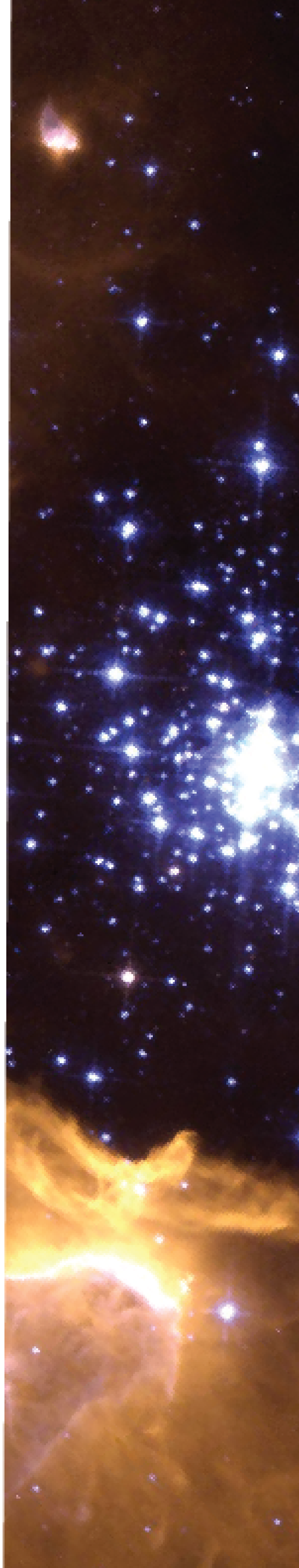
EDWARD BROWN

TO BUILD A STAR



About the cover: This Hubble image of the nebula NGC 3603 shows stars at different life stages, from birth to main-sequence and beyond.
Credit: Wolfgang Brandner (JPL/IPAC), Eva K. Grebel (U. Wash.), You-Hua Chu (UIUC), NASA

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Preface

These notes were written as I taught the junior/senior undergraduate course on stars at Michigan State University in the autumn semesters of 2012, 2014, and 2016, and were finally put into manuscript form, with additional figures and exercises, during the spring and summer of 2018. The motivation for assembling the notes was to make a self-contained package that could be inexpensively distributed to students instead of a textbook.

In addition to deriving a basic physical description of how stars work, a secondary goal of the course is to train students to make simple physical models and order-of-magnitude estimates. This is a crucial skill that is not incorporated enough into the typical undergraduate physics courses. In keeping with this goal, many of the exercises ask the students to make estimates or to employ simple models, such as constant density throughout the star, rather than to perform elaborate calculations. There are some exercises in the text that must be solved numerically, and the course does include a group numerical project.

The text layout uses the `tufte-book` LaTeX class[1]: the main features are a large outer margin in which the students can take notes and the tight integration of text, figures, and sidenotes. Exercises are embedded throughout the text. The exercises range from comprehension checks to longer, more challenging problems. This layout is meant to encourage students to actively work through the notes, and it will be interesting to see if that in fact occurs. Because the exercises are spread throughout the text, there is a "List of Exercises" in the front. I've also added boxes containing more advanced material that I felt students should be exposed to, but were not essential to the main development of the course.

[1] https://tufte-latex.github.io/tufte-latex/

One evening I tried to liven up the chapter titles. I noticed that the first two chapters had titles that were also titles for pop songs. I then decided to find song titles that would fit for the remaining chapters. When selecting titles, I imposed a rule that they all could plausibly go together on a playlist. This was challenging since the chapters originally had titles such as "The equation of state" or "The radiative opacity". The credits for the chapter titles, in order, go to Muse, Queen/David Bowie, Greta van Fleet, Dio, Deep Purple, David Bowie, and the Traveling Wilburys.

Please be advised that these notes are under active development; to refer to a specific version, use the eight-character stamp labeled "git version" on the copyright page.

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1 Starlight

1.1 Introduction: Our Sun

Let's start by considering the star we know best: the Sun. We'll denote the Sun with the symbol $\odot$. From the orbits of the planets we can deduce the mass of the sun from Kepler's laws:

Astronomy in the US uses the CGS system of units, whereas physics tends to use MKS. In these notes I follow the lead of undergraduate physics texts and also use MKS.

$$M_\odot = 1.99 \times 10^{30}\,\mathrm{kg}.$$

This is roughly 10^6 times the mass of the Earth, and is 1 000 more massive than Jupiter. Radar ranging of the solar system, combined with orbital periods, gives us the mean Earth-Sun distance, known as an ASTRONOMICAL UNIT:

$$1\,\mathrm{AU} = 1.5 \times 10^{11}\,\mathrm{m}.$$

The sun subtends an angle of about 0.5° across its diameter. Knowing the Earth-Sun distance and the angular size of the sun then tells us its radius:

$$R_\odot = 6.96 \times 10^{8}\,\mathrm{m}.$$

From measurements of the radiant flux and the distance, we then can infer the sun's radiant power, or LUMINOSITY:

$$L_\odot = 3.86 \times 10^{26}\,\mathrm{W}.$$

EXERCISE 1.1— Suppose we wish to replace the Simon power plant with a grid of solar panels. Under ideal conditions (direct light and 100% efficient panels), how many square meters of solar panels are needed to generate 70 MW (70×10^6 W)?

