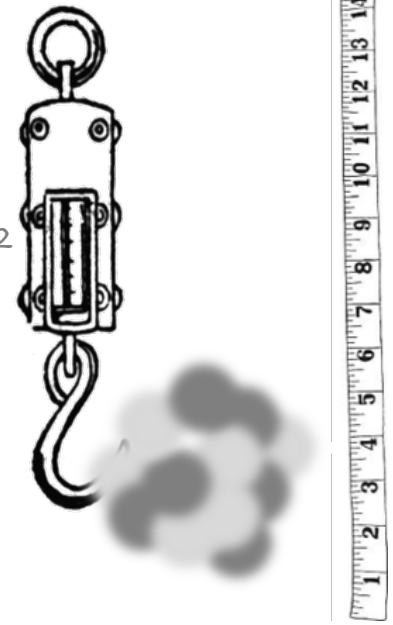
Lecture 12: Review of Structure & Decay

- Highlights from Lectures 1-10 of PHYS7501
- ***This doesn't necessarily cover everything that will be asked on the midterm, but it encompasses most of the main take-aways***

How big is a nucleus?

The definition of u being based on ^{12}C means it is a valuable tool for high-precision mass measurements, e.g. C.Scheidenberger et al., Nuc. Phys. A 2002



Phenomenological estimates:

- A nucleus's mass is roughly: M(Z,A) = A*amu
  - 1amu = atomic mass unit = u = 931.494MeV/c^2 ≈1.66x10^{-24}g
  - The amu is defined such that M(^{12}C) ≡ 12u
- A nucleus's $_{(charge)}$ radius is roughly: R(Z,A) = (1.2fm)*$A^{1/3}$
  - fm = femtometer (a.k.a. *fermi*) = 10^{-15}m
  - The radius of a nucleon is often referred to as r_0=1.2fm
  - For the RMS radius, multiply by $\sqrt{3/5}$
- Therefore, an estimate for the nuclear density is:
  - $\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{(1u)A}{\frac{4}{3}\pi(1.2fm)^3 A} \approx 0.14\ nucleons/fm^3$
  - For fun, in terms of mass-density: $\rho = \frac{(1.66\times10^{-24}g)A}{\frac{4}{3}\pi(1.2fm)^3 A} \approx 2.3 \times \frac{10^{-25}g}{fm^3}$

…which doesn't sound like much, but this is 2x10^{14} g/cm^3 (the Great Pyramid of Giza is only ~10^{12} grams)

Since A cancels in the ρ expression, the nuclear density is independent of the nuclear size, much as a liquid's density is independent of the size of the drop.
Partly inspired by this property, some basic nuclear calculations are based on this liquid drop analogy.
(G. Gamow, Proc. Roy. Soc. A 1929,1930)

Nuclear Transmutation

- Rules for converting one nuclide (or nuclides) to another (or others)
  - *Charge conservation*: $\sum q_{before} = \sum q_{after}$ (q from protons + positrons + electrons)
  - *Baryon conservation*: $\sum A_{before} = \sum A_{after}$ (A from neutrons + protons)
  - *Lepton number conservation*: $[N_{leptons} - N_{anti-leptons}]_{before} = [N_{leptons} - N_{anti-leptons}]_{after}$

  * Transmutation likelihoods are impacted by energetics and spin/parity selection rules

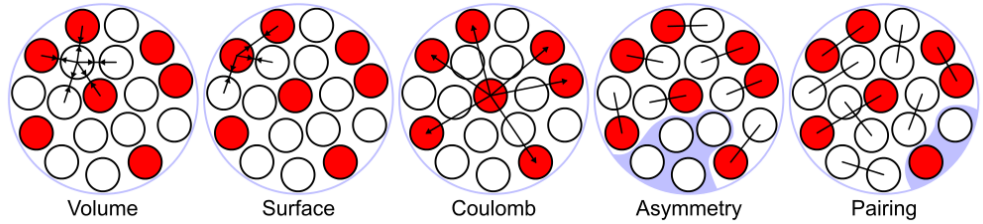
The first nuclear reaction intentionally made in the laboratory was $^{14}N(\alpha,p)$ in 1919. (E. Rutherford, Nature 1935).

The first measured radioactive decay was α decay from uranium. (H. Becquerel, Comptes Rendus 1896).

Two Types:

- Reactions
  - Multiple reactants create one or more products
  - Notation: A+b → c+D is written as *A(b,c)D*, where M(b)<M(A) and M(c)<M(D)
    - E.g. $^{12}C+\alpha \rightarrow {}^{16}O+\gamma$ is $^{12}C(\alpha,\gamma)^{16}O$ or even just $^{12}C(\alpha,\gamma)$ and is called "carbon-twelve alpha gamma"
- Decays
  - α, β^+, β^-, e^--capture, β-delayed γ/p/α/n emission, fission, cluster emission, prompt γ
  - Lose nucleons for all above except β decay, e^--capture, and prompt γ (following a reaction)

The Semi-Empirical Mass Formula



- BE(Z,A) = Volume - Surface - Coulomb - Asymmetry ± Pairing
- One mathematical parameterization* (*of many!*):

*from B. Martin, Nuclear and Particle Physics (2009)

  - $BE(Z,A) = a_v f_v(A) - a_s f_s(A) - a_c f_c(Z,A) - a_a f_a(Z,A) + i a_p f_p(A)$
    - **Volume**: Nucleons cohesively bind, so: $\boldsymbol{f_v(A) = A}$
    - **Surface**: Since radius goes as $R \propto A^{1/3}$ and surface area goes as $SA \propto R^2$, $\boldsymbol{f_s(A) = A^{2/3}}$
    - **Coulomb**: Energy for a charged sphere goes as $\frac{q^2}{R}$ and $R \propto A^{1/3}$, so $\boldsymbol{f_c(Z,A) = \frac{Z(Z-1)}{A^{1/3}}}$
    - **Asymmetry**: Z=N favored (want Z=A/2) but lesser problem for large A, so $\boldsymbol{f_a(Z,A) = \frac{\left(Z-\frac{A}{2}\right)^2}{A}}$
    - **Pairing**: Favor spin-0 nucleon pairs & disfavor unpaired nucleons, empirically $\boldsymbol{f_p(A) = \left(\sqrt{A}\right)^{-1}}$
      - Even-Z, Even-N: $\boldsymbol{i = +1}$
      - Odd-Z, Odd-N: $\boldsymbol{i = -1}$
      - Even-Odd: $\boldsymbol{i = 0}$
    - a_i are fit to data

A mnemonic for remembering SEMF contributions is "VSCAP".

Nuclear Mass Differences

- The energy released in a nuclear reaction is the "Q-value"
  - $Q = \sum_{reactants} ME(Z,A) - \sum_{products} ME(Z,A)$,
  - For example, $Q_{^{68}Se(p,\gamma)^{69}Br} = ME(^{68}Se) + ME(p) - ME(^{69}Br)$
  - $= (-54.189MeV) + (7.288MeV) - (-46.260MeV)$
  - $= -0.641MeV$
- Considering the case above, we calculated the energy released by adding one proton to ^{68}Se, which corresponds to the energy it takes to remove one proton from ^{69}Br, a.k.a. the "proton separation energy", S_p
- Similarly, can calculate the energy to remove 1-neutron S_n, two-protons S_{2p}, or two-neutrons S_{2n}
  - $S_p(Z,N) = ME(Z-1,N) + ME(p) - ME(Z,N)$
  - $S_n(Z,N) = ME(Z,N-1) + ME(n) - ME(Z,N)$
  - $S_{2p}(Z,N) = ME(Z-2,N) + 2 * ME(p) - ME(Z,N)$
  - $S_{2n}(Z,N) = ME(Z,N-2) + 2 * ME(n) - ME(Z,N)$

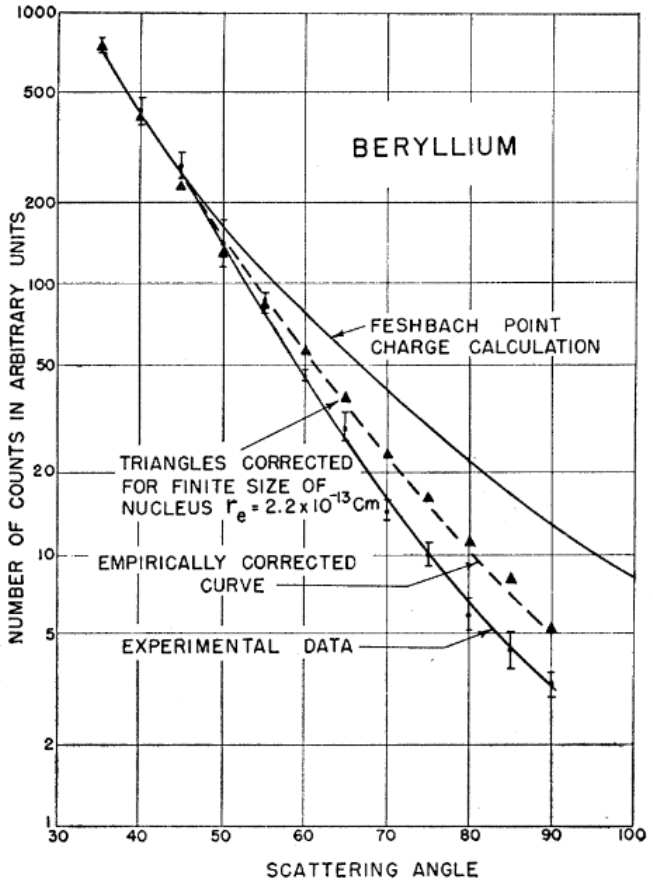
Nuclear charge distribution

Review: R. Hofstadter, Rev. Mod. Phys. (1956)

- Comparing measured $\sigma_{scatt}(\theta)$ to calculations w/ various $F(q)$ reveals a Fermi-like distribution

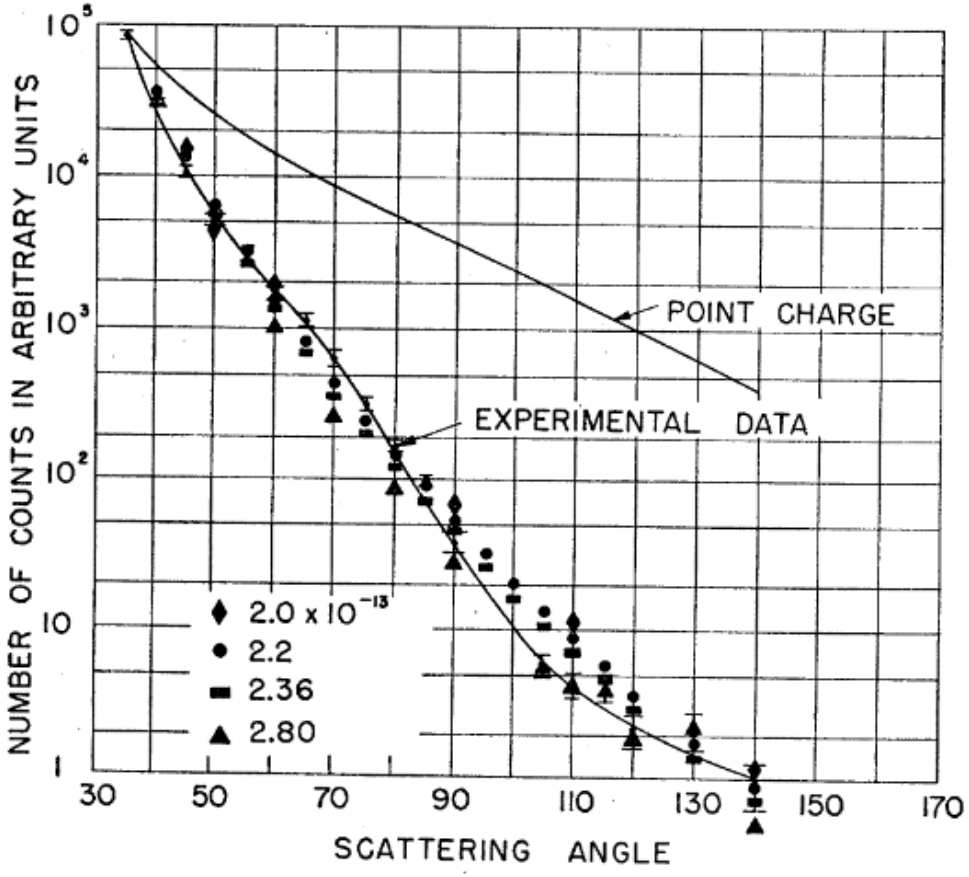
(R. Woods & D. Saxon, Phys. Rev. (1954))

125MeV e⁻ on Be

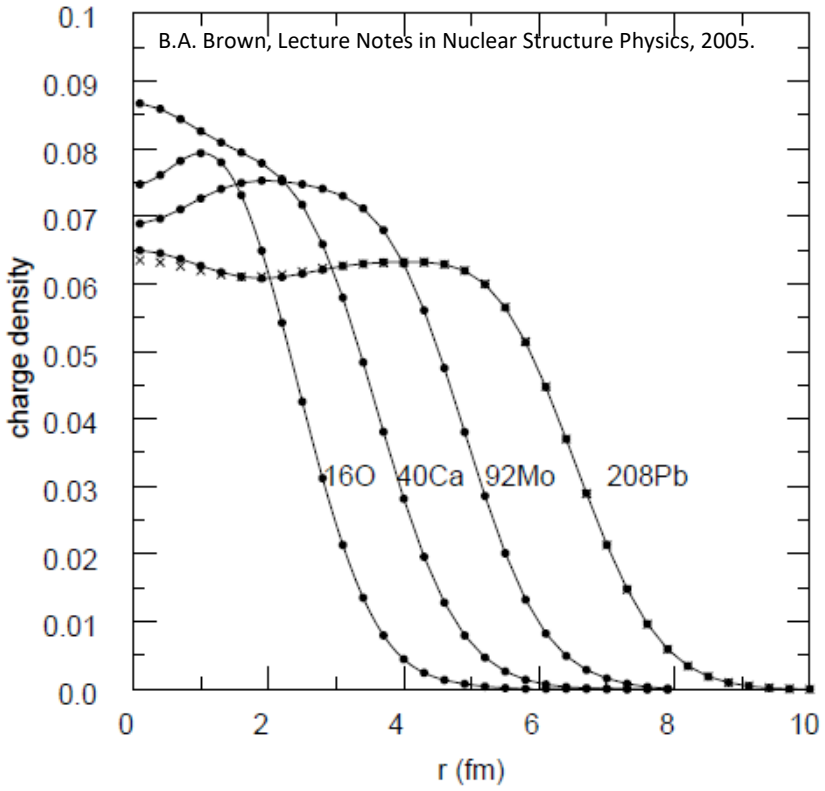


125MeV e⁻ on Au

R. Hofstadter, H. Fechter, & J. McIntyre, Phys. Rev. (1953)



Measured ρ_{ch} examples



Electric & Magnetic Moments

In general, a moment is a distance multiplied by a physical quantity. For distributions you integrate the quantity's distribution with respect to distance.

- The nuclear *magnetic* moments describe the distribution of electric currents in the nucleus
  - Causes nuclei to align along an external magnetic field, which can be exploited using NMR, MRI, etc.

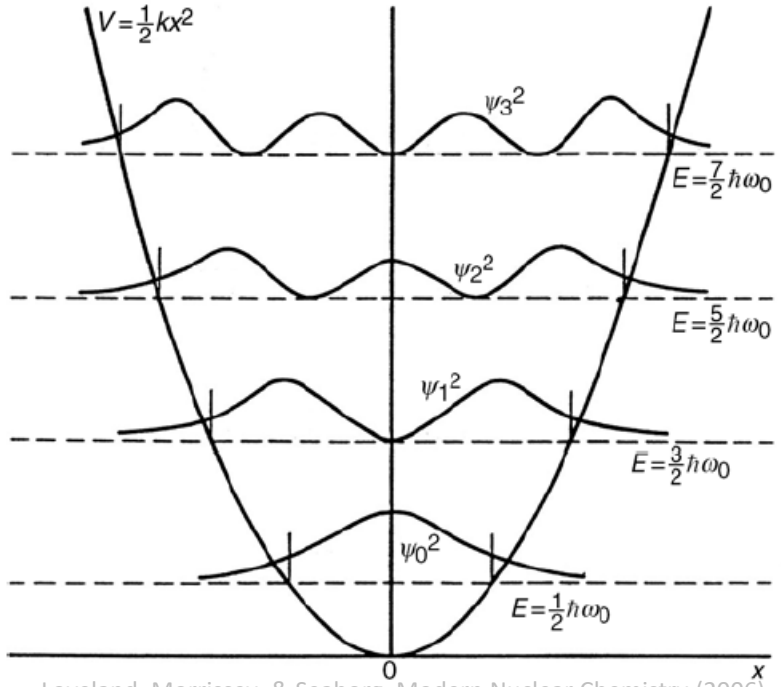
- The nuclear *electric* moments describe the distribution of electric charges in the nucleus
  - Used as a measure of the nuclear shape

First stab at the potential, V: The Harmonic Oscillator

- Based on some evidence (and logic) that nuclei aren't perfectly constant in density, Heisenberg (Z. Phys. 1935) posited that a parabolic potential could be assumed, conveniently allowing the adoption of the harmonic oscillator solutions
  (one of the few analytically solved systems!)
- This provides evenly spaced energy levels n, with $E_n = \left(n + \frac{1}{2}\right)h\upsilon$.
- The corresponding angular momenta are $l = n - 1, n - 3, \ldots \geq 0$.

i.e. only odd or even functions are allowed for each oscillator shell

- The number of particles per angular momentum is $2(2l + 1)$ for $2l + 1$ projections & 2 spins
- So, the number of particles per level is:



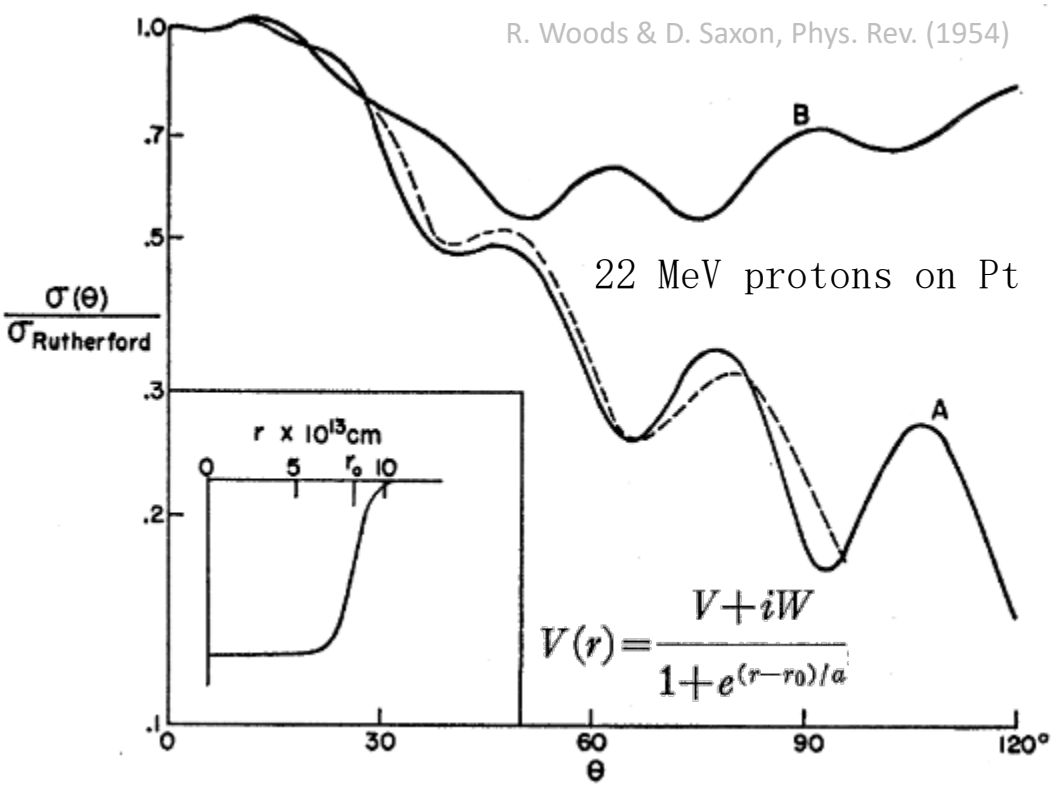
Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

n	l	# per level	Cumulative
1	0	2(2*0+1) = 2	2
2	1	2(2*1+1) = 6	8
3	0,2	2(2*0+1) = 2 + 2*(2*2+1) = 10 = 12	20
4	1,3	2*(2*1+1) = 6 + 2*(2*3+1) = 14 = 20	40
5	0,2,4	2*(2*0+1) = 2 + 2*(2*2+1) = 10 + 2*(2*4+1) = 18 = 30	70

