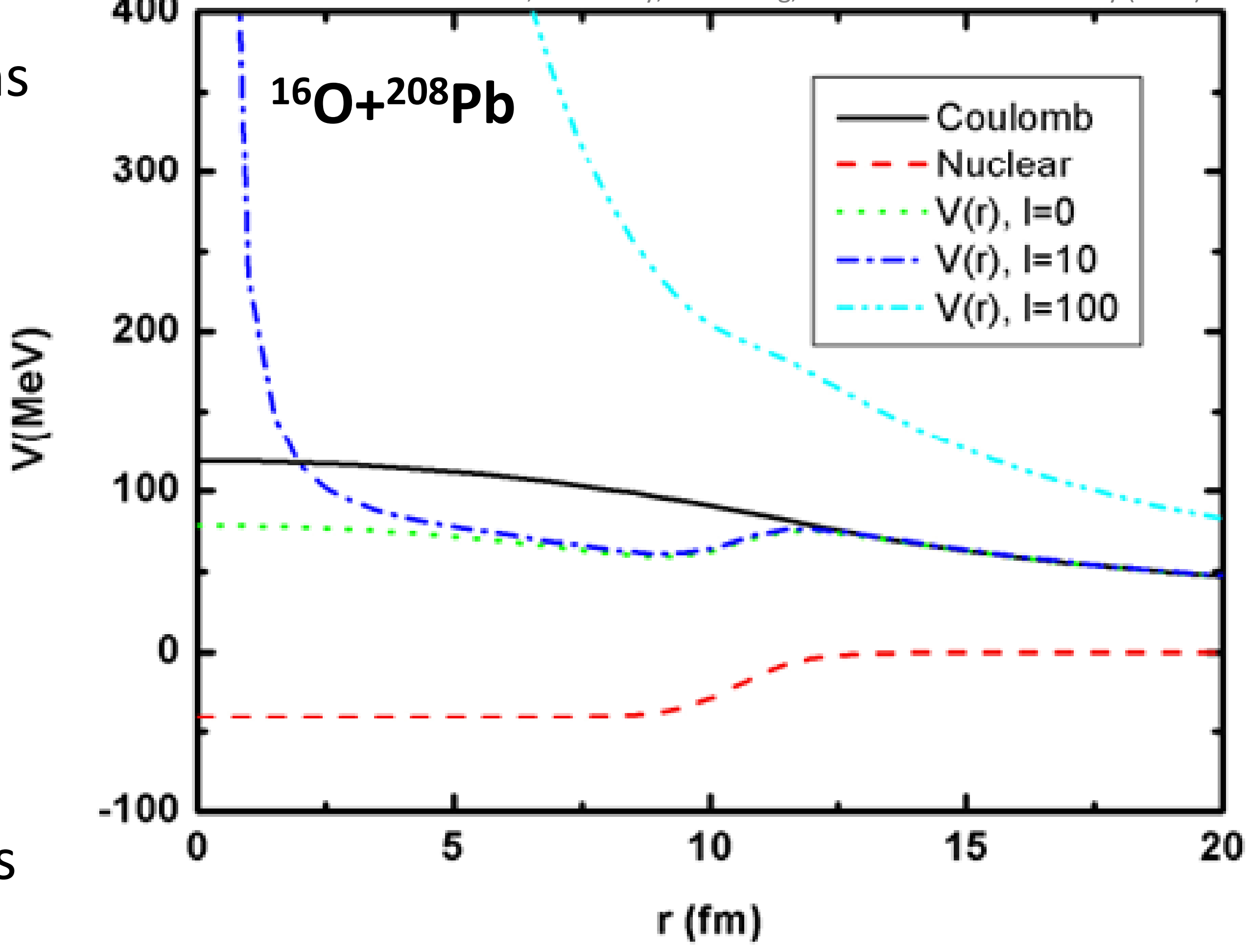


Ehmann & Vance, Radiochemistry and Nuclear Methods of Analysis (1991)

Cross Sections & Barriers

- For a more accurate prescription, just taking into account the Coulomb barrier won't do
- The full barrier has, nuclear, Coulomb, and centrifugal contributions
  - Coulomb: $V_C = \begin{cases} \frac{Z_p Z_t e^2}{r} & r > R \\ \frac{Z_p Z_t e^2}{R}\left(\frac{3}{2} - \frac{1}{2}\left(\frac{r^2}{R^2}\right)\right) & r < R \end{cases}$
  - Nuclear: $V_{WS} = \frac{V_0}{1+\exp([r-R]/a)}$
  - Centrifugal: $\frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}$
- Their sum is the "interaction barrier"
- You could find the distance of closest approach for this barrier and then play the same game as we did for the Coulomb barrier

Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)



Terminology *…because we have to feel special somehow!*

- The total interaction probability is described by the *cross section*
- The cross section as a function of projectile energy is called an *excitation function* (though I refuse to use this)
- The probability of an interaction ejecting an outgoing particle at a given angle is referred to as the *differential cross section*.
  - Rather than use the logical $\sigma(\theta)$, this is often instead written as $\frac{d\sigma}{d\Omega}$ or $\frac{d\sigma}{d\Omega}(\theta)$
- The probability of an interaction ejecting an outgoing particle at a given angle with a given energy is referred to as the *double differential cross section*
  - Rather than use the logical $\sigma(\theta)|_E$, this is often instead written as $\frac{d^2\sigma}{d\Omega dE}$
- The probability of an interaction ejecting an outgoing particle at a given energy is just described by the *particle spectrum*, because the good guys have to win sometimes

Further Reading

- Chapter 10: Modern Nuclear Chemistry (Loveland, Morrissey, Seaborg)
- Chapter 11: Introductory Nuclear Physics (K.S. Krane)
- Chapter 17: Introduction to Special Relativity, Quantum Mechanics, and Nuclear Physics for Nuclear Engineers (A. Bielajew)
- Chapters 1 & 2: Nuclear Reactions (H. Schieck)
- Chapters 3 & 4: Cauldrons in the Cosmos (Rolfs & Rodney)