When we observe a star, we collect only a small fraction of this power: if a telescope (or our eye) has a collecting area $\mathcal{A}$ and is a distance d from the star, then it intercepts a fraction $\mathcal{A}/(4\pi d^2)$ of the star's light. We call $F = L/(4\pi d^2)$ the FLUX. The units of flux are $\mathrm{W\,m^{-2}}$.

EXERCISE 1.2— What would the flux be from a star with $L = 0.1\,L_\odot$ at a distance of 10 pc? Recall that a parsec (pc) is defined by the relation

$$\frac{1\,\mathrm{AU}}{1\,\mathrm{pc}} = 1'' = \frac{1}{206\,265}.$$

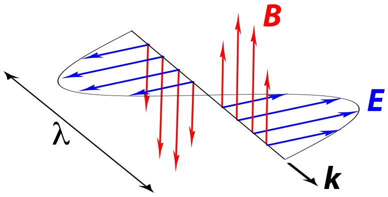

Figure 1.1: Schematic of the electric field (blue arrows) and magnetic field (red arrows) for a wave traveling along direction $\boldsymbol{k}$ with wavelength λ.

1.2 *The nature of light*

Charges feel an electric force. When we detect light, what happens at the atomic level is that the charges in our detector (antenna, CCD, eye) feel an electric (and magnetic) force that oscillates with frequency ν. Imagine setting up a grid of detectors and measure the electric force per unit charge at each point in space and at each instant of time. We call this force per charge the electric field $\boldsymbol{E}(\boldsymbol{x}, t)$. If a ray of light was beamed through this grid, we would notice a sinusoidal pattern traveling at speed[1] $c = 299\,792\,458\,\mathrm{m/s}$ with a wavelength $\lambda = c/\nu$. The amplitude of the light at our detector is proportional to $|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2$.

[1] This velocity is exact; the meter is defined in terms of the speed of light.

Suppose we put a filter in front of our detector that only accepted light in a narrow range of wavelengths $(\lambda, \lambda + \Delta\lambda)$. We would find that energy is deposited into our detector in discrete quanta of magnitude $hc/\lambda = h\nu$. We call these quanta PHOTONS. The light emitted by our sun (or any other source) consists of a huge number of photons distributed over a wide range of wavelengths, known as a SPECTRUM.

The symbol $h = 6.63 \times 10^{-34}\,\mathrm{J\,s}$ denotes PLANCK'S CONSTANT. It sets the scale for quantum mechanics.

EXERCISE 1.3— The peak of the sun's spectrum is at a wavelength of approximately 500 nm. Estimate the number of photons from the sun striking $1\,\mathrm{m}^2$ of Earth each second.

1.3 *Intensity and specific flux*

Take a detector (a CCD, an eye, a photographic emulsion) and place in front of the detector a filter that only lets through light with wavelengths in a range $\Delta\lambda$. Place a mask over the detector with a small pinhole of area ΔA that restricts the light falling on the detector to fall in a narrow cone of solid angle $\Delta\Omega$ about the normal to the detector (see Fig. 1.2). Then measure the energy ΔE incident on the detector in a time Δt. The quantity

$$I_\lambda \equiv \frac{\Delta E}{\Delta t\, \Delta A\, \Delta\lambda\, \Delta\Omega} \tag{1.1}$$

is known as the INTENSITY. It is the basic quantity describing radiation.

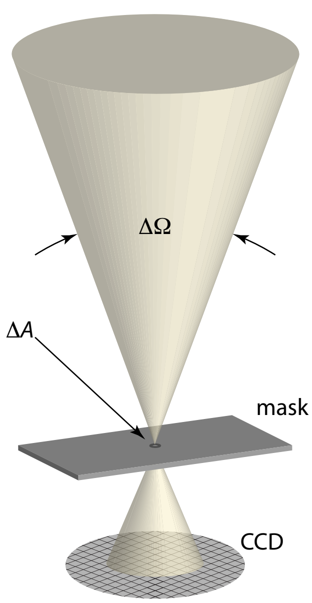

Figure 1.2: Schematic of radiative intensity

In situations in which the wavelength is small (relative to the system in question), light propagates along RAYS. By a ray of light, we mean the light emitted into a small cone of opening solid angle $d\Omega$ about a direction $\hat{\boldsymbol{k}}$. In the absence of any interactions with matter, the intensity is conserved along a ray if both source and receiver are stationary with respect to one another (Exercise 1.4).

Box 1.1 Solid angles

Imagine that your are at the center of a great sphere of radius R, and you shine a light that emits rays into some solid angle. Orient your coordinates so that the rays are traveling along the z-axis. The light will illuminate an area

$$A = R^2 \int_0^{2\pi} \int_0^{\theta} \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\varphi.$$

Here θ is the opening half-angle of the cone. The solid angle into which the light is emitted is $\Omega = A/R^2$. Astronomers often express the integral by changing variables to $\mu = \cos\theta$, so that the solid angle is

$$\Delta\Omega = \int_0^{2\pi} \int_{1-\Delta\mu}^{1} \mathrm{d}\mu \,\mathrm{d}\varphi.$$

If we integrate over all angles ($0 \le \theta \le \pi$, or $-1 \le \mu \le 1$, then we get the area of a sphere, $A = 4\pi R^2$.