Could the HO potential still be useful for some cases?
...can get the job done for light nuclei (e.g. H. Guo et al. PRC 2017)
...but need to be careful, because can impact results (B.Kay et al arXiv 2017)

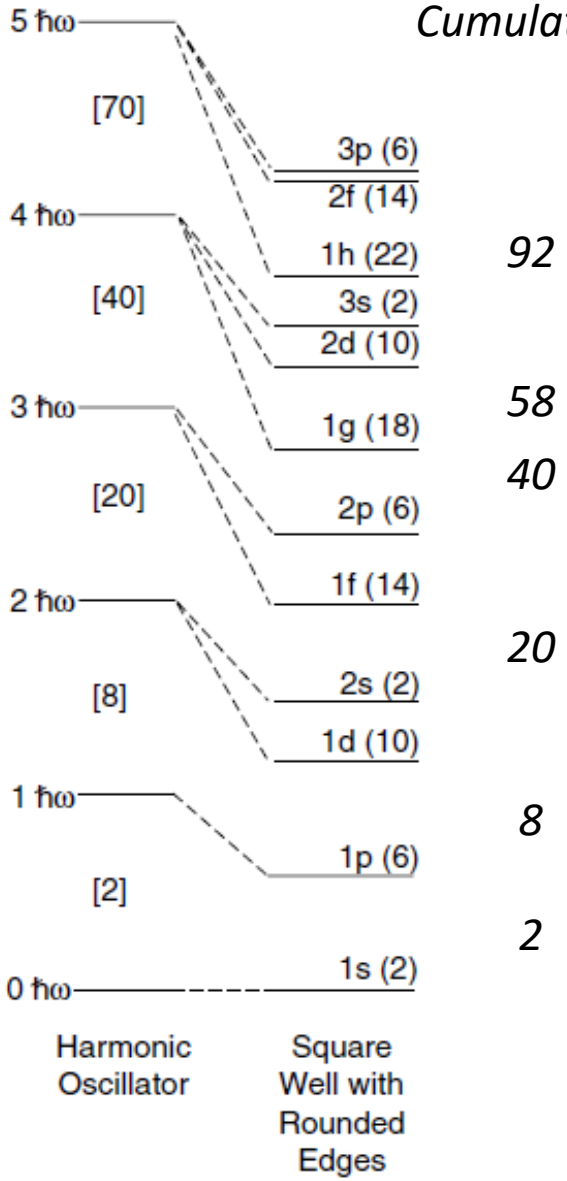
Move to an empirical potential: **Woods-Saxon**

- Since the nuclear interaction is short-range, a natural improvement would be to adopt a central potential mimicking the empirical density distribution
- This is basically a square well with soft edges, as described by the Woods-Saxon potential:



- Using the Woods-Saxon is a good idea because of commitment to reality… but we're no wiser as to the origin of the magic numbers

Was this step completely useless? No! It broke the degeneracy in ℓ

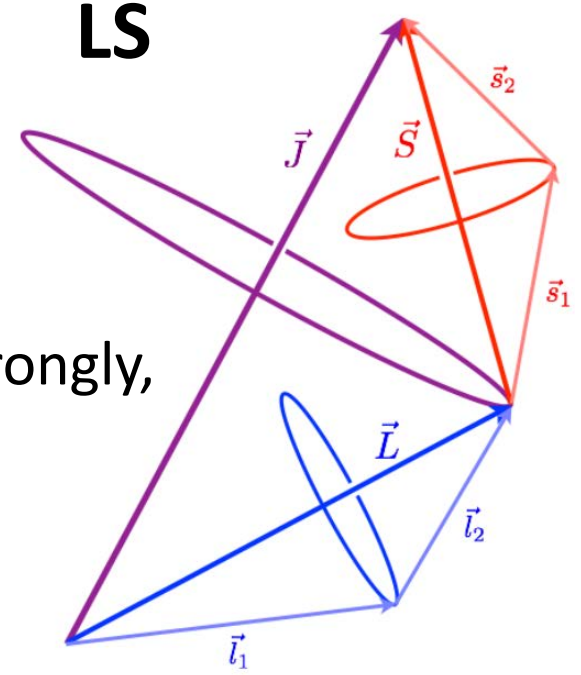


Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

The missing link: **the spin-orbit interaction**

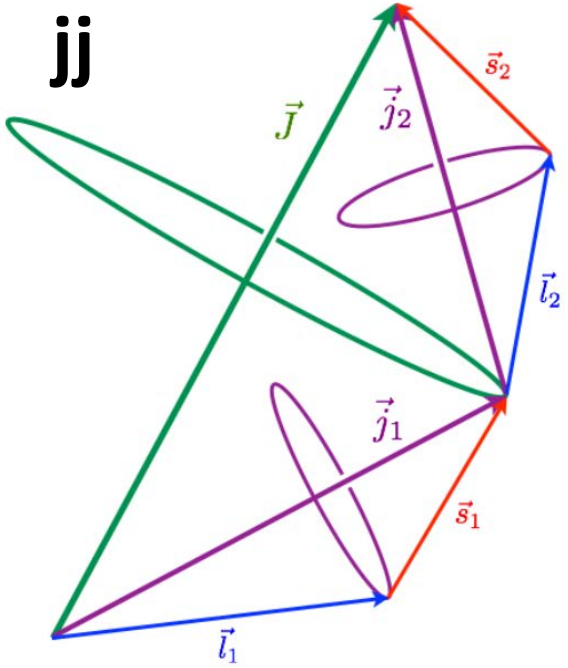
- Due to desperation or genius (or both) Maria Göppert-Mayer
  [Phys. Rev. February 1949] (and nearly simultaneously Haxel, Jensen, & Suess [Phys. Rev. April 1949])
  posited that nucleon spin and orbital angular momentum interacted strongly, making j the good quantum number for a nucleon: $\vec{j} = \vec{l} + \vec{s}$
- Prior to this approach,
  angular momentum was coupled as is typically done for atoms,
  where $\vec{J} = \vec{L} + \vec{S}$, $\vec{L} = \sum_{nucleons} \vec{l}$, and $\vec{S} = \sum_{nucleons} \vec{s}$
  - This is "LS coupling"
- Positing that the spin-orbit interaction is stronger than spin-spin or orbit-orbit means that instead, $\vec{J} = \sum_{nucleons} \vec{j}$ and $\vec{j} = \vec{l} + \vec{s}$
  - This is "jj coupling"

LS



A. Kastberg, Lecture Notes Physique Atomique

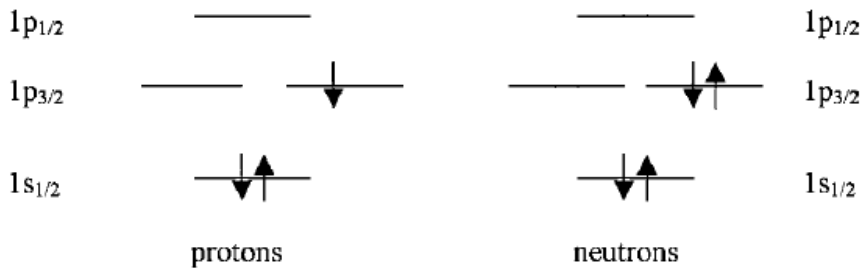
jj



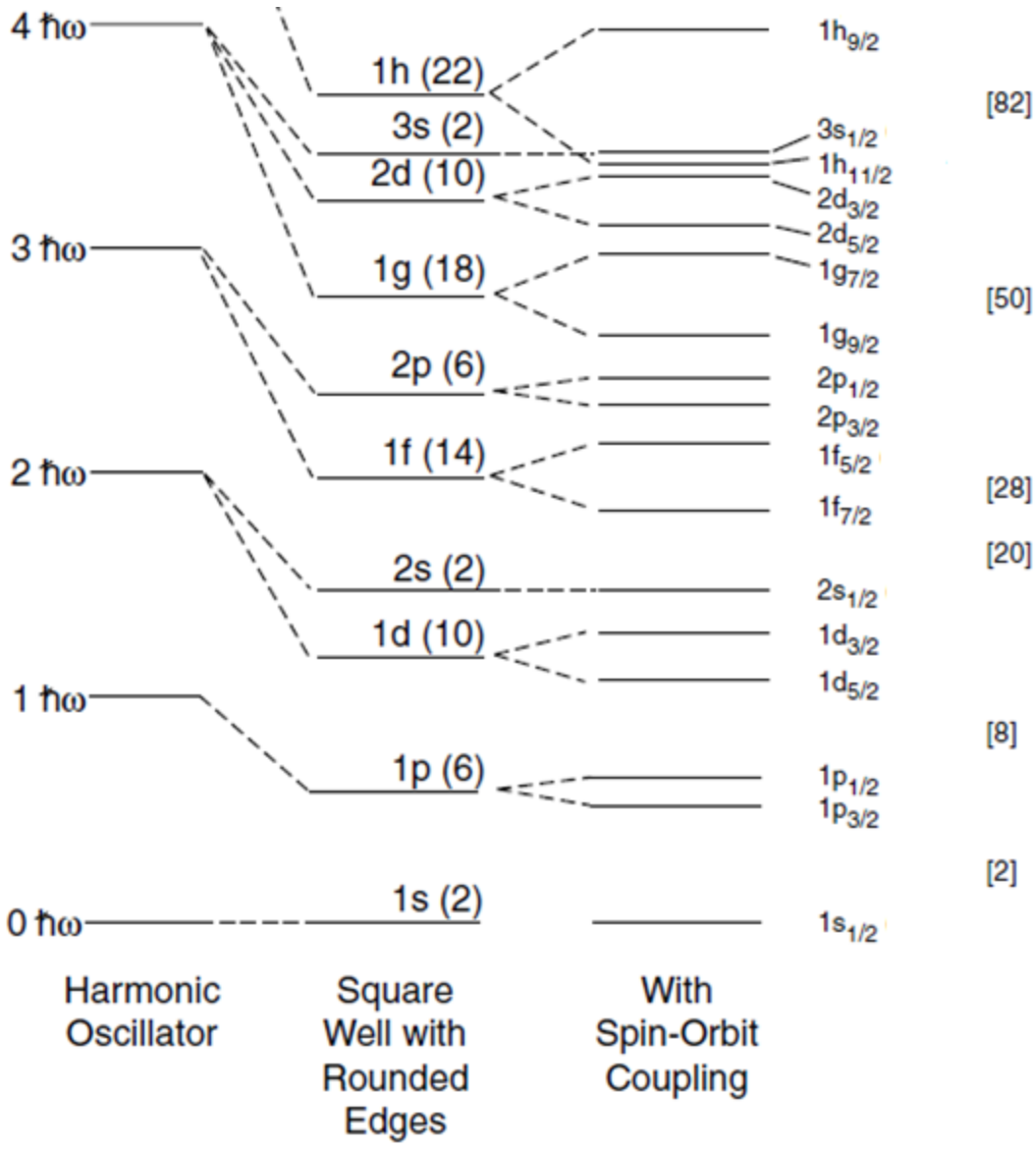
Filling the shells

- We can construct a nucleus using our "shell model":
  - A nucleon will go in the lowest-energy level which isn't already filled, i.e.
    - the largest angular momentum, j
    - for the lowest orbital angular momentum, l
    - for the lowest oscillator shell, n
  - $2j + 1$ protons or neutrons are allowed per level
  - Each level is referred to by its nlj
    - n by the # for the oscillator shell (convention either starts with 0 or 1)
    - l by spectroscopic notation (s=0,p=1,d=2,f=3,…)
    - j by the half-integer corresponding to the spin

- For example: ^{7}Li (Z=3, N=4)

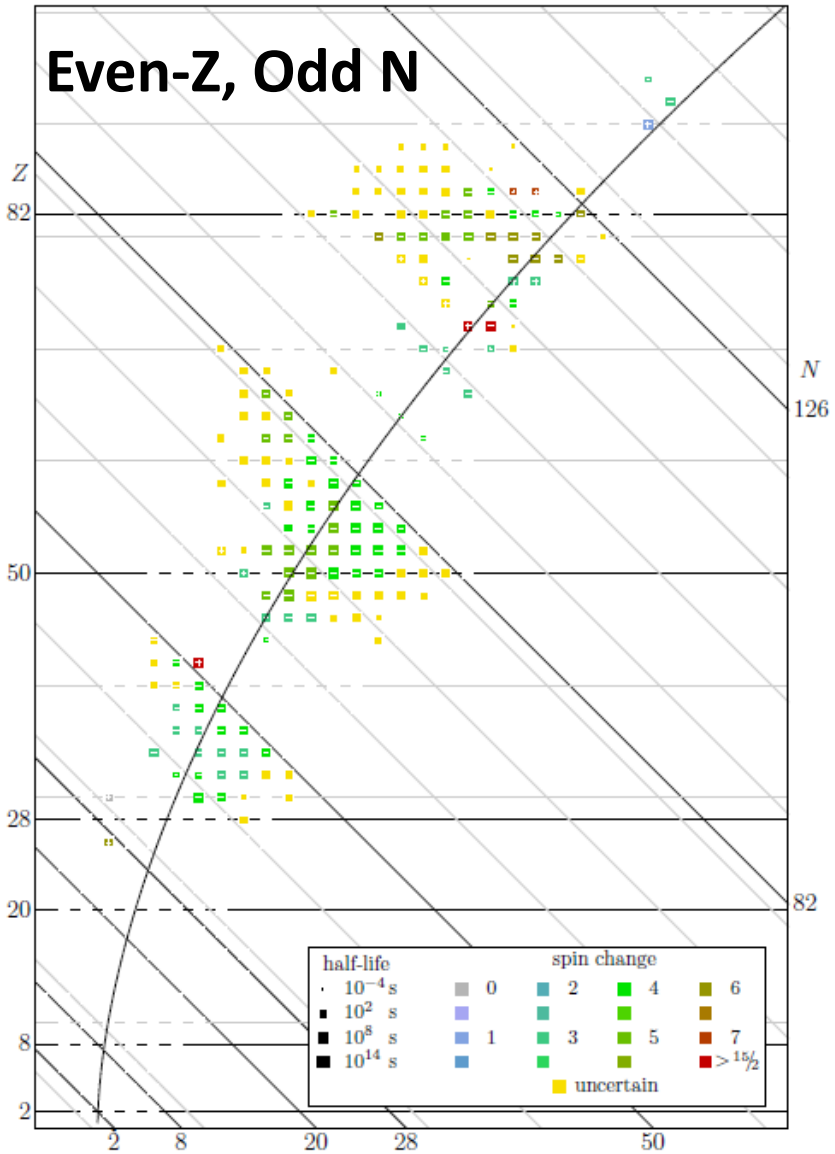


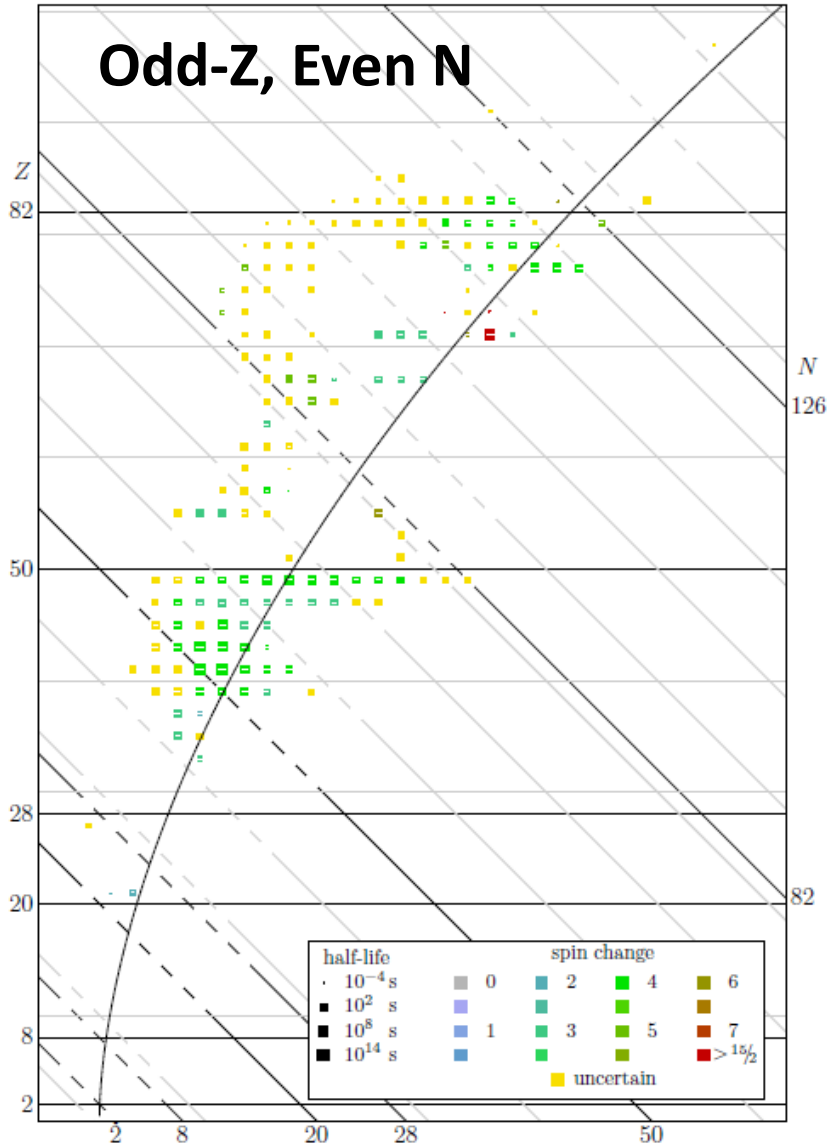
Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

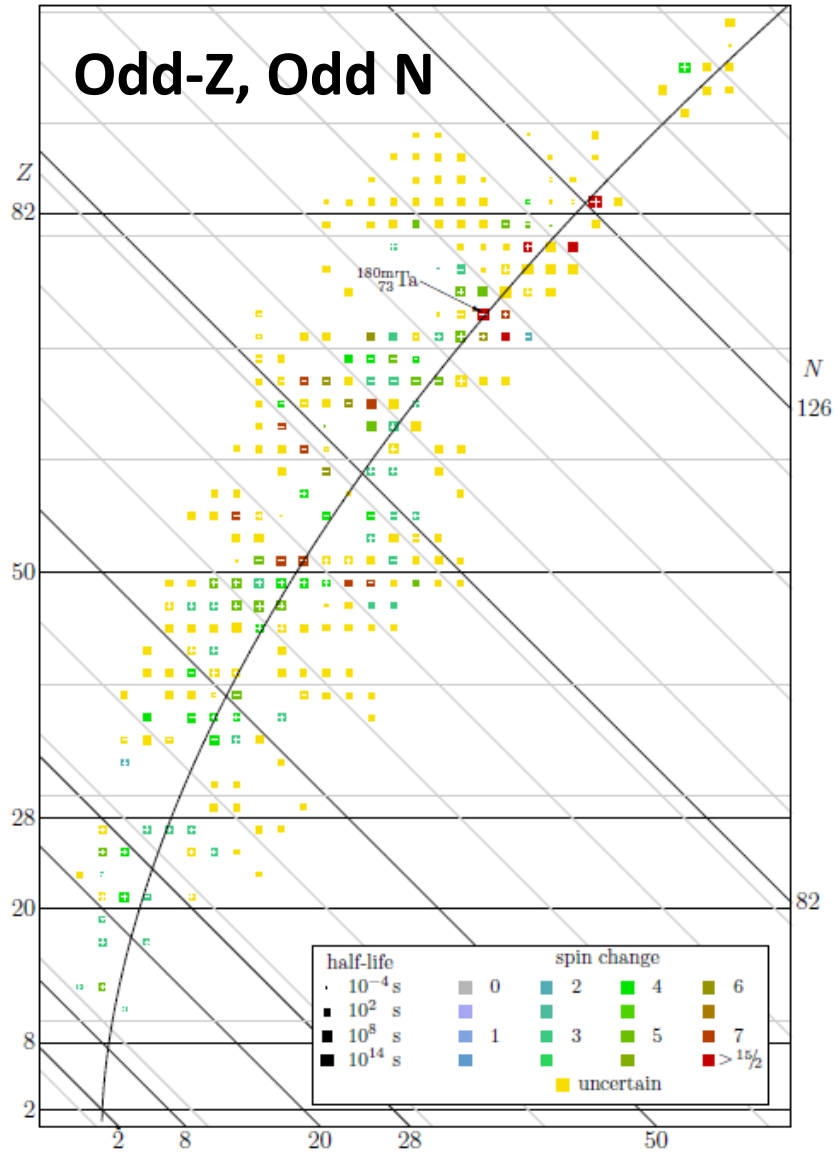


Isomers on the Nuclear Chart

L. van Dommelen, Quantum Mechanics for Engineers (2012)







Special cases exist (mostly for higher-A nuclides) where even-even nuclei have isomers
(e.g. M. Müller-Veggian et al., Z.Phys.A (1979))

Collective Model

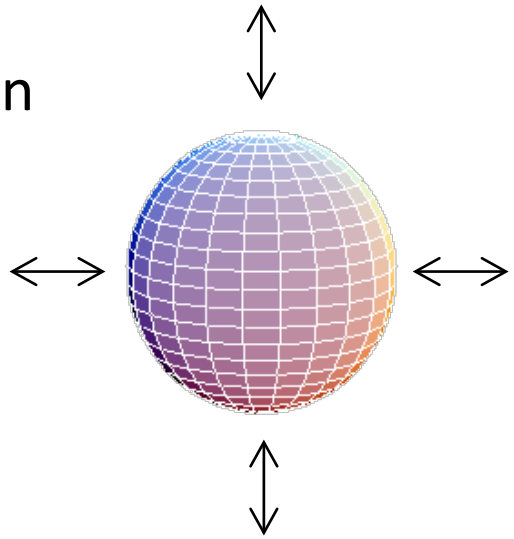
- There are compelling reasons to think that our nucleus isn't a rigid sphere
  - The liquid drop model gives a pretty successful description of some nuclear properties.

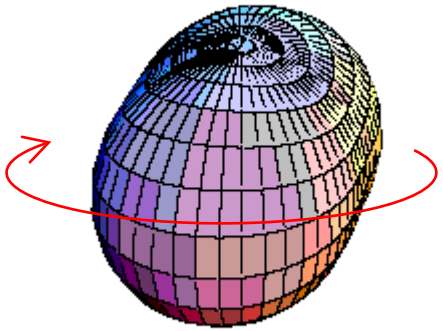
    …can't liquids slosh around?
  - Many nuclei have non-zero electric quadrupole moments (charge distributions)

    …this means there's a non-spherical shape.

    …can't non-spherical things rotate?
- Then, we expect nuclei to be able to be excited rotationally & vibrationally
  - We should (and do) see the signature in the nuclear excited states

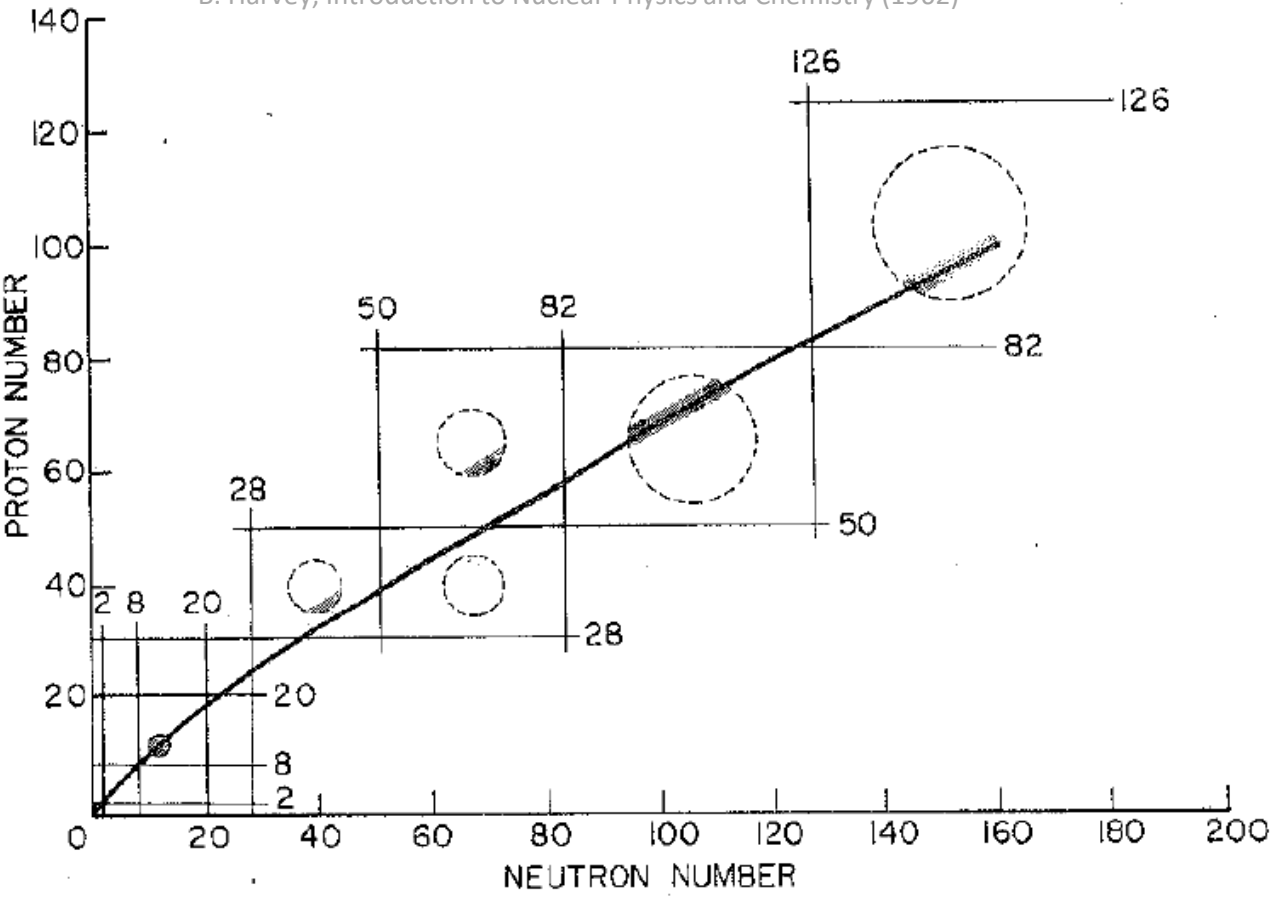
- The relative energetics of rotation vs vibration can be inferred from geometry
  - The rotational frequency should go as $\omega_r \propto \frac{1}{R^2}$ (because $I \equiv \frac{L}{\omega}$ and $I \propto MR^2$)
  - The vibrational frequency should go as $\omega_v \propto \frac{1}{(\Delta R)^2}$ (because it's like an oscillator)
  - So $\omega_r \ll \omega_v$



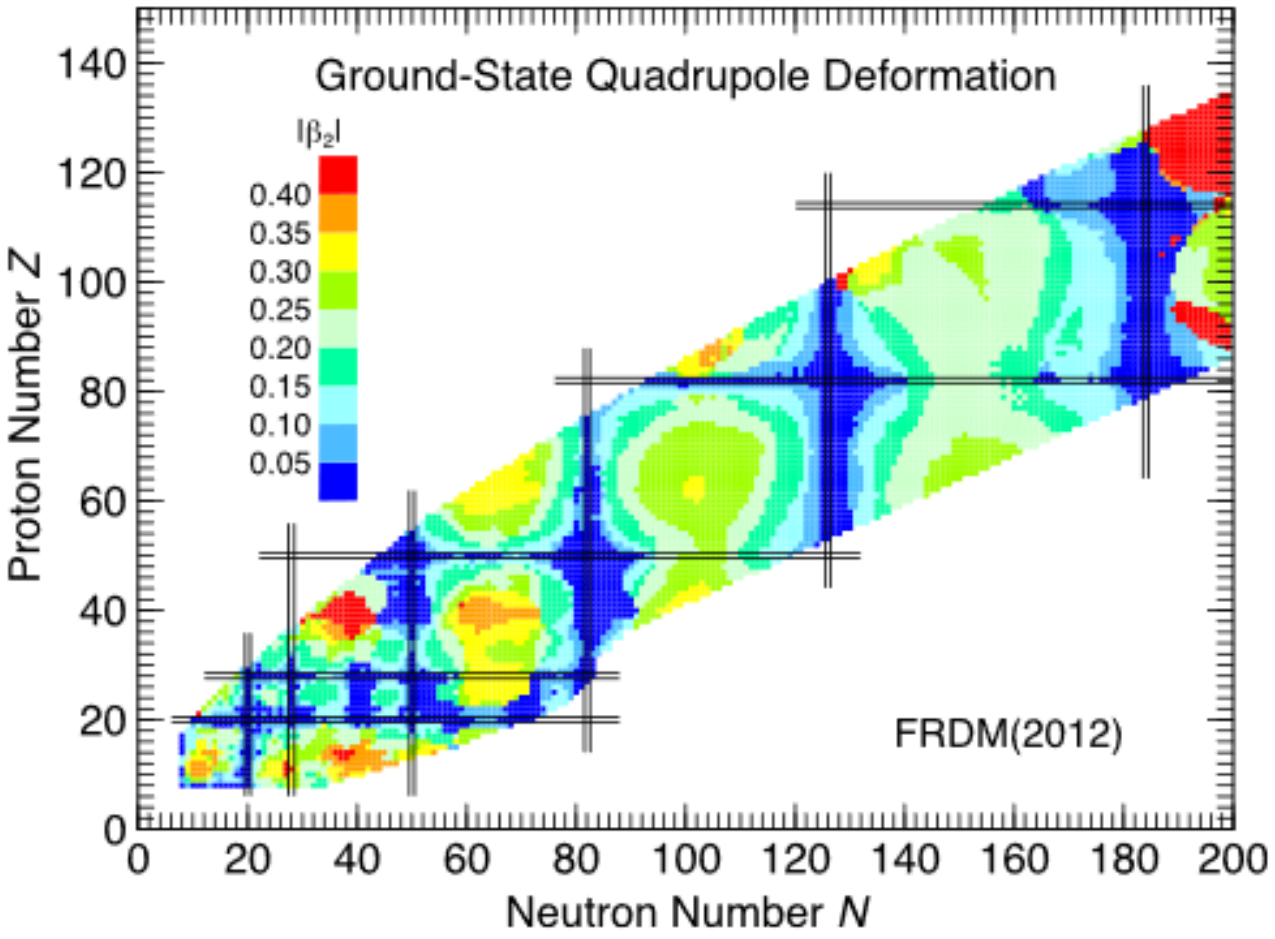


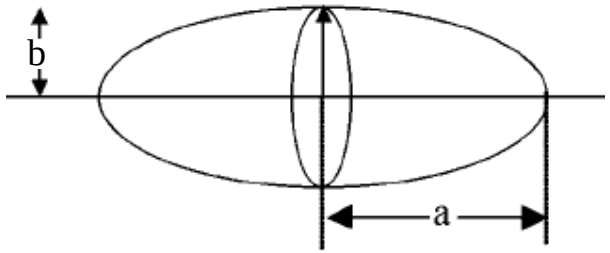
Predicted regions of deformation

B. Harvey, Introduction to Nuclear Physics and Chemistry (1962)



[P. Möller et al. ADNDT (2016)](#)



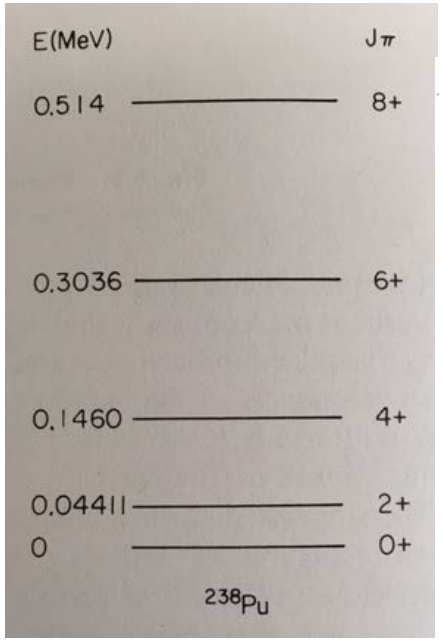


$$\beta_2 \equiv \alpha_{2,0} = \frac{4}{3}\sqrt{\frac{\pi}{5}}\frac{(a-b)}{R_{sph}}$$

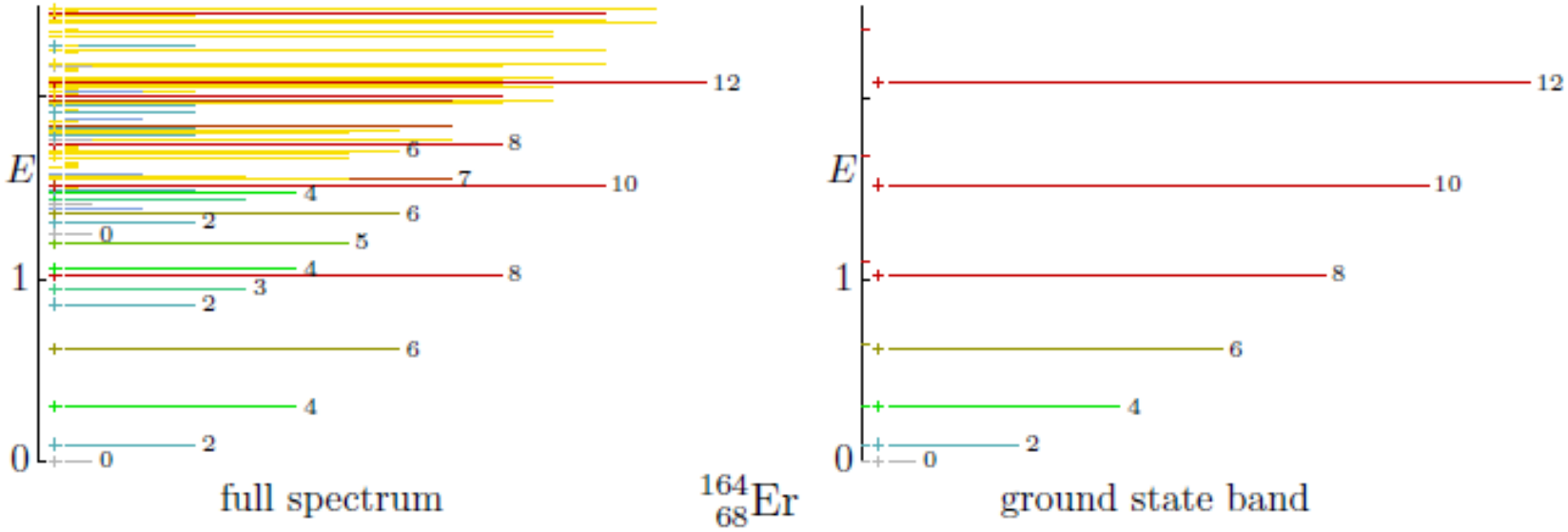
Loveland, Morrissey, Seaborg, Modern Nuclear Chemistry (2006)

Rotational bands: sequences of excited states

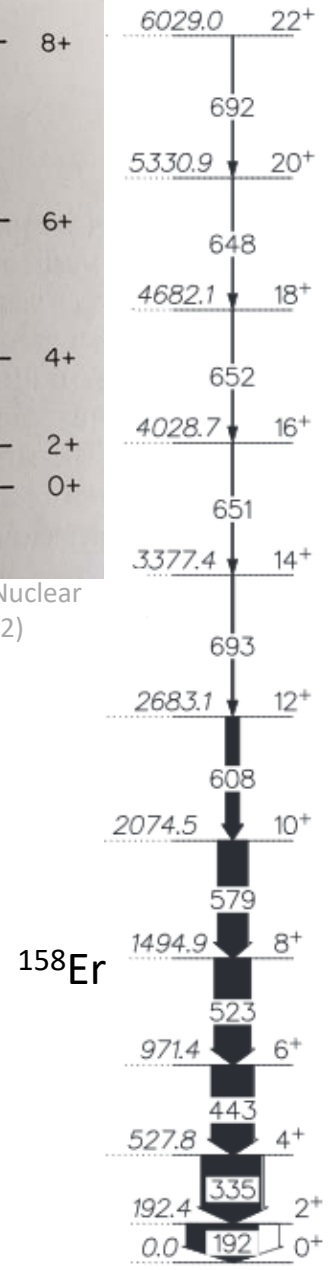
- $E_{rot} = \frac{\hbar^2 j(j+1)}{2I}$, so for a given I, $\Delta E \propto j(j+1)$
- Note that parity needs to be maintained because rotation is symmetric upon reflection and so 0^+ ground-states can only have j=0,2,4,… (because $\pi = (-1)^J$)
- Without observing the decay scheme, picking-out associated rotational states could be pretty difficult
- Experimentally, coincidence measurements allow schemes to be mapped



B. Harvey, Introduction to Nuclear Physics and Chemistry (1962)



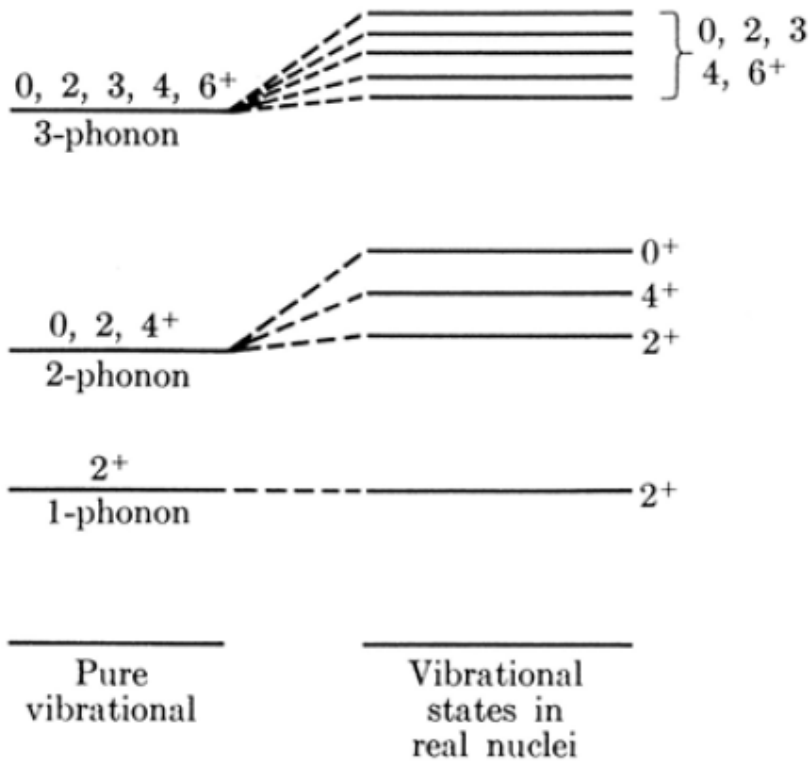
L. van Dommelen, Quantum Mechanics for Engineers (2012)



T.Dinoko et al. EPJ Web Conf. (2013)

Vibrational energy levels

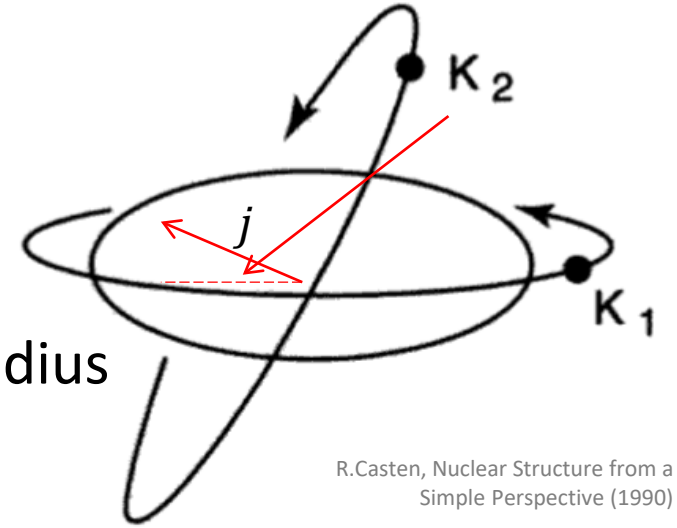
- Just as the quantum harmonic oscillator eigenvalues are quantized, so too will the energy levels for different quanta (*phonons*) of a vibrational mode.
- Similarly, the energy levels have an even spacing, $E_n = (n + \frac{1}{2})\hbar\omega$
- Even-even nuclides have 0+ ground states, and thus, for a $\lambda = 2$ vibration, $n = 2$ excitations will maintain the symmetry of the wave-function (i.e. $n = 1$ excitations would violate parity)
- Therefore, the 1st vibrational state will be 2^+