EXERCISE 1.4— Your friend flashes a light: in a time Δt it emits a energy ΔE_{emit} in a waveband $\Delta\lambda$. The opening through which the light passes has area ΔA_{emit}, and the light goes into a cone of opening solid angle $\Delta\Omega_{\mathrm{emit}}$ (see Fig. 1.3). Your friend therefore calculates her intensity as

$$I_{\lambda,\mathrm{emit}} = \frac{\Delta E_{\mathrm{emit}}}{\Delta t \,\Delta A_{\mathrm{emit}} \,\Delta\lambda \,\Delta\Omega_{\mathrm{emit}}}.$$

You stand a distance d ($d^2 \gg \Delta A_{\mathrm{emit}}, \Delta A_{\mathrm{obs}}$) from your friend with a camera. The aperture on your camera has area $\Delta A_{\mathrm{obs.}}$. Show that the intensity you receive is $I_{\lambda,\mathrm{obs}} = I_{\lambda,\mathrm{emit}}$.

1. Calculate the incident energy that falls on your camera aperture ΔE_{obs}.
2. What solid angle $\Delta\Omega_{\mathrm{obs}}$ is subtended by the rays entering the aperture?
3. Now compute your intensity

$$I_{\lambda,\mathrm{obs}} = \frac{\Delta E_{\mathrm{obs}}}{\Delta t \,\Delta A_{\mathrm{obs}} \,\Delta\lambda \,\Delta\Omega_{\mathrm{obs}}}.$$

and show that this is the same as what your friend calculated.

To compute the SPECIFIC FLUX F_λ, we multiply the intensity by $\cos\theta$, where θ is the angle between the ray and the normal of our area[2] and

[2] this gives the projected area

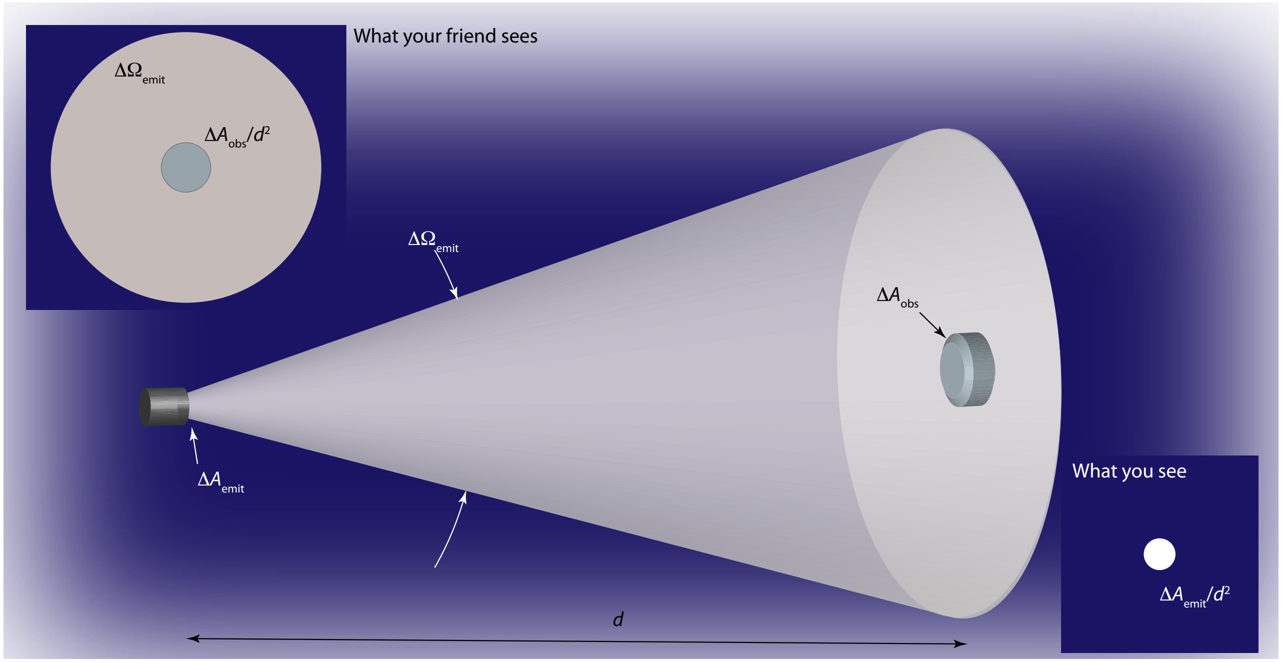

Figure 1.3: Schematic for exercise 1.4.

integrate over angle:

$$F_\lambda = \int I_\lambda \cos\theta \sin\theta \, d\theta \, d\varphi. \tag{1.2}$$

The specific flux has dimensions

$$[F_\lambda] \sim \frac{\text{energy}}{\text{time} \cdot \text{area} \cdot \text{wavelength}}.$$

1.4 *Thermal emission*

Imagine we had a material that emits and absorbs equally well at all wavelengths. We then made from this material a hollow box, and we heated this box to a temperature T. The hot atoms in the walls of the box would emit (and absorb) photons bouncing around in the cavity in this box, until the photons were in thermal equilibrium[3] with the walls of the box. If we then drilled a small hole in the side of the box, some photons would escape (but not so many as to disturb the thermal equilibrium). The intensity emerging from such a box is known as the PLANCK SPECTRUM:

[3] Meaning that the radiation field is on average neither gaining or losing energy from the walls of the box

$$I_\lambda(\text{Planck}) \equiv B_\lambda(T) = \frac{2hc^2}{\lambda^5}\left[\exp\left(\frac{hc}{\lambda k_B T}\right) - 1\right]^{-1}. \tag{1.3}$$

Here $k_B = 1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}$ denotes BOLTZMANN'S CONSTANT. This spectrum is also known as a BLACKBODY spectrum, because it is emitted from a material that absorbs (and therefore emits) equally well at all wavelengths. The emission is peaked at a wavelength $\lambda_{\rm pk} \sim hc/k_{\rm B}T$. Fig. 1.4 displays Planck's spectra for various temperatures. Note that B_λ increases at all wavelengths as the temperature increases.