Loveland, Morrissey, Seaborg, Modern Nuclear Chemistry (2006)

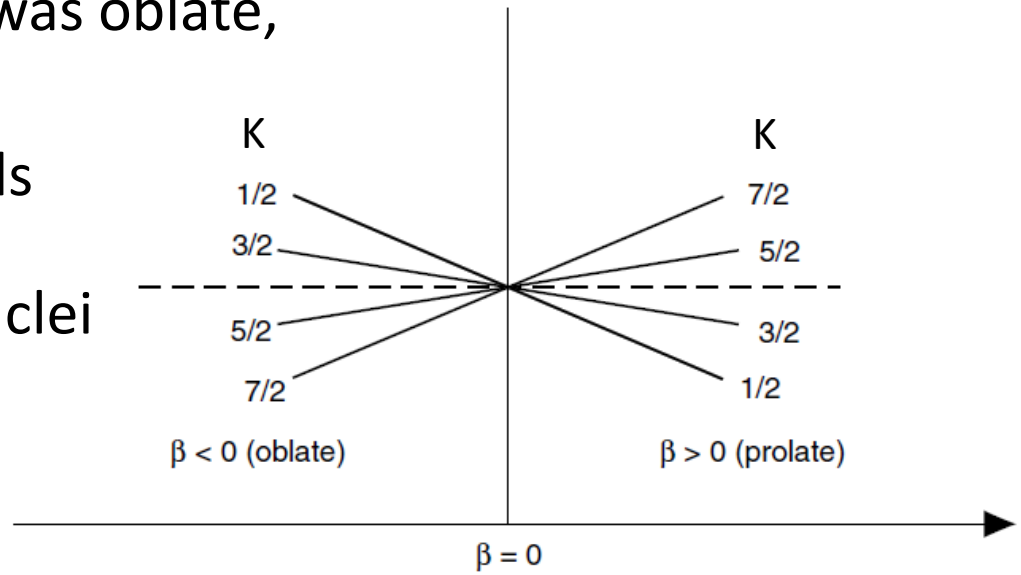
- We can excite an independent quadrupole vibration by adding a second phonon
- The second phonon will build excitations on the first, coupling to either 0^+,2^+, or 4^+
- Employing a nuclear potential instead winds up breaking the degeneracy for states associated with a given number of phonons

Nilsson model: single-particle level splitting



R.Casten, Nuclear Structure from a Simple Perspective (1990)

- Consider the options for our nucleon's orbit around the nucleus
- Orbits with the same principle quantum number will have the same radius
- Notice that the orbit with the smaller projection of j (K_1) sticks closer to the bulk of the nucleus during its orbit
- Since the nuclear force is attractive, the K_1 orbit will be more bound (i.e. lower energy) than the K_2 orbit
- The opposite would be true if the nucleus in our picture was oblate, squishing out toward the K_2 orbit
- Therefore, for prolate nuclei, lower K single-particle levels will be more bound (lower-energy), whereas larger K states will be more bound for oblate nuclei

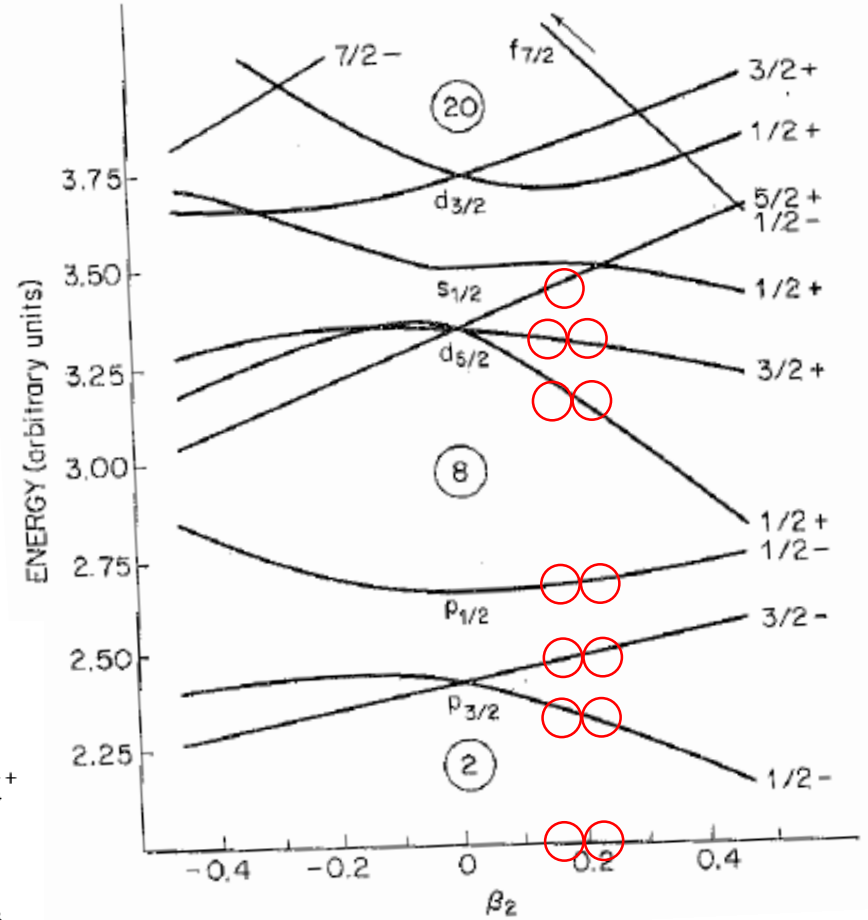


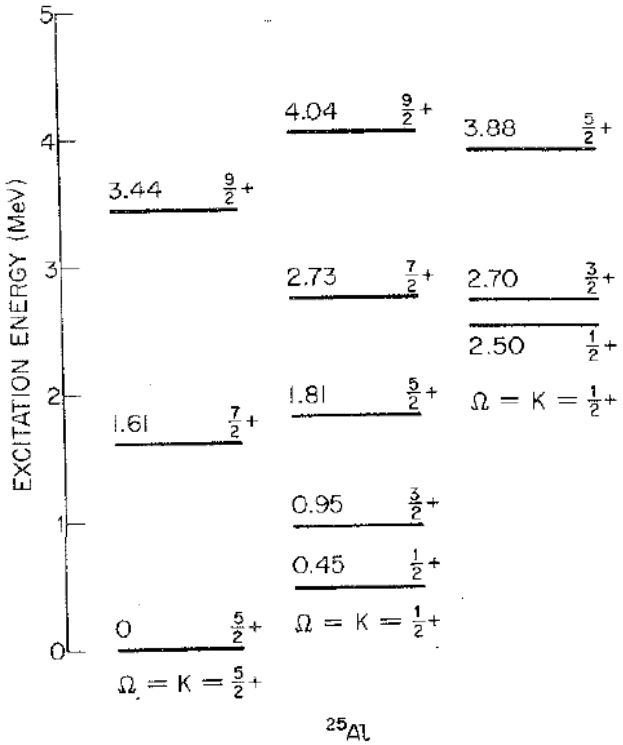
Loveland, Morrissey, Seaborg, Modern Nuclear Chemistry (2006)

Nilsson Model: Example

- Consider ^{25}Al, for which we expect $\beta_2 \approx 0.2$, like ^{27}Al
- There are 13 protons and 12 neutrons, so the unpaired proton will be responsible for J^π
- Filling the single-particle levels,
  - We place two protons in the $1s_{1/2}$ level, which isn't shown
  - Then two more in $1/2^-$, two more in $3/2^-$, two more in $1/2^-$, two more in the $1/2^+$, two more in the $3/2^+$
  - And the last one winds up in the $5/2^+$ level
- So, we predict $J^\pi_{g.s} = {}^5/_2{}^+$
- For the first excited sate, it seems likely the proton will hop up to the nearby $1/2^+$ level
- Agrees with data
- Since ^{25}Al is deformed, we should see rotational bands with states that have (integer)+j and $\propto j(j+1)$ spacing

B. Harvey, Introduction to Nuclear Physics and Chemistry (1962)





Connection to thermodynamics

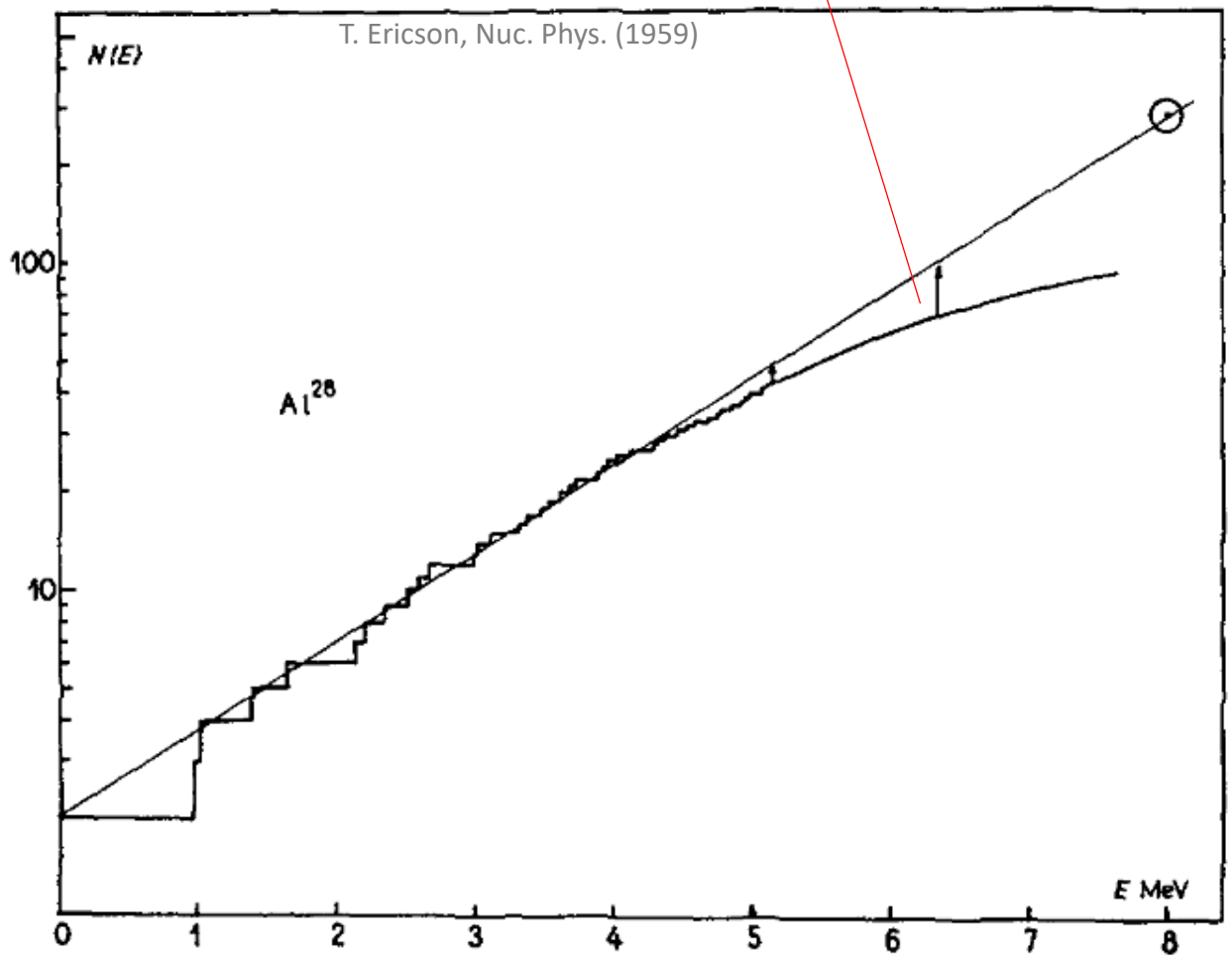
- Now we can solve for our entropy: $S(E) = \int \frac{dE}{T(E^*)} + constant \approx \int k_B \sqrt{\frac{a}{E^*}} dE + constant$
- Since a zero-temperature system has zero entropy, $S(E) \approx k_B 2\sqrt{aE^*}$
- Recall from the microscopic picture, $S = k_B \ln(g)$
- So, the number of accessible configurations (a.k.a. nuclear states) for our system is $g \approx \exp\left(2\sqrt{aE^*}\right)$
- The density of states is going to be proportional to the total number of states
- So, the state density $\rho(E^*) = C \exp\left(2\sqrt{aE^*}\right)$, where C is a constant
- A more careful treatment using partition functions and other statistical mechanics tools yields: $\rho(E^*) = \frac{\sqrt{\pi}}{12a^{1/4}E^{*5/4}} \exp\left(2\sqrt{aE^*}\right)$

H. Bethe, Phys. Rev. (1936)

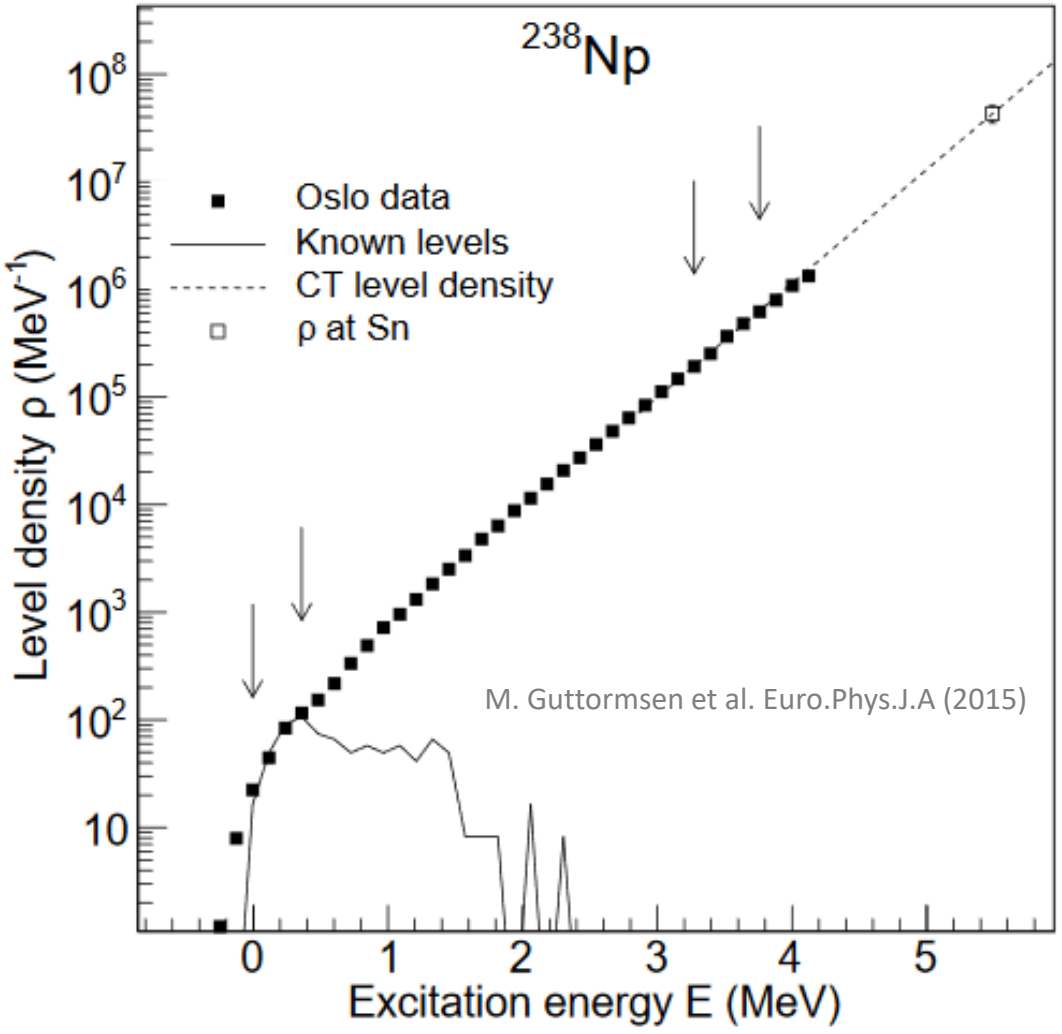
- Going back to our estimate for ^{238}U and using $a = 1/d$ and $E^* = S_n$, we get $\rho \approx 3 \times 10^4 MeV^{-1}$
- In practice, C and a are usually fit to data
- C in particular isn't so relevant, since we can normalize $\rho(E^*)$ to the region at low excitation energy where individual levels can be counted and ideally also to $\rho(E^* = S_n)$

Experimental results confirm the exponential behavior of $\rho(E^*)$

One challenge in comparing to counts of discrete states is knowing if your measurement missed any levels

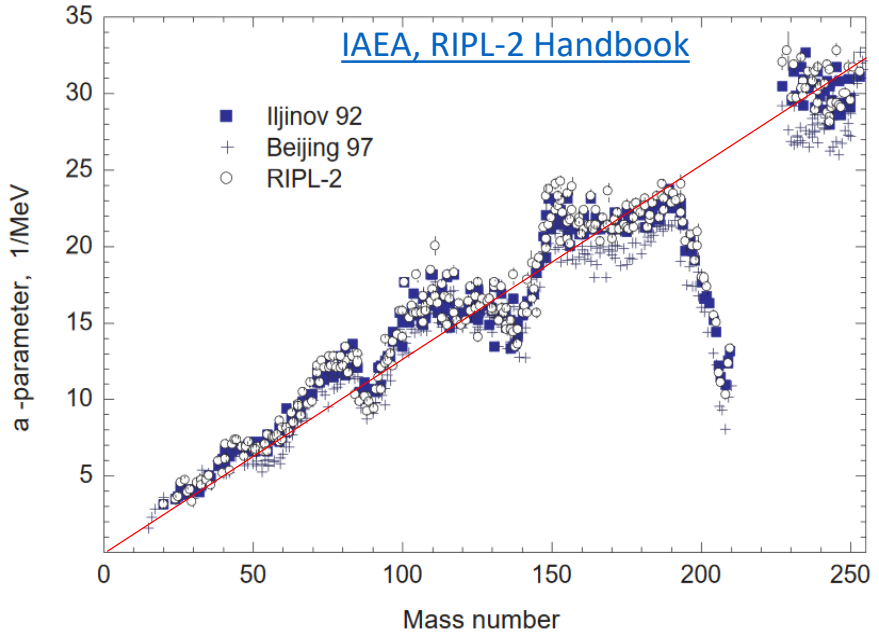


Techniques which are sensitive to the integrated number of levels can overcome this …though with assistance from models

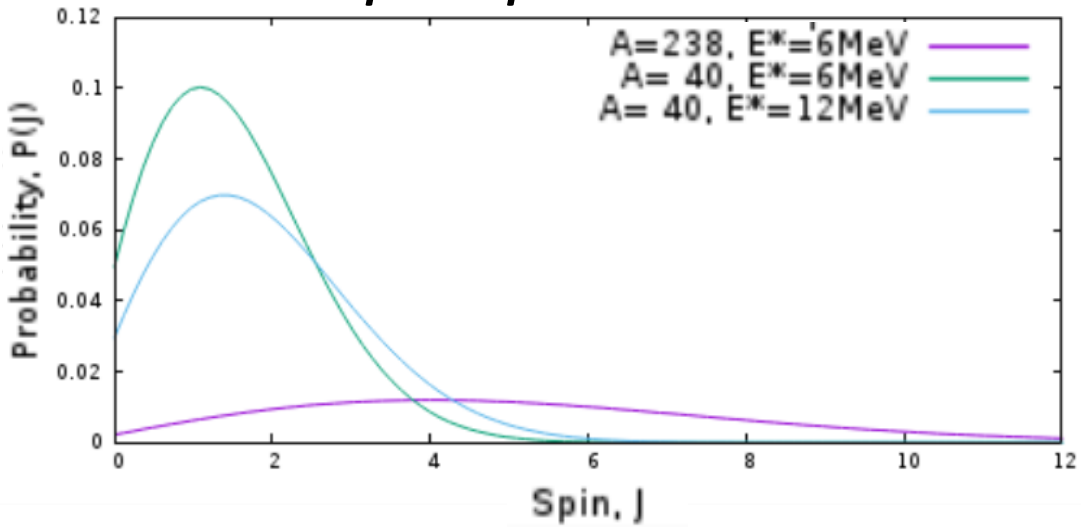


σ^2 and the spin distribution

- Having found $\sigma^2 = \frac{Ik_BT}{\hbar^2}$, now we need to estimate the moment of inertia I
- Since we assumed a spherical nucleus earlier to justify the degeneracy of E^* in M, we'll double-down and use I for a rigid sphere: $I = \frac{2}{5}MR^2$
- Using this and the previously derived formula for the nuclear temperature $T \approx \frac{1}{k_B}\sqrt{E^*/a}$, and the standard estimate for the nuclear radius $R = r_0 A^{1/3}$: $\sigma^2 = \frac{2Mr_0^2 A^{2/3}}{5\hbar^2}\sqrt{\frac{E^*}{a}}$
- Using $M = Am_u \approx \left(931.5\,^{MeV}/_{c^2}\right)\mathrm{A}$, $\hbar c \approx 197 MeV\ fm, r_0 \approx 1.2 fm$: $\sigma^2 \approx 0.014 A^{5/3}\sqrt{\frac{E^*}{a}}$
- It turns out, fits to neutron resonances yield $a \approx {}^{A}/_{8}\,MeV^{-1}$
- Therefore, $\sigma^2 \approx 0.014 A^{7/6}\sqrt{E^*}$



Implied spin distributions



Decay Equations for One Nuclear Species

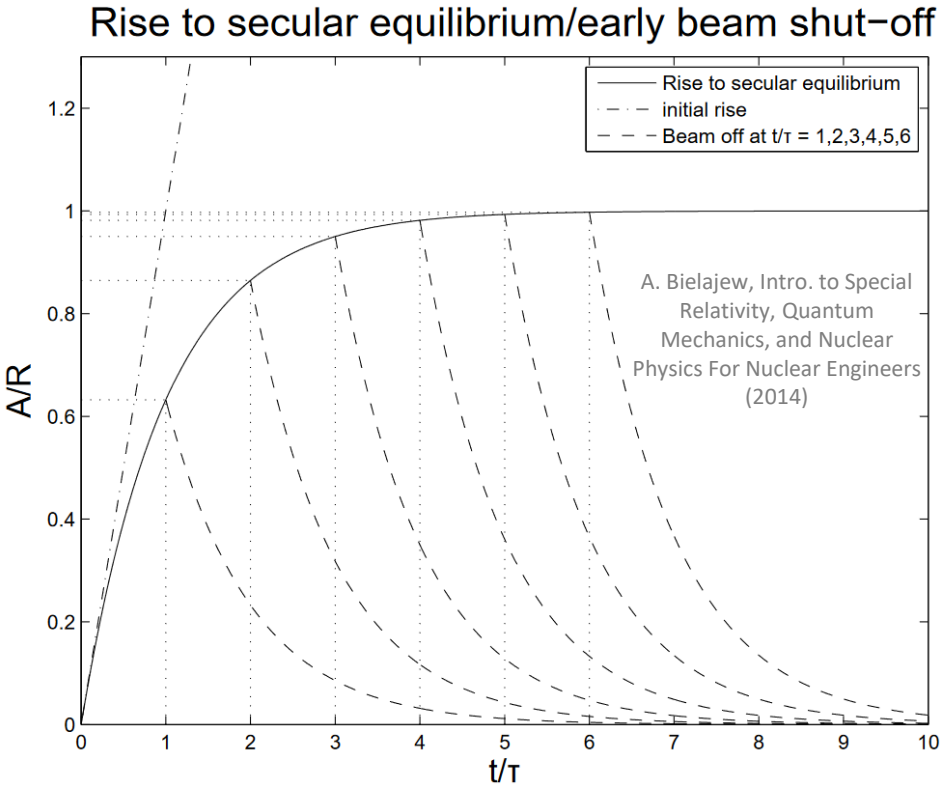
- Solving for $N(t)$ from the seemingly innocuous equation $\frac{dN}{dt} = -\lambda N(t)$, is actually pretty hard, unless one employs the Laplace transform, which turns our differential equation into an algebraic one
- For the LHS, we assume $N(t)$ to be an exponential function (empirically a safe bet) and use the derivative property of Laplace transform:
  - $\frac{dN}{dt} \rightarrow sN(s) - N(0)$
- For the RHS, simply swap-in s for time: $-\lambda N(t) \rightarrow -\lambda N(s)$
- Therefore, $sN(s) - N(0) = -\lambda N(s)$
- Which is re-written as: $N(s) = \frac{N(0)}{(s+\lambda)}$
- Using one of several different methods (See E.g. D.Pressyanov, Am.J.Phys. 2002, or a Math Methods book), the inverse Laplace transform can be employed, yielding the familiar relation:
  - $N(t) = N(0)e^{-\lambda t}$, the number of nuclei existing at time t
- Since the Activity (decays/second) $A = \lambda N$,
  - $A(t) = A(0)e^{-\lambda t}$

Common Descriptors for Decay

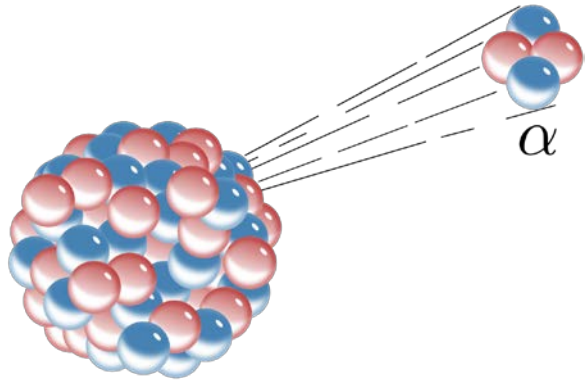
- Aside from the decay constant λ, other more intuitive quantities are often used
- It is common to state the time at which half of the nuclei in a radioactive sample will have undergone decay, a.k.a. the half-life: $t_{½}$
  - By definition: $\frac{N(t_{½})}{N(0)} = \frac{1}{2}$ … so, $\frac{1}{2} = e^{-\lambda t_{½}}$
  - Re-write as $2 = e^{\lambda t_{½}}$, which makes it apparent that $t_{½} = \frac{\ln(2)}{\lambda}$
- An alternative piece of trivia is the mean lifetime for the nuclei in the sample, τ
  - By definition: $\tau = \frac{\int_0^\infty tN(t)dt}{\int_0^\infty N(t)dt} = \frac{\int_0^\infty tN(t)dt}{N(0)/\lambda} = \frac{N(0)/\lambda^2}{N(0)/\lambda} = \frac{1}{\lambda}$
  - Therefore $t_{½} = \ln(2)\tau \approx 0.693\tau$
- We can re-state the lifetime in terms of the equivalent energy width using the Heisenberg uncertainty principle:
  - $\Delta E \cdot \Delta t \geq \hbar$
  - Taking the mean lifetime as the time uncertainty, $\Delta E = \frac{\hbar}{\tau} \approx \frac{6.582\times10^{-22} MeV}{\tau\ (in\ seconds)}$

Secular Equilibrium & Radioactivity from Nuclear Reactions

- If a radioactive nucleus is made in a nuclear reaction (e.g. in a star, from cosmic rays, or using an accelerator), the constant replenishment of the daughter is similar to the case of a very long-lived parent
- In particular this is true of the reaction rate is much slower than the daughter decay …which is pretty much always the case
- Starting with $N_2(0) = 0$ and recasting "decay" of $1 \rightarrow 2$ as a reaction, $R_{12} = \lambda_1 N_1(0) = A_1(0)$, $A_2(t) = \frac{\lambda_2}{\lambda_2 - \lambda_1} A_1(0)\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) + A_2(0)e^{-\lambda_2 t}$ becomes $A_2(t) = \frac{\lambda_2}{\lambda_2 - \lambda_1} R_{12}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$
- Since $\lambda_1 \approx 0, \quad A_2(t) \approx R_{12}\left(1 - e^{-\lambda_2 t}\right)$, which is known as the "activation equation"
- Linear growth of the daughter at short times, but have diminishing returns as time increases
- If you're making a custom radioactive source on-site, this lets you know how long to bother performing the production reaction
- This relation allows nuclear reaction cross sections to be measured using the daughter decay, and it lets you know how long to perform the reaction

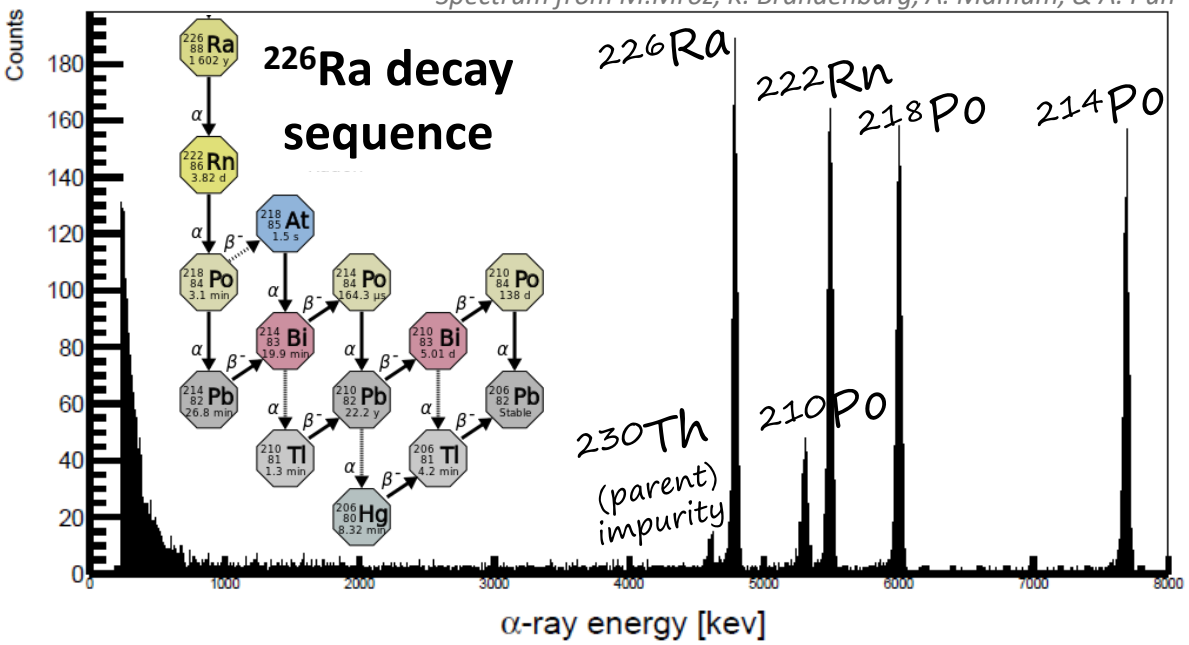


α energy from Q_α



- When an α is emitted, it will share some energy with the heavy recoil, so KE_α isn't quite equal to Q_α
- We just need to employ conservation of momentum and energy
  - $\vec{p}_{parent} = \vec{p}_{daughter} + \vec{p}_\alpha$
  - Conveniently $p_{parent} = 0$, so the daughter and α will move in opposite directions and $p_{daughter} = -p_\alpha$
  - $\frac{p_{parent}^2}{2A_{parent}} + Q_\alpha = \frac{p_{daughter}^2}{2A_{daughter}} + \frac{p_\alpha^2}{2A_\alpha} = \frac{p_\alpha^2}{2A_{daughter}} + \frac{p_\alpha^2}{2A_\alpha} = \frac{A_\alpha}{A_{daughter}} KE_\alpha + KE_\alpha$
  - $Q_\alpha = \frac{A_\alpha + A_{daughter}}{A_{daughter}} KE_\alpha = \frac{A_{parent}}{A_{daughter}} KE_\alpha$
  - $KE_\alpha = \frac{A_{daughter}}{A_{parent}} Q_\alpha$
  - So it's a pretty small effect
    (though not so for β-delayed particle emission in lighter nuclei)
- Conveniently, α sources typically have several E_α from the decay chain, and so they provide several energy calibration points

Spectrum from M.Mroz, K. Brandenburg, A. Mamum, & A. Pun

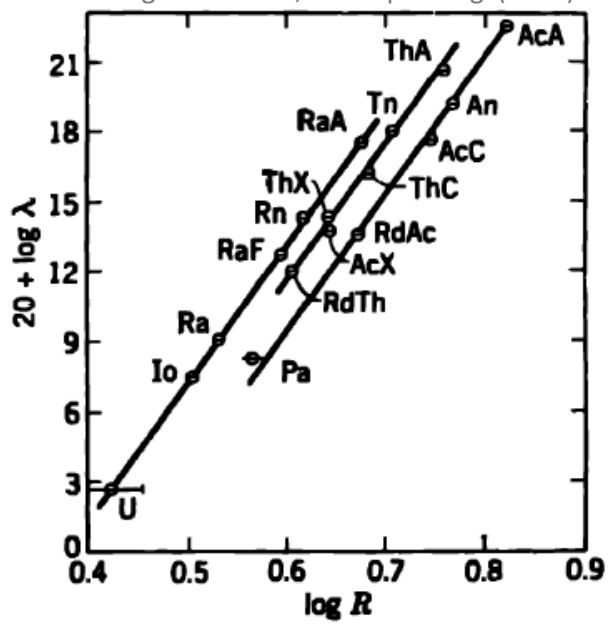


Geiger-Nuttall relation

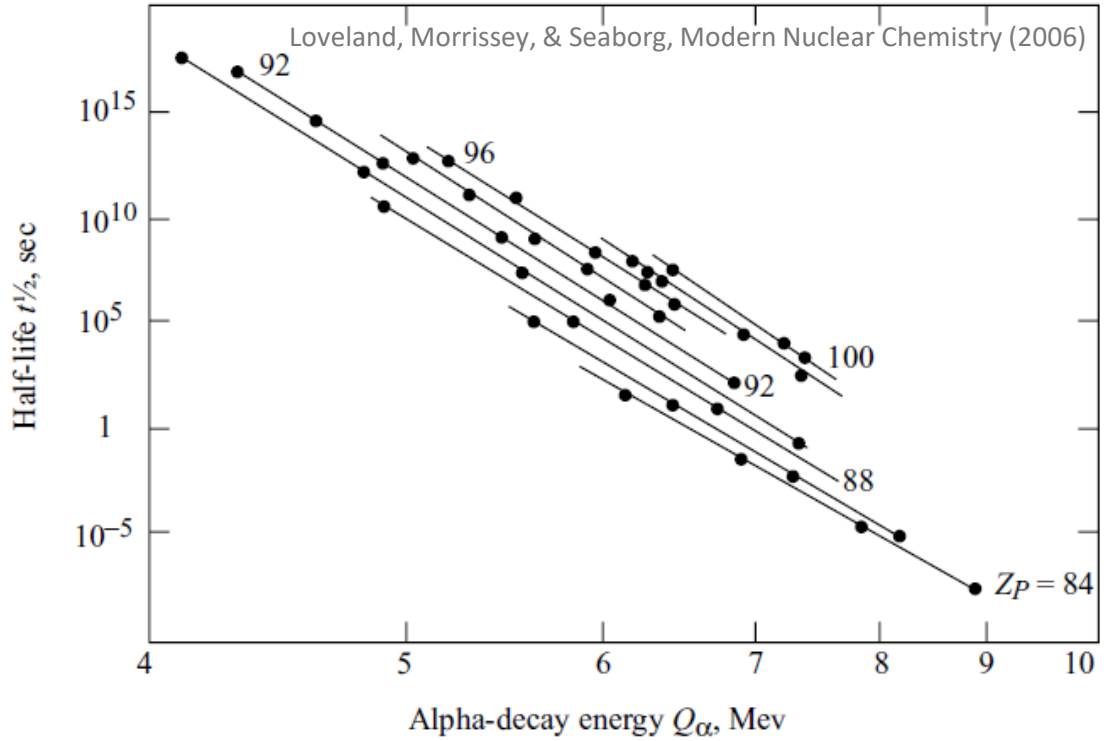
- In an early effort to characterize α-decay, Geiger & Nuttall
  (H. Geiger & J.M. Nuttall, Philisoph. Mag. (1911, 1912))
  compared the range of α particles in a material vs $t_{½}$ of the α-source and found a linear relationship in log-log space
- In modern terms, using Q_α instead of range, we get the Geiger-Nuttall relation: $\log_{10}(t_{½}) = a + bZQ_\alpha^{-½}$
- Obviously the α energy somehow impacts $t_{½}$

  …incredibly strongly
- For $\sim \times 2$ increase in Q_α, nearly 20 orders of magnitude decrease in $t_{½}$!!

Geiger & Nuttall, Philisoph. Mag. (1912)



Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)



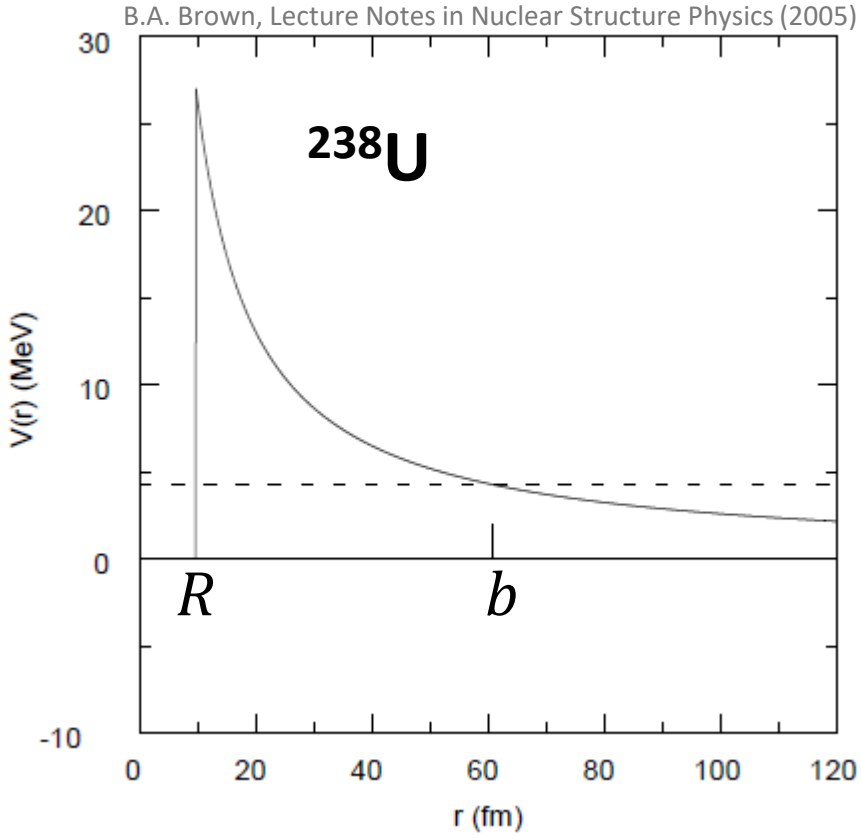
What does this imply about useful α sources?
There's a relatively limited range of E_α available.
- *Large E_α sources aren't active for long enough,*
- *while low E_α sources require huge amounts to have an appreciable activity ($A=\lambda N$).*

Tunneling through a thick barrier

- $T = e^{-2G}$, where $2G = 2\frac{e^2}{\hbar c} Z_\alpha Z_{daughter} \sqrt{\frac{2\mu c^2}{Q_\alpha}} \left[cos^{-1}\left(\frac{R}{b}\right) - \sqrt{\frac{R}{b}\left(1 - \frac{R}{b}\right)} \right]$ is pretty ugly
- Conveniently, for most cases $b \gg R$,

  so the Gamow factor $2G \approx \frac{\pi}{2}\frac{e^2}{\hbar c} Z_\alpha Z_{daughter} \sqrt{\frac{2\mu c^2}{Q_\alpha}}$
- Since the decay half-life will be inversely proportional to the tunneling probability, $t_{½} \propto e^{2G} \propto e^{Z/\sqrt{Q_\alpha}}$
- You may notice this is what Geiger & Nuttall told us all along $\log_{10}(t_{½}) = a + bZQ_\alpha^{-½}$

B.A. Brown, Lecture Notes in Nuclear Structure Physics (2005)

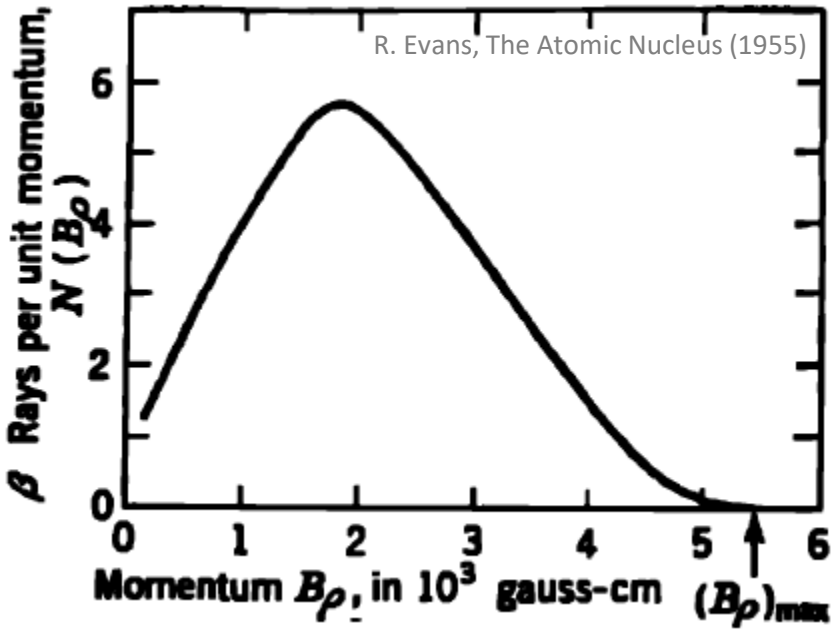


Aside:
Gamow realized this formalism would work just as well for a charged-particle tunneling in (i.e. for nuclear fusion). For nuclear fusion, 2G is often written instead as 2πη, where η is the Sommerfeld parameter.
In nuclear astrophysics, T=P=exp(-2πη) is multiplied by the Maxwell-Boltzmann distribution to get the Gamow window.