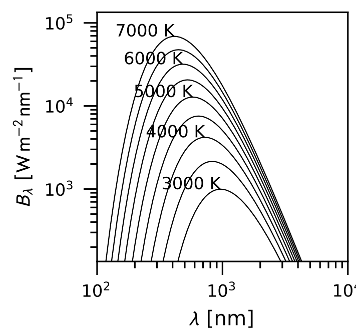

Figure 1.4: Thermal spectra for temperatures ranging from 3000 K to 7000 K.

EXERCISE 1.5 — Show that the peak of the thermal spectrum, temperature T, occurs (i.e., where B_λ is maximum) at a wavelength

$$\lambda_{\rm pk} = 290\,{\rm nm}\left(\frac{10\,000\,{\rm K}}{T}\right).$$

This result is known as WIEN'S LAW. Check this: what is the peak wavelength of the sun's emission? What is the peak wavelength for the cosmic microwave background ($T_{\rm CMB} = 2.73\,{\rm K}$)?

The Planck spectrum, expressed in terms of frequency, is

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\left[\exp\left(\frac{h\nu}{k_{\rm B}T}\right) - 1\right]^{-1}. \tag{1.4}$$

EXERCISE 1.6 — What is the frequency corresponding to $\lambda_{\rm pk}$ in Exercise 1.5? Compute the frequency $\nu_{\rm pk}$ at which B_ν is maximum. Is $\nu_{\rm pk}$ the same as the frequency corresponding to $\lambda_{\rm pk}$?

Suppose we try to compute the specific flux using Eq. (1.2). Since B_λ doesn't depend on angle, the integral is easy:

$$F_\lambda = B_\lambda \int_0^{2\pi}\int_0^{\pi} \cos\theta\,\sin\theta\,{\rm d}\theta\,{\rm d}\varphi = 0.$$

EXERCISE 1.7 — Explain, without using mathematical expressions, why there is no net flux for thermal emission.

Although the net flux is zero, if we just want the radiation escaping from our cavity, we only want to integrate over the angles $0 \le \theta \le \pi/2$. If we do this, then our *outward-going* specific flux is

$$F_\lambda({\rm outward}) = B_\lambda \int_0^{2\pi}\int_0^{\pi/2} \cos\theta\,\sin\theta\,{\rm d}\theta\,{\rm d}\varphi = \pi B_\lambda. \tag{1.5}$$

To find the total power emitted per area for thermal radiation, we need to integrate F_λ over wavelength:

$$F = \int_0^\infty F_\lambda({\rm outward})\,{\rm d}\lambda = \int_0^\infty \frac{2\pi hc^2}{\lambda^5}\frac{{\rm d}\lambda}{\exp\left(hc/\lambda k_{\rm B}T\right) - 1}. \tag{1.6}$$

By changing variables to $x = hc/\lambda k_{\rm B}T$, we can write this integral as

The integral over x can be converted into a Riemann zeta function; for further information, consult a text on mathematical methods in physics.

$$F = \frac{2\pi k_B^4}{h^2 c} T^4 \times \underbrace{\int_0^\infty \frac{x^3}{e^x - 1}\,dx}_{=\pi^4/15} = \left[\frac{2\pi^5}{15}\frac{k_B^4}{h^2 c}\right] T^4.$$

The quantity in $[\cdot]$ is called the STEFAN-BOLTZMANN CONSTANT:

$$\sigma_{SB} = 5.7 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}};$$

The total energy radiated per second per area from a thermal emitter of temperature T is thus $\sigma_{SB} T^4$.

REAL STARS ARE *NOT* BLACKBODIES! That being said, their spectra are roughly thermal, so we can define an EFFECTIVE SURFACE TEMPERATURE

$$T_{eff} = \left[\frac{F}{\sigma_{SB}}\right]^{1/4}.$$

The total power output, or *luminosity*, of a star of radius R is thus

$$L = 4\pi R^2 \sigma_{SB} T_{eff}^4.$$

For the sun, $T_{eff} = 5780\,\mathrm{K}$.