Why is it α particles that are being emitted?

- So far we've been smugly pleased with ourselves about our ability to describe α decay …but why α decay? Why not proton decay, or ^{3}He decay, or ^{12}C decay?
- The short answer is Q-values, Coulomb barriers, and clustering probabilities
  - *Q-value*: The cluster decay must be energetically favorable
  - *Coulomb barrier*: Higher-Z particles will have a larger barrier to tunnel through
  - *Clustering probability:* It's less likely for more nucleons to congregate within a nucleus

β decay spectrum, spin conservation, and the neutrino

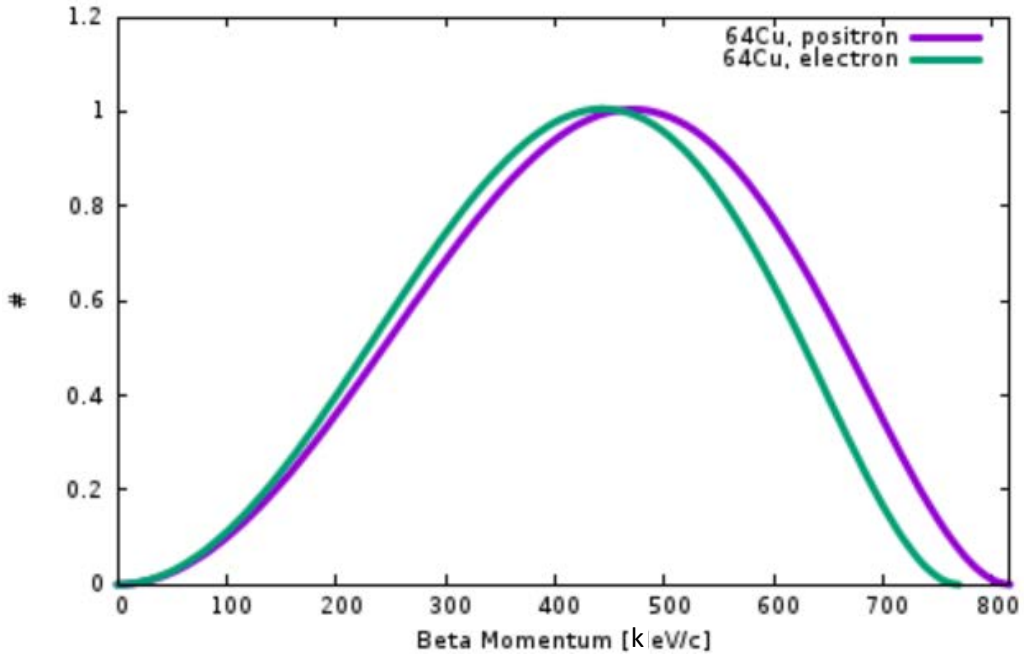


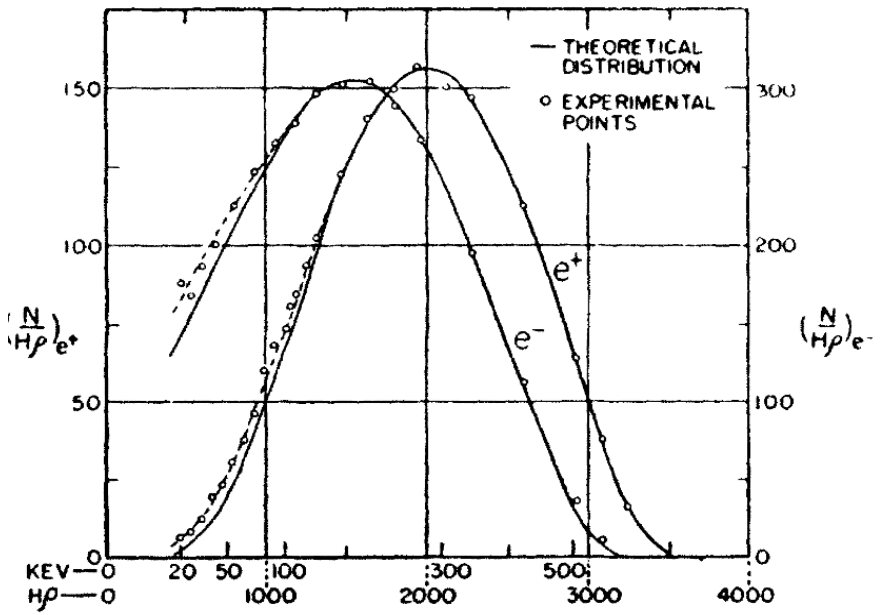
- Early experiments investigating the "β ray" showed that it was not emitted with a singular energy, like the "α ray", but rather in a continuum of energies
  - Though the maximum energy is equal to the decay Q-value
- Furthermore, the reaction $n \rightarrow p + e^-$ doesn't conserve spin!
  - $J_n = J_p = J_e = \frac{1}{2}$ … so $0 \leq J_p + J_e \leq 1 \quad \neq \frac{1}{2}$
- To remedy this issue, Pauli proposed the involvement of a 3rd hypothetical particle, the neutrino ν *Like a proper old-timey physicist, he made this proposal not in a paper, but in a letter to physicist Lisa Meitner*
- Given the above considerations, it was postulated that ν is a spin-½ particle ("fermion") that it is massless* and electrically neutral
  (of course this isn't quite true, but true enough for our purposes)
- In one of his last works before switching to primarily performing experimental work, Fermi postulated (E. Fermi, Z. Phys. 1934) that nucleons could act as sources & sinks of electrons and neutrinos, in analogy to charged particles acting as sources and sinks of photons in quantum electrodynamics (the only successful theory of interactions between quantum particles at that point)

For what it's worth, Nature rejected Fermi's paper for being "too remote from physical reality"

β decay phase space factor & the β energy spectrum

- Considering the β decay rate for an electron momentum within $p_e + dp_e$,
  $\lambda(p_e)dp_e = \frac{2\pi}{\hbar}\left|\langle\Psi_{final}|H'|\Psi_{initial}\rangle\right|^2 \rho(E)$
- The matrix element is just some number, so the functional form is from $\rho(E)$
- Therefore, we expect $\lambda(p_e)dp_e \propto \left(Q - KE_p\right)^2 p_e^2 = (Q - KE_e)^2(2m_e KE_e) = \left(Q - \frac{p_e^2}{2m_e}\right)^2 p_e^2$





Wu & Albert, Phys. Rev (1949)

FIG. 2. Momentum spectra of Cu^{64} negatrons and positrons.

Not too bad, but what effect are we forgetting that will cause positrons and electrons to behave differently?

Coulomb repulsion!

Decay selection rules and "forbidden" decays $\lambda = \frac{2\pi}{\hbar}\left|\int \Psi^*_{final} H' \Psi_{initial} d\tau\right|^2 \rho(E_f)$

- As was just alluded to, ignoring higher-order terms of the $e^{-i(\vec{p}_\nu + \vec{p}_e)\cdot\frac{\vec{r}}{\hbar}}$ Taylor expansion omits the possibility for angular momentum transfer
- If angular momentum transfer is to occur, higher-order terms need to be included and it will no longer be the case that $\left|M_{fi}\right|^2$ is independent of p_e
- In fact, for these cases $\Delta J > 1$ and/or $\Delta\pi = yes$, the leading-order overlap $\left|M_{fi}\right|^2 = 0$ and so a higher-order term will be necessary
- The order that's required will correspond to the angular momentum transfer of the decay ΔJ
- This combined with whether or not parity is changed is referred to as how "forbidden" a transition is…even though it's just a hindrance
  - $0^+ - 0^+ \rightarrow "super - allowed"$
  - $0^+ - 1^+ \; or \; \Delta J = 0 \; or \; 1 \; and \; \Delta\pi = no \rightarrow "allowed"$
  - $\Delta J = 0 \; or \; 1, \Delta\pi = yes \; \rightarrow "first \; forbidden"$
  - $\Delta J = 2, \Delta\pi = no \rightarrow "second \; forbidden"$
- For a given transition type, ft will typically be within an order of magnitude of some value

λ for Electron capture

$$\lambda = \frac{2\pi}{\hbar}\left|\int \Psi^*_{final} H' \Psi_{initial} d\tau\right|^2 \rho(E_f)$$

- Rather than a nucleon undergoing transmutation by its lonesome, instead e^--capture can occur
- This is either due to a capture of a low-lying (usually the "K-shell") electron or due to the electron Fermi energy in an electron-degenerate environment being high enough to overcome the electron-capture Q-value
- The decay constant for electron-capture decay is a bit different than for β decay, because the final state only consists of a nucleon and a neutrino … i.e. $KE_e = 0$ and $\Psi_f = \psi_{d,k}\varphi_\nu$
- The decay constant is then: $\lambda = \frac{G_F^2}{2\pi^3\hbar^3c^3}\left|M_{fi}\right|^2 T_\nu^2 |\varphi_K(0)|^2$,
  where $\varphi_K(0)$ is the wave-function for the inner-most atomic electron (the one in the "K-shell")
- You may recall from your Quantum class, $\varphi_K(0) = \frac{1}{\sqrt{\pi}}\left(\frac{Zm_e e^2}{4\pi\varepsilon_0\hbar^2}\right)^{3/2}$
- As such, the ratio of electron-capture to β^+ decay for a nucleus goes as $\frac{\lambda_K}{\lambda_{\beta^+}} \propto Z^3$

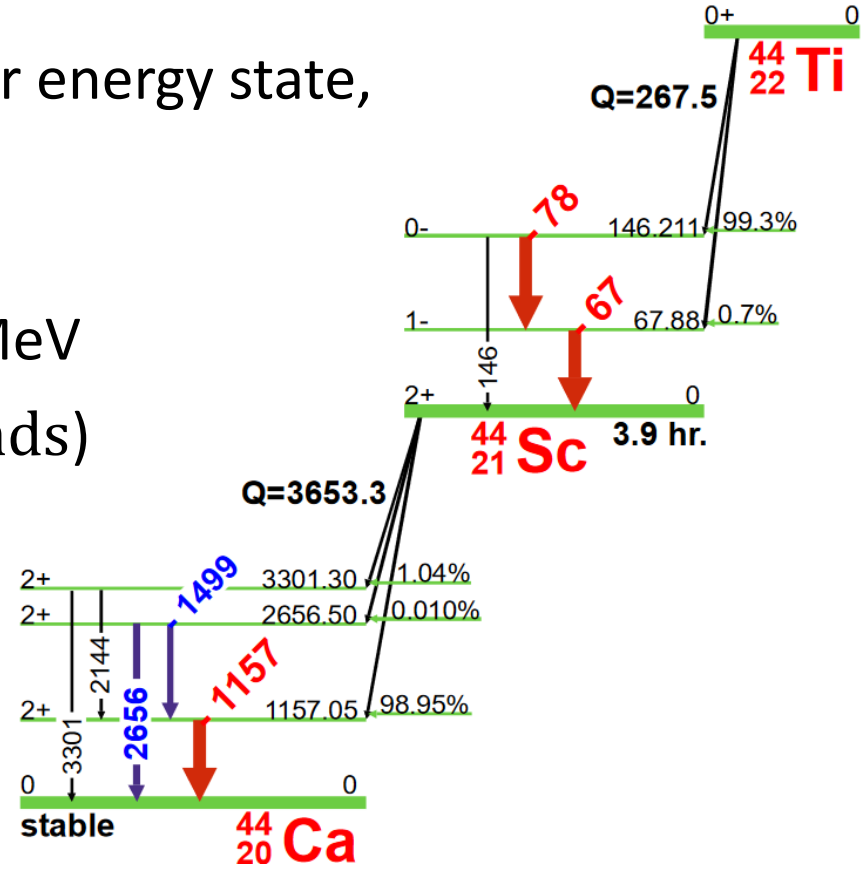
  (of course, $Q_\beta > 2m_e$ is a requirement for β^+ decay to be possible in the first place)

Since EC decay only emits a neutrino, which will be almost impossible for us to detect, how do you figure EC decay is usually detected?

X-ray and Auger electron emission due to atomic electrons filling the vacated orbital

γ decay basics

- γ decay is a de-excitation from an excited bound state to a lower energy state, preceded by some decay or reaction
- *[Just to be clear]* Z & A are unchanged
- γ ray energies can span anywhere from several keV to several MeV
- γ decay lifetimes are typically extremely short ($\tau \lesssim$ femtoseconds)
  [with the exception of isomeric states]



INEL γ Spectrum Catalog

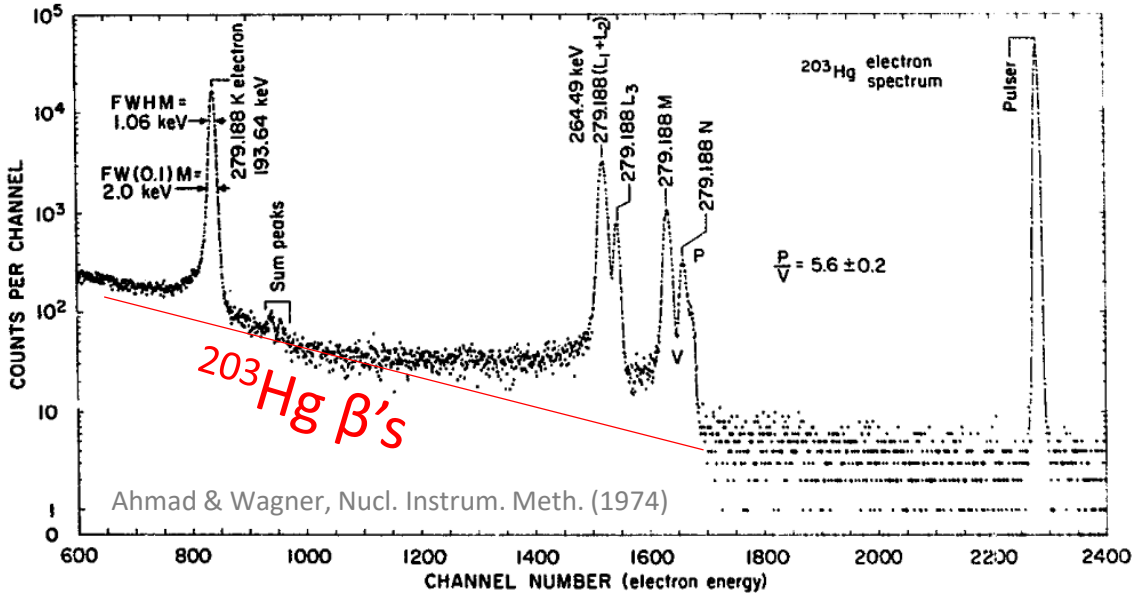
γ decay types

- Parity and angular momentum are conserved during γ decay
- Photons carry some integer angular momentum with a minimum $l = 1$, where l is referred to by the multipole 2^l
  - $l = 1: dipole, l = 2\ quadrupole, \cdots$
- A photon's parity depends not only on l, but also on the decay type
- A photon decay corresponds to shift in the nucleus's charge and matter distribution
  - Shift in the charge distribution = change in electric field = *Electric*
  - Shift in the current distribution [i.e. orbitals of protons]= change in magnetic field = *Magnetic*
- The selection rules corresponding to a particular decay type are:

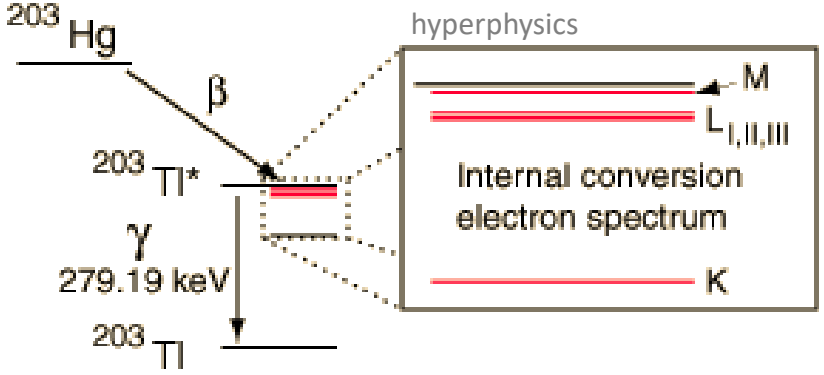
Radiation Type	Name	$l = \Delta I$	$\Delta\pi$
E1	Electric dipole	1	Yes
M1	Magnetic dipole	1	No
E2	Electric quadrupole	2	No
M2	Magnetic quadrupole	2	Yes
E3	Electric octupole	3	Yes
M3	Magnetic octupole	3	No
E4	Electric hexadecapole	4	No
M4	Magnetic hexadecapole	4	Yes

How does $0^+ \rightarrow 0^+$ happen? *Internal conversion*

- Since $l_{min} = 1$ for a photon, de-excitation by photon emission isn't possible
- Instead the process of internal conversion can happen, whereby a nucleus interacts electromagnetically with an orbital electron and de-excites by ejecting that orbital electron
- This process operates in competition with γ decay for any transition, not just $0^+ \rightarrow 0^+$
- The energy of the emitted electron is: $E_{IC} = E_{xs} - E_{BE,e^-}$, where E_{xs} is the decay transition energy, and E_{BE,e^-} is the electron binding energy



Ahmad & Wagner, Nucl. Instrum. Meth. (1974)



A similar, but different phenomenon is Internal Pair Conversion, where a photon with $E_\gamma > 2m_e c^2$ interacts with the coulomb field of the nucleus to create an e^+-e^- pair. See e.g. A. Wuosmaa et al. Phys. Rev. C Rapid Comm. 1998

Weisskopf (a.k.a. single-particle) estimates for λ

- $\lambda(l_\gamma, J_i, \pi \to J_f, \pi) = \frac{8\pi(l_\gamma+1)}{l_\gamma[(2l_\gamma+1)!!]^2}\frac{\left(E_\gamma/\hbar c\right)^{2l_\gamma+1}}{\hbar} B(l_\gamma, J_i, \pi \to J_f, \pi),$
- Reduced transition probabilities assuming the initial to final state transition is due to a single nucleon re-orienting itself within a nucleus of uniform density with $R = r_0 A^{1/3}$ are:
  - $B_{s.p.}(E, l_\gamma) = \frac{1}{4\pi}\left[\frac{3}{l_\gamma+3}\right]^2 r_0^{2l_\gamma} A^{2l_\gamma/3} e^2 (fm)^{2l_\gamma}$
  - $B_{s.p.}(M, l_\gamma) = \frac{10}{\pi}\left[\frac{3}{l_\gamma+3}\right]^2 r_0^{(2l_\gamma-2)/2} \mu_n^2 (fm)^{2l_\gamma-2}$, where the nuclear magneton $\mu_n = \frac{e\hbar}{2m_p c}$

The units for B change with l_γ!