1.5 *The radiation energy density*

We introduced

$$I_\lambda \equiv \frac{dE}{dt\,dA\,d\lambda\,d\Omega}$$

as the radiant energy dE crossing an area dA in a time dt, directed into a solid angle $d\Omega$, and carried by photons with wavelengths in a range $d\lambda$. Notice that in time dt, these photons will fill a volume $dV = c dt \times dA$. Hence we can write the intensity as

$$I_\lambda = c\frac{dE}{dV\,d\lambda\,d\Omega}.$$

Using this expressions, we define the radiant energy density per wavelength as

$$U_\lambda \equiv \frac{dE}{dV\,d\lambda} = \frac{1}{c}\int I_\lambda\,d\Omega. \quad (1.7)$$

If the radiation is thermal, that is, if $I_\lambda = B_\lambda$, then

$$U_\lambda = \frac{B_\lambda}{c}\int_0^{2\pi}\int_0^\pi \sin\theta\,d\theta\,d\varphi = \frac{4\pi}{c}B_\lambda,$$

and the total radiant energy density is

$$U = \int_0^\infty U_\lambda\,d\lambda = \frac{4}{c}\pi\int_0^\infty B_\lambda\,d\lambda = \left[\frac{4\sigma_{SB}}{c}\right] T^4.$$

In this expression,

$$a = \left[\frac{4\sigma_{\mathrm{SB}}}{c}\right] = 7.566 \times 10^{-16}\,\mathrm{J\,m^{-3}\,K^{-4}}.$$

and we have used equations (1.5) and (1.6). The energy density of thermal radiation is $U = aT^4$.

It is common to denote the average (over angle) intensity as

$$J_\lambda = \frac{1}{4\pi}\int I_\lambda\,\mathrm{d}\Omega; \tag{1.8}$$

the specific energy density is thus

$$U_\lambda = \frac{4\pi}{c}J_\lambda.$$

Box 1.2 Momentum transport and radiation pressure

In addition to transporting energy, photon also carry momentum. You will learn in your quantum mechanics course that the momentum of a photon of energy $h\nu$ traveling along direction $\hat{\boldsymbol{k}}$ is

$$\boldsymbol{p} = \frac{h\nu}{c}\hat{\boldsymbol{k}} = \frac{h}{\lambda}\hat{\boldsymbol{k}}.$$

Here ν and $\lambda = c/\nu$ are the frequency and wavelength of the photon. Hence the momentum carried by photons of energy E_ν along direction $\hat{\boldsymbol{k}}$ is E/c. Since I_ν is the amount of energy carried by photons per area per time along the direction $\hat{\boldsymbol{k}}$, the momentum transported by those photons per area per time along direction $\hat{\boldsymbol{k}}$ must be $(I_\nu/c)\hat{\boldsymbol{k}}$.

TO RELATE THIS MOMENTUM TRANSPORT TO THE RADIATION PRESSURE, suppose we have a sheet of absorbing material with a normal $\hat{\boldsymbol{n}}$ being impinged by a ray of photon traveling along $\hat{\boldsymbol{k}}$. As the photons are absorbed, they transfer momentum (along direction $\hat{\boldsymbol{n}}$) of $(E_\nu/c)\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}}$ to the matter. The projected area of the ray on the matter is $\mathrm{d}A\,\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}}$. The rate of momentum transfer along $\hat{\boldsymbol{n}}$ per area per frequency is therefore

$$\begin{aligned} P_\nu &= \frac{1}{c}\int I_\nu \underbrace{\left(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}}\right)}_{\text{proj. area}} \underbrace{\left(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}}\right)}_{\text{comp. of } \boldsymbol{p} \text{ along } \hat{\boldsymbol{n}}} \mathrm{d}\Omega \\ &= \frac{1}{c}\int_0^{2\varphi}\int_{-1}^{1} I_\nu \mu^2\,\mathrm{d}\mu\,\mathrm{d}\varphi. \end{aligned} \tag{1.9}$$

A change in momentum per time is a force; hence equation (1.9) represents the force per area, or PRESSURE, exerted by photons

Box 1.2 continued

with frequencies in $[\nu, \nu + d\nu]$. The two factors of $\mu = \cos\theta$ account for the projected area and the component of momentum along the normal to the surface $\hat{\boldsymbol{n}}$.

If the radiation is thermal, so that $I_\nu = B_\nu$ and is independent of angle, then

$$P_\nu = \frac{4\pi}{3c} B_\nu.$$

We can integrate P_ν over frequency to get the total radiation pressure,

$$P_{\text{rad}} = \frac{4\pi}{3c} \int_0^\infty B_\nu \, d\nu = \frac{4}{3c} \sigma_{\text{SB}} T^4 = \frac{1}{3} a T^4. \tag{1.10}$$

Note that the pressure is 1/3 of the energy density for thermal radiation. This is in general true for a gas of relativistic particles that have momentum proportional to energy.

1.6 *Magnitudes*

When observing a star, astronomers are collecting light over a range of frequencies. To compare observations, astronomers typically pass the light through standard filters and measure the transmitted flux. The flux in a given band is then

$$F_{\text{band}} = \int F_\lambda \, T(\lambda) \, d\lambda.$$

Here $T(\lambda)$ is the TRANSMISSION FUNCTION for that filter and specifies how much light is let through as a function of wavelength. The transmission functions for some common UV/optical/IR filters are shown in Figure 1.5. For example, the V-band filter is centered at $\lambda = 551$ nm and has a width at half-max of 88 nm.

EXERCISE 1.8— Suppose you wished to observe a sun-like star, and you wanted to observe wavelengths near the peak of the spectrum. Which filter would you choose, and why? What about for a star with a surface effective temperature $T_{\text{eff}} = 8\,000$ K?

When making observations, it is common to compare the fluxes in a particular band between two stars. Optical astronomers therefore define the *apparent magnitude* as

$$m(A) - m(B) = -2.5 \log \left[\frac{F(A)}{F(B)} \right]. \tag{1.11}$$

NB. Throughout this text, $\log \equiv \text{lg}$ denotes $\log_{10}$ and ln denotes $\log_e$.

Here $F(A)$ and $F(B)$ are two different measurements of flux (from two different stars, for example) in a particular waveband. It is common to

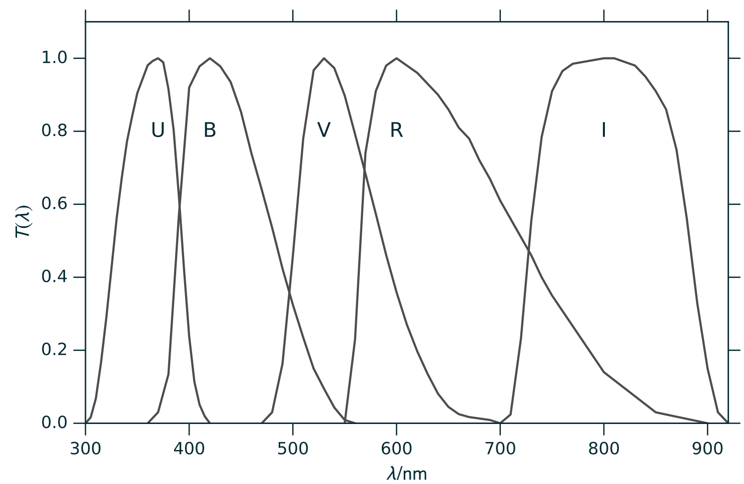

Figure 1.5: Some standard UV/optical/IR filters. The $T(\lambda)$ are normalized so that $\max(T) = 1$.

use the label of the waveband in place of m. Thus, for example, when an astronomer says, "The V-magnitude is 16.6", what she means is that the apparent magnitude measured with a standard V-filter is 16.6.

As an application of this, imagine comparing the flux from a star, at a distance d, with that from an imaginary identical star located at a distance of 10 pc. We'll call the magnitude of this imaginary star at 10 pc the ABSOLUTE MAGNITUDE M and define the DISTANCE MODULUS as

$$\begin{aligned} \mathrm{DM} \equiv m - M &= m(d) - m(10\,\mathrm{pc}) \\ &= -2.5 \log \left[\frac{L/4\pi d^2}{L/4\pi (10\,\mathrm{pc})^2} \right] \\ &= -2.5 \log \left[\left(\frac{10\,\mathrm{pc}}{d} \right)^2 \right] \\ &= 5 \log \left(\frac{d}{\mathrm{pc}} \right) - 5. \end{aligned} \tag{1.12}$$

Since the absolute magnitude is a measure of the flux from the star *if it were* at a specified distance, the absolute magnitude is a proxy for the luminosity *measured in a given filter*.

WE CAN ALSO COMPARE THE FLUX FROM TWO DIFFERENT FILTERS FOR THE SAME STAR. The difference in magnitudes for two different filters defines a COLOR INDEX, which is a measure of the star's spectrum (and, roughly, its temperature). For example,

$$B - V \equiv m_B - m_V = -2.5 \log \left[\frac{\int_{B-\mathrm{band}} F_\lambda \, \mathrm{d}\lambda}{\int_{V-\mathrm{band}} F_\lambda \, \mathrm{d}\lambda} \right]$$

measures the ratio of fluxes in the B and V bands for a particular star.

EXERCISE 1.9 — How would the $B - V$ index of the sun compare to that of a hotter star, e.g., one with $T_{\mathrm{eff}} = 8\,000\,\mathrm{K}$?

2
Under Pressure

2.1 Hydrostatic equilibrium

Consider a fluid at rest in a gravitational field. By a FLUID, we mean that the pressure is isotropic[1] and directed perpendicular to any given surface. Let's now imagine a small fluid element, as depicted in Fig. 2.1. The gravitational acceleration is in the direction $-\hat{\boldsymbol{r}}$; the fluid element has thickness Δr along the direction of the gravitational force and cross-sectional area ΔA.

[1] Meaning the pressure is the same in all directions.

Since our fluid is at rest, the forces must balance. This implies that the pressure only depends on r, so that there is no net sideways force on our fluid element. If the fluid has a density (mass per unit volume) ρ, then the mass of the fluid element is $\rho \Delta A \Delta r$, and the gravitational force on the fluid element is $-(\rho \Delta A \Delta r) g(r) \hat{\boldsymbol{r}}$. This gravitational force is balanced by the *difference* in pressure $P(r)$ between the upper and lower faces of the element.

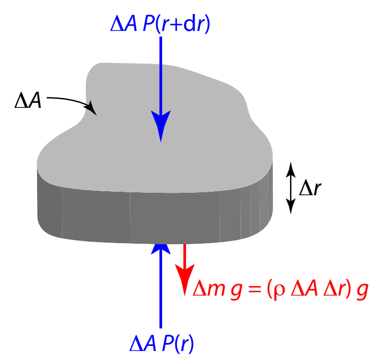

Figure 2.1: A fluid element in hydrostatic equilibrium.

The pressure force on the upper face is $-\Delta A \times P(r + \Delta r)\hat{\boldsymbol{r}}$; on the lower face, $\Delta A \times P(r)\hat{\boldsymbol{r}}$. For the element to be in hydrostatic equilibrium the forces along $\hat{\boldsymbol{r}}$ must balance,

$$\Delta A \left[-P(r + \Delta r) + P(r) - \Delta r \rho g(r) \right] = 0.$$

Dividing by Δr and taking the limit $\Delta r \to 0$ gives us the equation of hydrostatic equilibrium:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho g(r). \tag{2.1}$$

This is a differential equation describing how the pressure varies in the star. We don't have enough information yet to solve it, because we haven't specified either the gravity $g(r)$ or the density ρ.