- Note: There is a steep dependency of λ on l, so only one multipole of a decay type will matter
- The equations above are still a huge pain to work with and more noble souls have worked-out the decay constant for various situations.
- Using Q in MeV, λ_γ in s^{-1} for a nucleus with mass number A is given by:

L. van Dommelen, Quantum Mechanics for Engineers (2012)

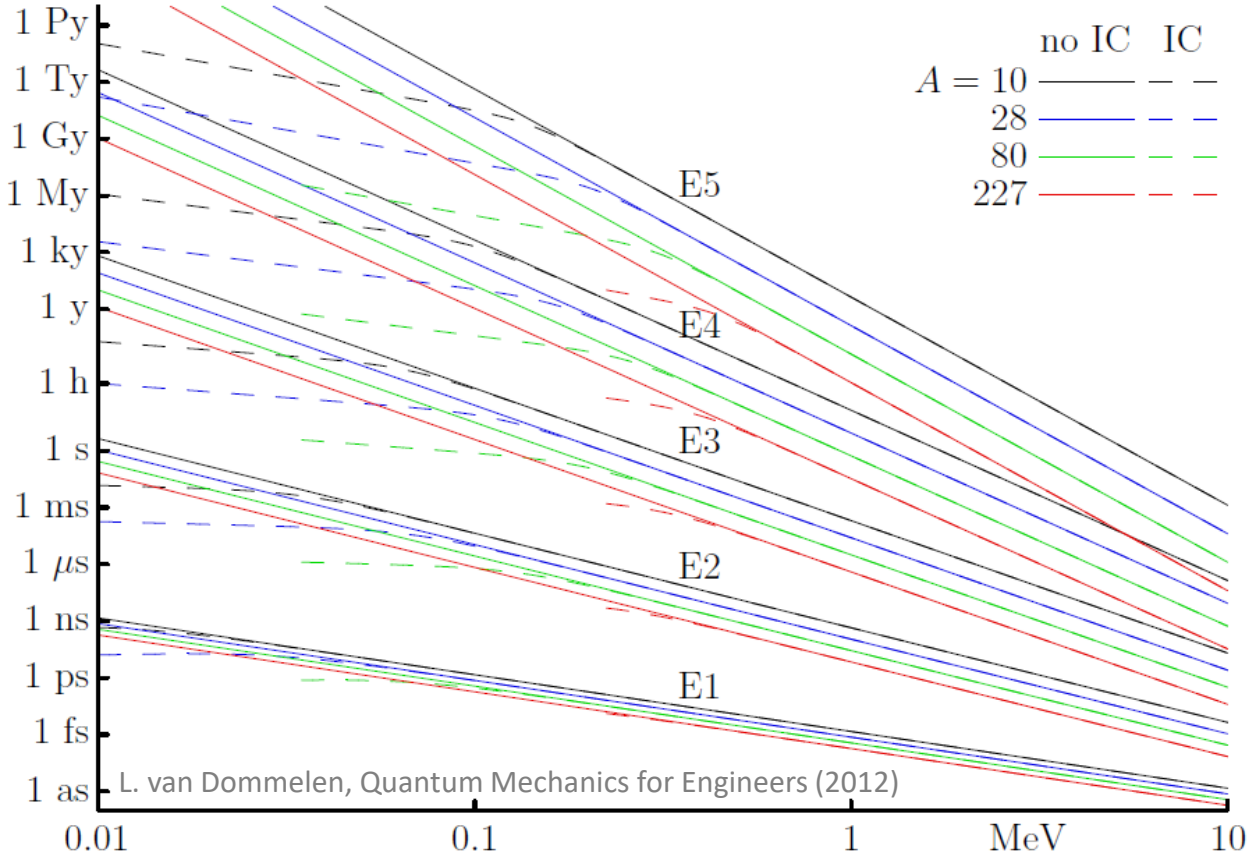
$$\lambda^{\mathrm{E}\ell} = C_{\mathrm{E}\ell} A^{2\ell/3} Q^{2\ell+1} \qquad \lambda^{\mathrm{M}\ell} = C_{\mathrm{M}\ell} A^{(2\ell-2)/3} Q^{2\ell+1}$$

ℓ :	1	2	3	4	5
$C_{\mathrm{E}\ell}$:	$1.0\,10^{14}$	$7.3\,10^{7}$	34	$1.1\,10^{-5}$	$2.4\,10^{-12}$
$C_{\mathrm{M}\ell}$:	$3.1\,10^{13}$	$2.2\,10^{7}$	10	$3.3\,10^{-6}$	$7.4\,10^{-13}$

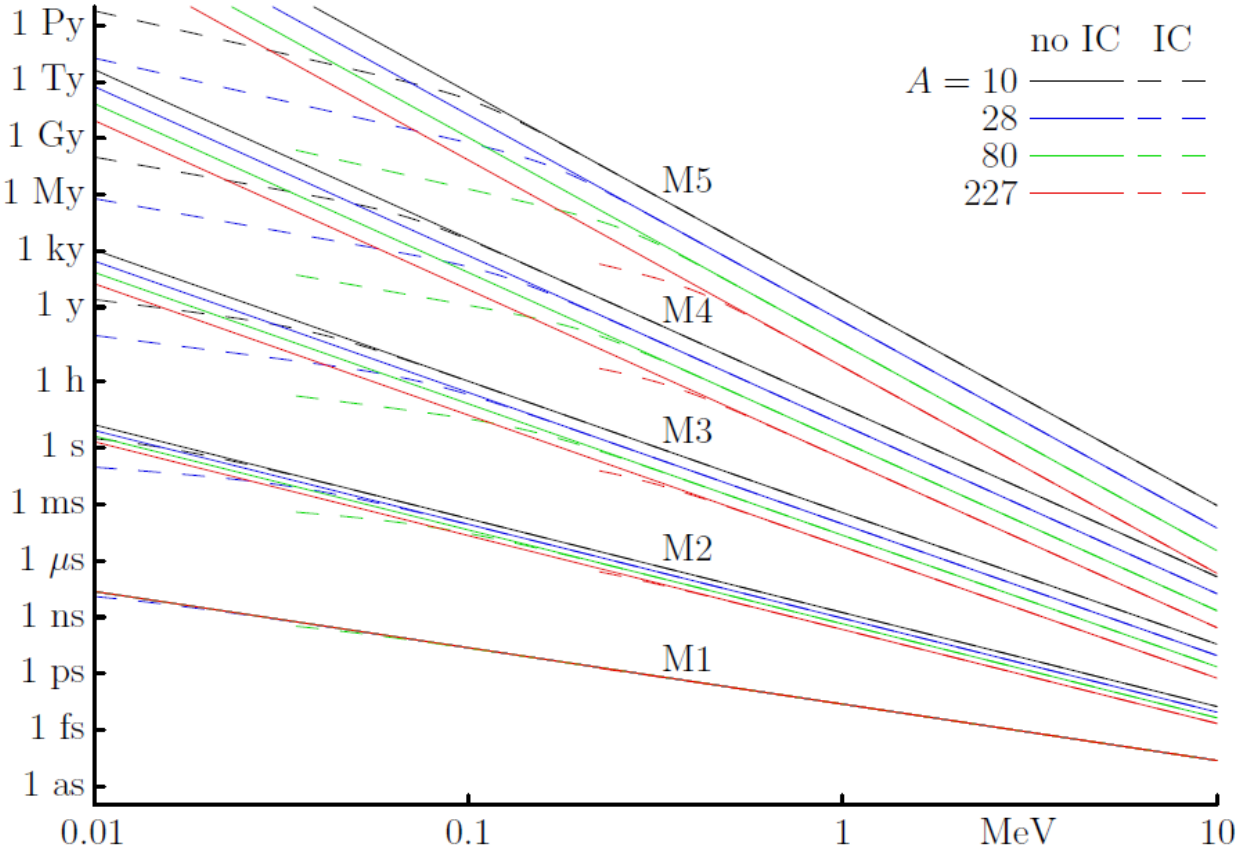
- Weisskopf estimates are generally within an order of magnitude of the real answer, so γ decay constants are often quoted as the ratio to this estimate in “Weisskopf Units” [w.u.]

Weisskopf (a.k.a. single-particle) estimates for $t_{1/2}$

$t_{1/2}(E_\gamma)$ *for* ***Electric*** *Transitions* *(from Weisskopf)*



$t_{1/2}(E_\gamma)$ *for* ***Magnetic*** *Transitions* *(from Moszkowski)*



Note that E transitions of a given multipole and E_γ are ~100X faster than M transitions with the same E_γ,ℓ

Now we see how it is that low-energy high-spin states exist as isomeric states.

Internal conversion coefficient, α

- Don't forget about our old friend, internal conversion, which competes with γ decay
- Competition between the two is described by the internal conversion coefficient $\alpha = \frac{\lambda_{IC}}{\lambda_\gamma}$, so $\lambda = \lambda_{IC} + \lambda_\gamma = \lambda_\gamma(1+\alpha)$
- α depends on the density of electrons near the nucleus, and so some friendly atomic physicists have done the dirty work of calculating the following approximate formulas:
  - $\alpha(El) = \frac{Z^3}{n^3}\left(\frac{l}{l+1}\right)\alpha_{f.s.}^4\left(\frac{2m_ec^2}{Q}\right)^{l+5/2}$; $\alpha(Ml) = \frac{Z^3}{n^3}\alpha_{f.s.}^4\left(\frac{2m_ec^2}{Q}\right)^{l+3/2}$, where $\alpha_{f.s.} \approx \frac{1}{137}$, Q is the transition Q-value, and n is the principal quantum number of the orbital electron being ejected

  These rely on the Born approximation, so Z<<137 ought to apply
- The atomic orbitals $K, L, M, N, O, \cdots$ correspond to $n = 1, 2, 3, 4, 5, \cdots$
- Clearly this process is favored for high-Z nuclei, …but also for $Q < 1.022MeV$ $l = 0$ transitions
- For $0^+ \rightarrow 0^+$ transitions, $\lambda_{E0} = 3.8 \cdot Z^3 A^{4/3} Q^{1/2}$, with Q in MeV and λ in s^{-1}

γ angular correlations, general case

- Generally speaking, $W(\theta)$ for any γ-γ coincidence is defined by a sum of Legendre polynomials:
  - $W(\theta) = \sum_{i=0}^{i=l} a_{2i}\, P_{2i}(\cos\theta)$
  - i.e. $W(\theta) = 1 + a_2\cos^2(\theta) + a_4\cos^4(\theta) + \cdots a_{2l}\cos^{2l}(\theta)$, where the normalization is such that $W(90°) = 1$
- The coefficients a_i are fit to data and the results are checked against the expected results for particular combinations of J_i, J_i, J_f, l_1, l_2
- For common cases, pre-tabulated values are available to compare to

R.Evans, The Atomic Nucleus (1955)

γ-γ cascade $I_A(l_1)I_B(l_2)I_C$	$W(\vartheta)\,d\Omega = (1 + a_2\cos^2\vartheta + a_4\cos^4\vartheta)\,d\Omega$	
	a_2	a_4
0(1)1(1)0	1	0
1(1)1(1)0	$-\frac{1}{8}$	0
1(2)1(1)0	$-\frac{1}{8}$	0
2(1)1(1)0	$+\frac{1}{13}$	0
3(2)1(1)0	$-\frac{3}{29}$	0
0(2)2(2)0	−3	+4
1(1)2(2)0	$-\frac{1}{8}$	0
2(1)2(2)0	$+\frac{3}{7}$	0
2(2)2(2)0	$-\frac{15}{13}$	$+\frac{16}{13}$
3(1)2(2)0	$-\frac{3}{29}$	0
4(2)2(2)0	$+\frac{1}{8}$	$+\frac{1}{24}$

Arfken, Klema, & McGowan, Phys. Rev. Lett. (1952)

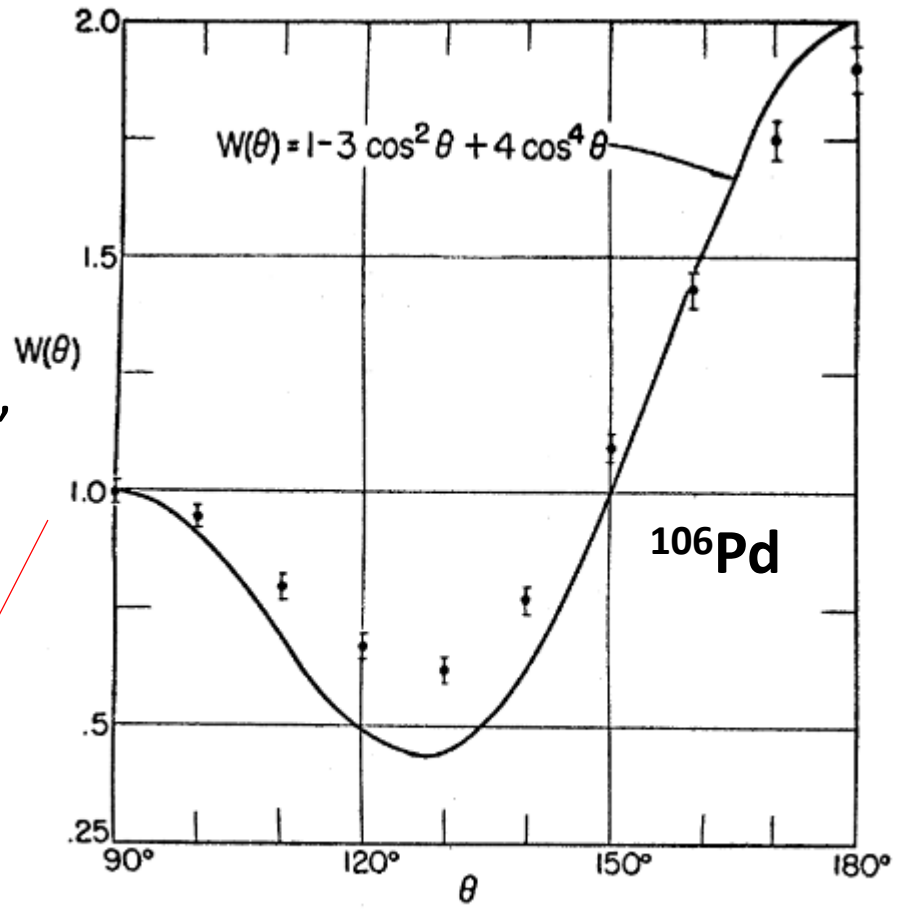
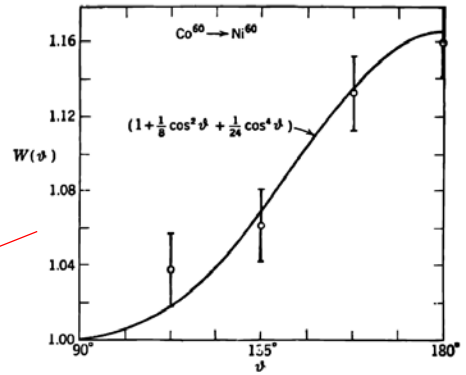
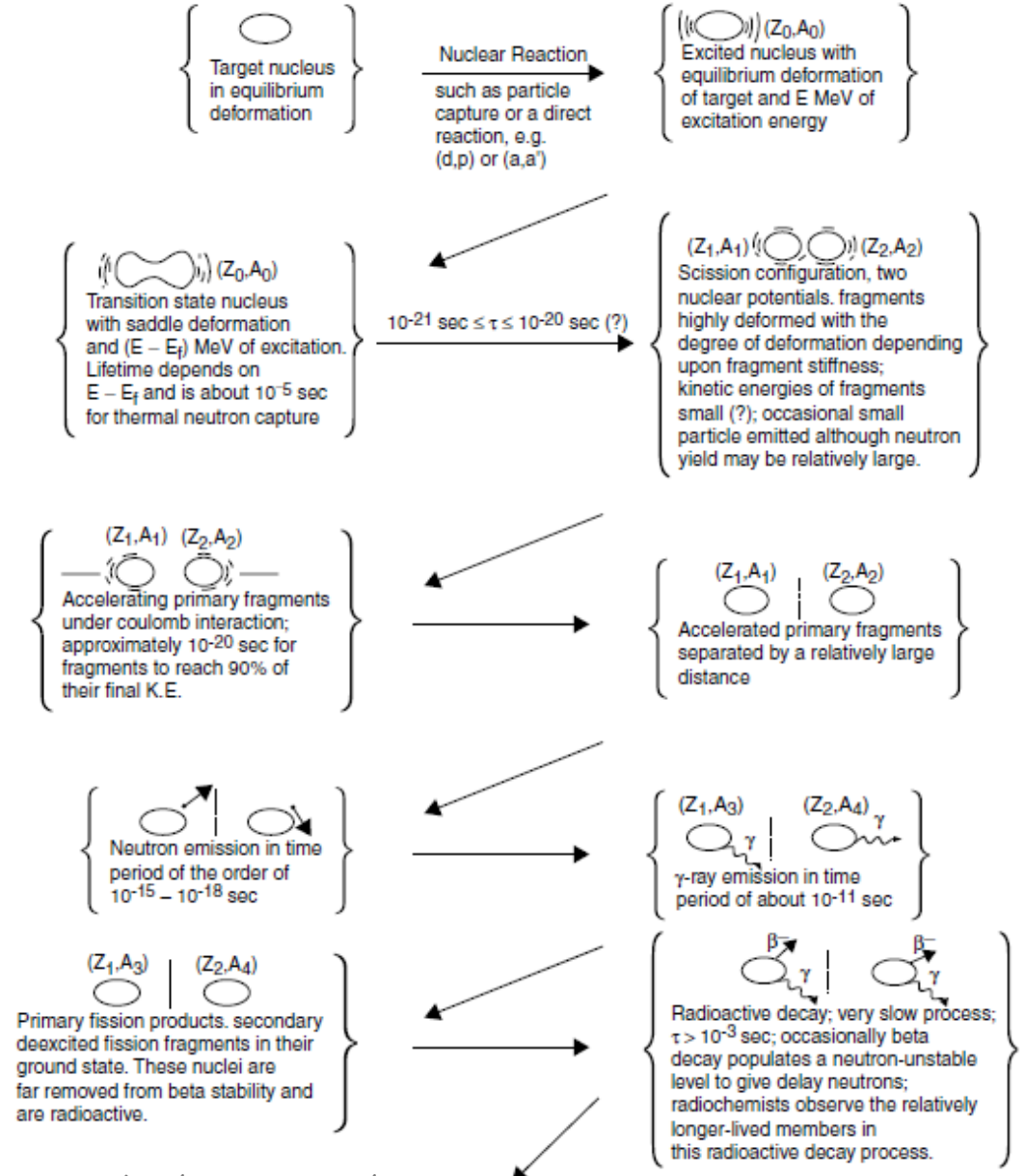


FIG. 1. Angular correlation of the 513- and 624-kev gamma-ray cascade in Pd^{106} (not corrected for finite solid angles).



Steps of fission

1. A nucleus becomes deformed either due to an external perturbation that brings in energy or an internal cluster rattling around within the potential well
2. The energy is absorbed as a collective excitation that manifests itself as a drastic shape change, elongating the nucleus into a peanut shape
3. The separation of the two lobes of the peanut becomes great enough that the two repel each other, splitting apart at the scission point
4. The coulomb repulsion accelerates the two fragments apart
5. The two fragments are each highly excited and de-excite initially via neutron emission, followed by γ emission
   (meaning prompt neutrons will be emitted along the direction of the fragments)
6. The neutron-rich fragments will then β decay back to stability, possibly emitting delayed neutrons via β-delayed neutron emission



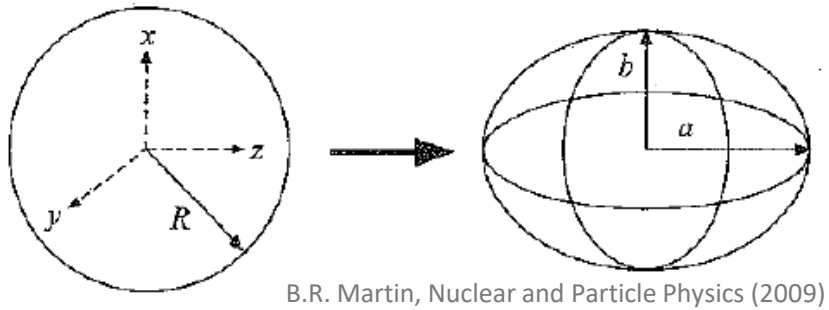
Why do you figure there is neutron emission at first and finally only γ emission?