Constant gravity, incompressible fluid

Let's try a simple case: an incompressible (density is fixed) fluid in constant gravity. This isn't a good approximation for a star, but it is a good approximation to Earth's oceans: the density of sea water changes by less than 5% between the surface and ocean floor.

The SI unit of pressure is the **Pascal**: $1\,\mathrm{Pa} = 1\,\mathrm{N\,m^{-2}}$. The mean pressure at terrestrial sea level is $1\,\mathrm{atm} = 1.013 \times 10^5\,\mathrm{Pa}$. Other common units of pressure are the **bar** ($1\,\mathrm{bar} = 10^5\,\mathrm{Pa}$) and the **Torr** ($760\,\mathrm{Torr} = 1\,\mathrm{atm}$).

EXERCISE 2.1— Water is nearly incompressible and has a density $\rho = 10^3\,\mathrm{kg\,m^{-3}}$. Solve eq. (2.1) to get an equation for the pressure as a function of depth in the ocean. How deep would you need to dive for the pressure to increase by $1\,\mathrm{atm} = 1.013 \times 10^5\,\mathrm{Pa}$? Does this agree with your experience?

Let's look at this in a bit more detail. Suppose we take our fluid layer to be thin, so that g is approximately constant. Then we can write equation (2.7) as

$$\int_{P_0}^{P(z)} \mathrm{d}P = -g \int_0^z \rho\,\mathrm{d}z.$$

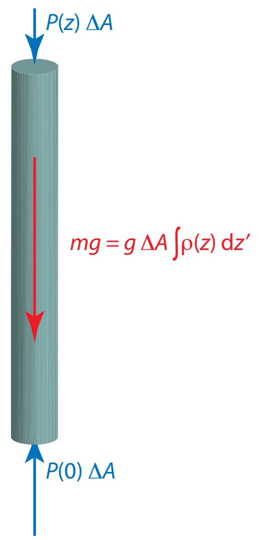

Figure 2.2: The mass of a column of fluid.

Now consider a cylinder of cross-section ΔA that extends from 0 to z. The mass of that cylinder is

$$m(z) = \Delta A \times \int_0^z \rho\,\mathrm{d}z.$$

and its weight is $m(z)g$.

The difference in pressure between the bottom and top of the cylinder is just

$$P_0 - P(z) = gm(z)/\Delta A,$$

that is, the weight per unit area of our column. Let's apply this to our atmosphere: if we take the top of our column to infinity and the pressure at the top to zero, then the pressure at the bottom (sea level) is just the weight of a column of atmosphere with a cross-sectional area of $1\,\mathrm{m}^2$.

The isothermal ideal gas

In general the density ρ depends on the pressure P and temperature T via an EQUATION OF STATE. Let's relax our condition of constant density, but keep gravity and temperature constant and assume the fluid is an ideal gas[2]. For N particles in a volume V at pressure and temperature P and T, the ideal gas equation of state is

$$PV = Nk_\mathrm{B}T. \tag{2.2}$$

In chemistry, it is convenient to count the number of particles by MOLES. One mole of a gas has $N_\mathrm{A} = 6.022 \times 10^{23}$ particles[3], and the number of moles in a sample is $n = N/N_\mathrm{A}$. If we divide and multiply equation (2.2) by N_A, then our ideal gas equation becomes

$$PV = n\,[N_\mathrm{A} k_\mathrm{B}]\,T \equiv nRT,$$

where $R = N_\mathrm{A} k_\mathrm{B} = 8.314\,\mathrm{J\,K^{-1}\,mol^{-1}}$ is the gas constant. This is perhaps the most familiar form of the ideal gas law—but it is not in a form useful to astronomers.

[2] By IDEAL gas, we mean that the particles are non-interacting; as a result, the energy of the gas only depends on the kinetic energy of the particles and in particular is independent of the volume.

[3] The constant N_A is known as AVOGADRO'S NUMBER.

We astronomers don't care about little beakers of fluid—we have whole stars to model! Put another way, volume isn't a useful quantity since we are working in the middle of a large mass of fluid. Instead, define the NUMBER DENSITY as the number of particles per volume, N/V. The mass of each particle is $\mathcal{A} \times m_u$, where m_u has a mass of one ATOMIC MASS UNIT.[4] Hence the mass per volume of our fluid is

$$\rho = \mathcal{A}m_u \times \frac{N}{V}.$$

We call ρ the MASS DENSITY, or density for short. This quantity appears in equation (2.1).

Starting with eq. (2.2), dividing by V and then multiplying and dividing by $\mathcal{A}m_u$ gives

$$P = \left(\frac{\mathcal{A}m_u N}{V}\right)\frac{k_B}{\mathcal{A}m_u}T \equiv \rho\frac{k_B}{\mathcal{A}m_u}T. \tag{2.3}$$

Equation (2.3) is the form most convenient for fluid dynamics, because it is in terms of an intrinsic fluid property—the density ρ—rather than in terms of the volume.