A massive particle is better suited to remove the large angular momentum present in high-lying excited states. At lower excitation energies, a particle has to tunnel out of the nucleus, while the γ doesn't.

Loveland, Morrissey, & Seaborg, Modern Nuclear Chemistry (2006)

Energetics of shape change: *Liquid drop picture*

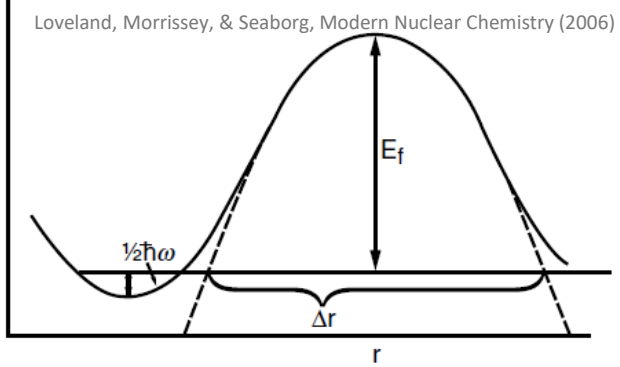
- For the deformed shape, which is an ellipsoid in this picture, the nuclear radius can be parameterized as an expansion in terms of Legendre polynomials
  [which for axial symmetry will only keep the $l = 2$ term]
  - $R(\theta) = R_0[1 + \alpha_2 P_2(\cos\theta)]$
  - α_2 is the quadrupole distortion parameter, which is related to the quadrupole deformation by $\alpha_2 = \sqrt{\frac{5}{4}\pi}\beta_2$, and the ellipsoid axes by: $a = R_0(1 + \alpha_2)$, $b = R_0 \frac{1}{\sqrt{1+\alpha_2}}$



B.R. Martin, Nuclear and Particle Physics (2009)

- It turns out, expanding the Coulomb and surface energy terms as a power series in α_2 yields
  - $E_c' \approx a_c \frac{Z^2}{A^{1/3}}\left(1 - \frac{1}{5}\alpha_2^2\right)$ and $E_s' \approx a_s A^{2/3}\left(1 + \frac{2}{5}\alpha_2^2\right)$
- Meaning the energy cost for deformation is $\Delta E = \frac{\alpha_2^2}{5}\left(2a_s A^{2/3} - a_c \frac{Z^2}{A^{1/3}}\right)$
- So, when the non-deformed Coulomb energy is twice the non-deformed surface energy or greater, there is zero energetic cost (or even an energetic gain) to deform (and ultimately fission!)
- The fissionability parameter $x = \frac{E_c}{2E_s} = \frac{a_c}{2a_s}\frac{Z^2}{A} \equiv \frac{Z^2/A}{\left(Z^2/A\right)_{critical}}$ is a measure of fission favorability

Spontaneous fission rate

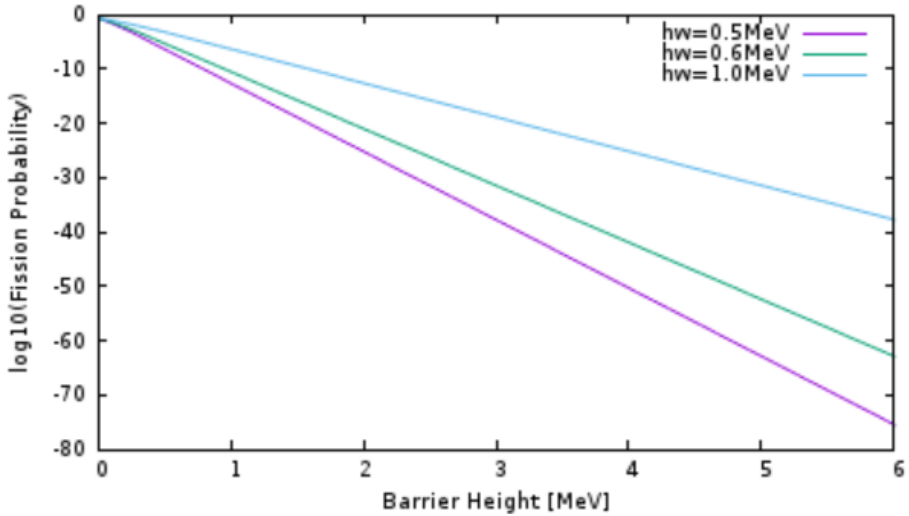


- Spontaneous fission is akin to α decay (or proton or cluster decay, for that matter), where a barrier is "assaulted" at some rate and there is probability for tunneling through the barrier
- Here, the difference is that the potential is not from the nucleus, but rather the potential energy surface for the landscape of possible shapes.
  The nucleus itself is tunneling through the barrier.
- $t_{1/2} = \frac{\ln(2)}{fP}$, where f is the assault frequency and P is the tunneling probability
- f corresponds to the rate at which the nuclear shape is changing,
  i.e. the frequency of surface oscillations, which is $\sim 10^{20} s^{-1}$ (Hill & Wheeler, Phys. Rev. 1953)
- For the simplest case of a one-humped barrier, approximated as a inverted parabola,
  (due to the liquid-drop +single particle potential for deformation, See Lec. 4),
  the tunneling probability is (Hill & Wheeler, Phys. Rev. 1953)

$$P = \frac{1}{1+\exp(2\pi E_f/\hbar\omega)}$$

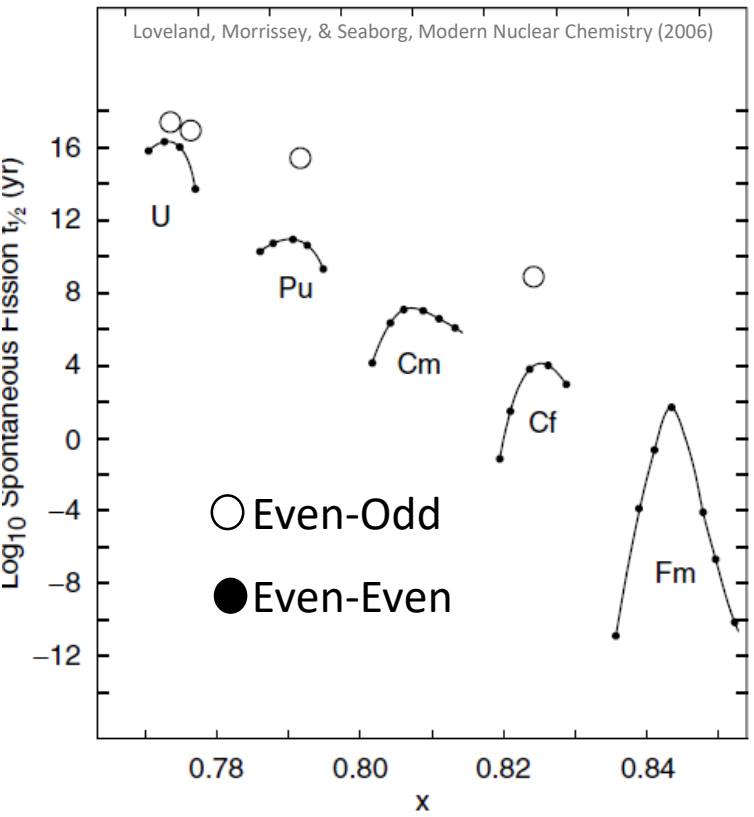
- So, $t_{1/2} \approx 10^{-21}\exp(2\pi E_f \,/\, \hbar\omega)$

Extremely sensitive to predictions of E_f

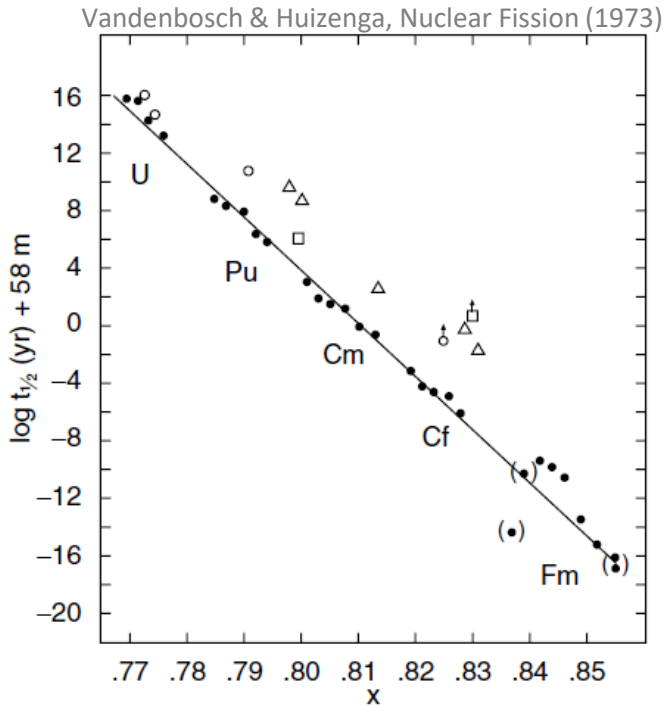


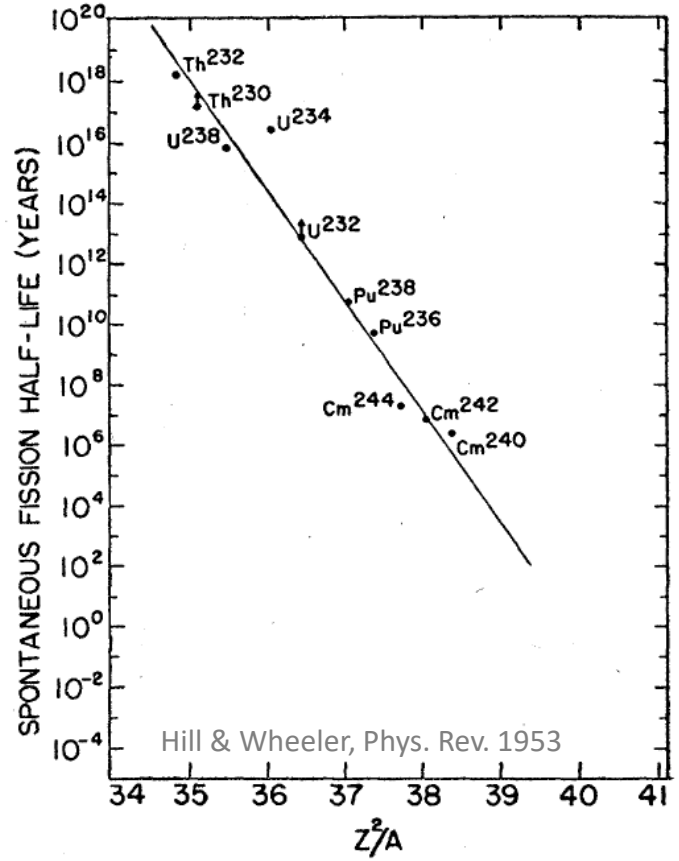
Spontaneous fission rate

- The height of the fission barrier is related to the fissionability parameter x *(recall ~48 means fission immediately)*
- So without resorting to fancy calculations of E_f, the half-life for spontaneous fission can be ball-parked
- But this method is extremely rough



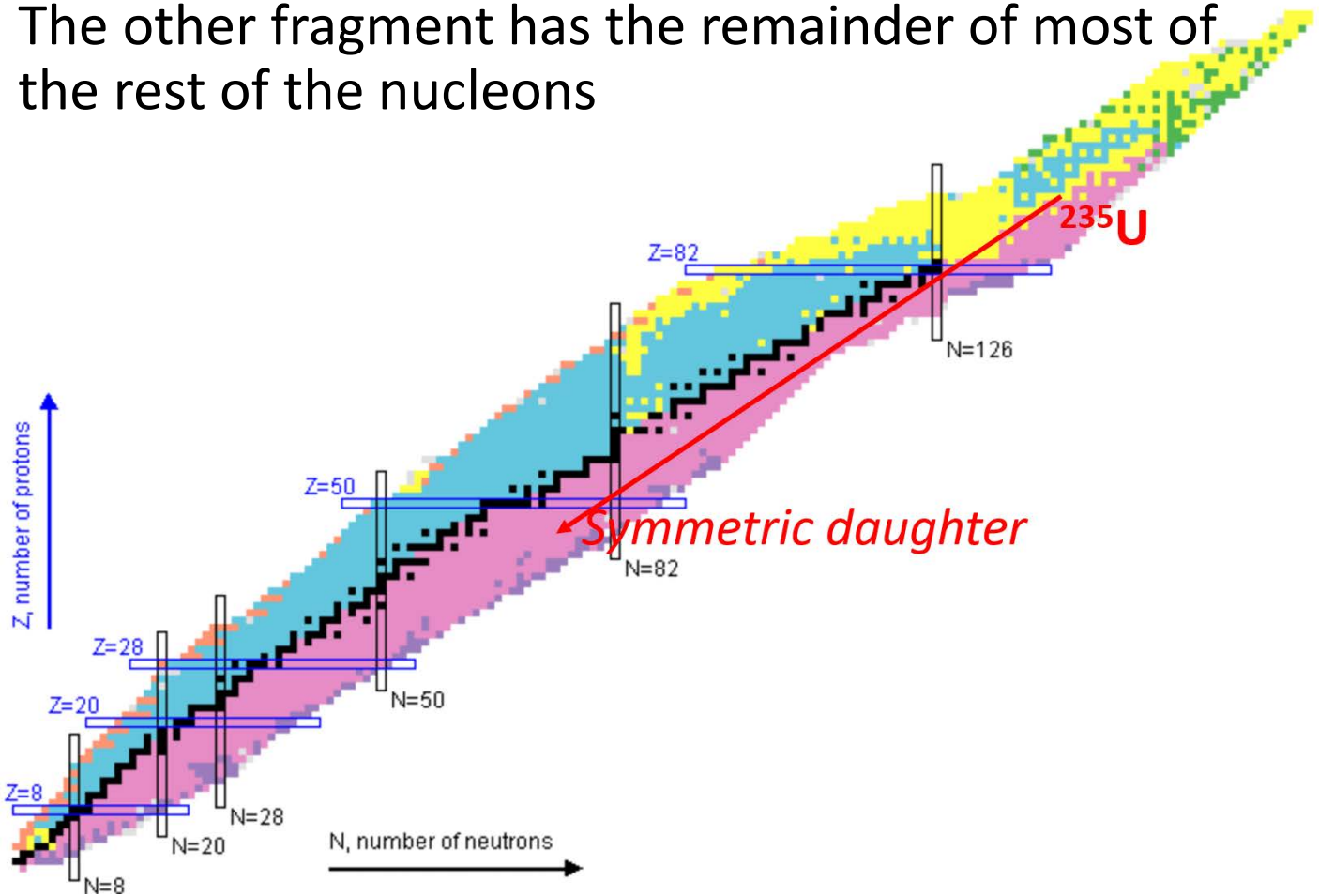
….using shell-corrected masses gives some improvement:



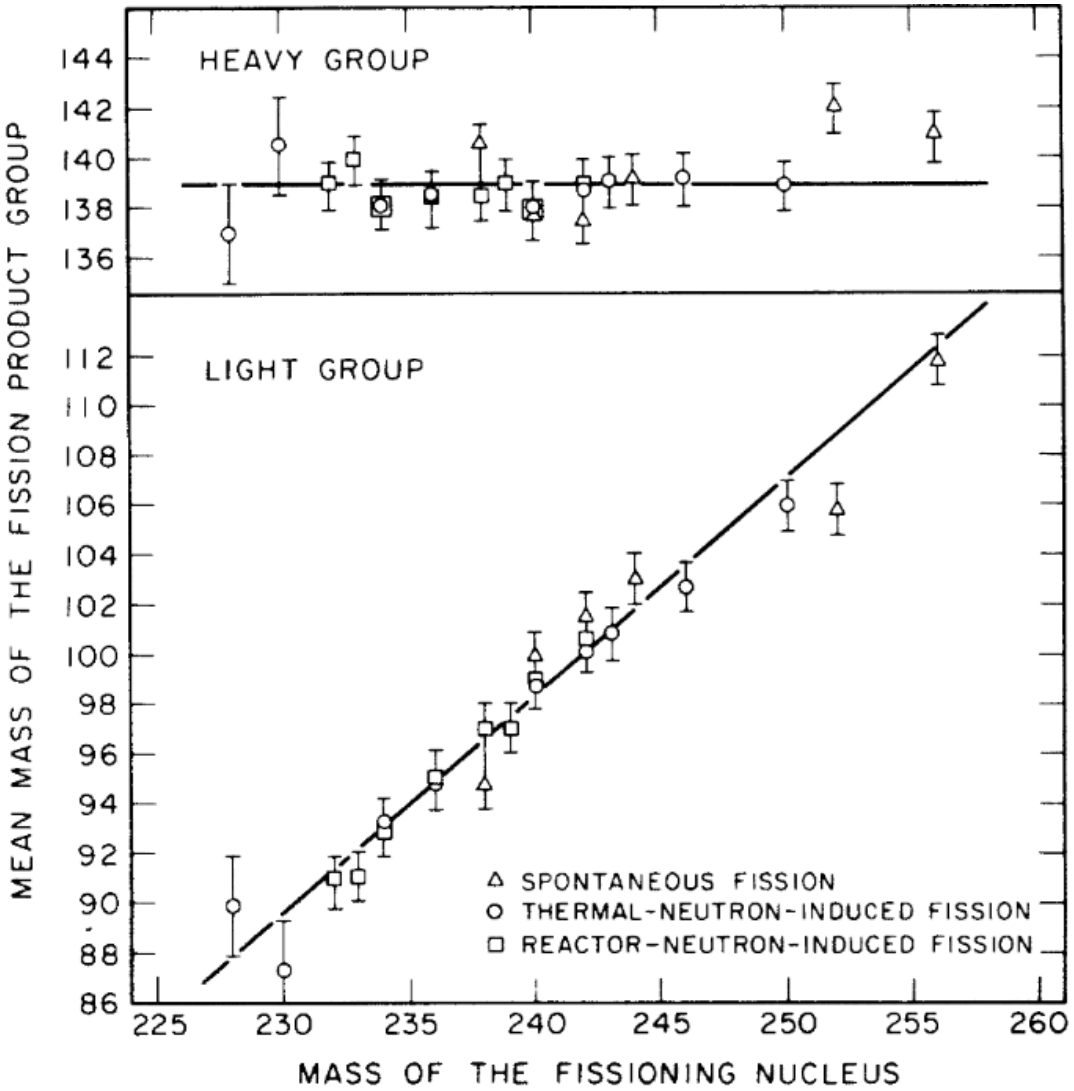


Fission fragment mass distribution

- The culprit for the asymmetric mass distribution are the $Z = 50$, $N = 82$ shell closures, which favors nuclei in this range for one of the fission fragments.
- The other fragment has the remainder of most of the rest of the nucleons



K. Flynn et al. PRC Vol 5 1972



Fission fragments: *kinetic energy*

- The total kinetic energy of fission products will roughly be the coulomb repulsion energy of the two main fission fragments,
  - $TKE = \frac{Z_1 Z_2 \alpha \hbar c}{1.8\left(A_1^{1/3} + A_2^{1/3}\right)}$ *…where r_0=1.8fm is used instead of 1.2 because of the strong deformation at scission*
- E.g. ^{240}Pu
  - $A/Z \approx 2.55$. If $Z_{high} = 50, N_{high} = 82$, then $Z_{low} = 94 - 50 = 44, N_{low} = 240 - 132 - 44 = 64$
  - $TKE = \frac{(50)(44)\alpha \hbar c}{1.8\left(132^{1/3} + 108^{1/3}\right)} \approx 178 MeV$

Schunk & Robledo, Rep. Prog. Phys. (2016)