[4] $1\,\mathrm{u} = 1.661 \times 10^{-27}\,\mathrm{kg}$ is 1/12 of the mass of 1 mol of ^{12}C atoms in their ground state

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EXERCISE 2.2— Let's take a stab at modeling Earth's atmosphere with equation (2.1). Take Earth's atmosphere to be dry (no water, so we don't have to worry about condensation) and model it as an ideal gas. Also assume the temperature doesn't change with altitude. The average molecular mass of dry air is $\mathcal{A} = 28.97$. Integrate eq. (2.1) from $z = 0$, where $P(z = 0) = P_0$, to a height z. Show that the solution is $P(z) = P_0 e^{-z/H_P}$, where H_P is the PRESSURE SCALE HEIGHT—the height over which the pressure decreases by a factor $1/e$. Evaluate H_P for dry air at a temperature of 288 K (15 °C). Is the answer reasonable, based on your experience? Is the assumption of an isothermal atmosphere a good one? Explain why or why not.

The mass $\mathcal{A}$ of an atom or nuclei, when expressed in atomic mass units, is approximately equal to the atomic number A (Table 2.1). The electron mass is $m_e = 0.0005485\,\mathrm{u}$. Unless we need high accuracy, we can neglect the electron mass and take the mass of an atom or nuclide to be $A \times m_u$.

We denote an atomic isotope or nuclide as $^{A}\mathrm{El}$, where A is the atomic number (total number of neutrons and protons in the nucleus) and *El* is the element abbreviation (corresponds to number of protons in the nucleus).

Table 2.1: Selected atomic masses

nuclide	A	$\mathcal{A}$	$(\lvert\mathcal{A} - A\rvert/A)$ (%)
n	1	1.00865	0.865
^{1}H	1	1.00783	0.783
^{4}He	4	4.00260	0.065
^{12}C	12	12.00000	0.000
^{16}O	16	15.99491	0.032
^{28}Si	28	27.97693	0.082
^{56}Fe	56	55.93494	0.116

2.2 *Mass density and the mean molecular weight*

For a mixture with different types of particles, it is useful to introduce the MEAN MOLECULAR WEIGHT μ. This is computed by taking the total mass of a sample of particles and dividing by the total number of particles, so that

$$\mu = \frac{\rho}{m_u n_{tot}} = \frac{1}{m_u}\frac{\sum_i m_i n_i}{\sum_i n_i}. \tag{2.4}$$

Some examples may make this clearer. Suppose we star with molecular hydrogen (H_2). The mass of a sample of n molecules is $\approx\ 2\,m_u\ \times\ n$, since each molecule has 2 nucleons. The total number of particles in our sample is n, so

$$\mu(\mathrm{H}_2) = \frac{2m_\mathrm{u} \times n}{nm_\mathrm{u}} = 2.$$

Now suppose we raise the temperature and dissociate those molecules into individual atoms. For n atoms, the mass is $1\,\mathrm{u} \times n$, so $\mu = 1$. It becomes a little trickier when the atoms are ionized. Let's raise the temperature further, so that the gas ionizes into electrons and nuclei (protons). The electrons contribute negligibly to the mass, so if we have n atoms, the mass is still $1\,\mathrm{u} \times n$. The total number of particles has *doubled*, however, since for each atom there are now 2 particles (electron and nucleus). The mean molecular weight is therefore

$$\mu(^1\mathrm{H} + e^-) = \frac{nm_\mathrm{u}}{2nm_\mathrm{u}} = \frac{1}{2}.$$

In terms of the mean molecular weight, the ideal gas EOS becomes

$$P = \rho \left(\frac{k_\mathrm{B}}{\mu m_\mathrm{u}} \right) T. \tag{2.5}$$

EXERCISE 2.3— What is μ for a fully ionized ^{4}He gas ($A = 4$, with 2 electrons per atom)?

2.3 *The mass distribution*

Now let's look at how gravity varies within a star. Suppose we are at a distance r from the stellar center. Newton discovered that the gravitational force inside a spherical shell vanished. This means that the net gravitational force arising from portions of the star exterior to our position vanishes. The gravitational force depends only on the amount of mass interior to our position. This mass is

$$m(r) = 4\pi \int_0^r \rho(r) r^2 \, \mathrm{d}r, \tag{2.6}$$

with ρ being the mass density.

Furthermore, the gravitational force from a spherically symmetric mass is identical to that of a point particle of the same mass. Thus, the gravitational force at a radius r from the center is simply

$$g(r) = \frac{Gm(r)}{r^2}.$$

Using this expression for g, we can write eq. (2.1) as

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho \frac{Gm(r)}{r^2}. \tag{2.7}$$

EXERCISE 2.4 — What happens to $m(r)$ and $g(r)$ at the center ($r \to 0$)? Before doing any calculation, see if you can argue that $g(r) \to 0$ at the center. If this is so, then what is $\mathrm{d}P/\mathrm{d}r$ at the center? Assuming that $\rho \to \rho_c \approx$ const. near the center, integrate eq. (2.6) over a small radius Δr to get $m(\Delta r)$ and hence $g(\Delta r)$. Show explicitly that $g(\Delta r) \to 0$ as $\Delta r \to 0$.

To recap, we now have two differential equations, (2.6) and (2.7), that describe the structure of a star. These equations are for the pressure $P(r)$ and density $\rho(r)$ in the star. We can't solve these equations yet, however, because we don't have a relation between P and ρ. For example, the ideal gas equation of state relates P and ρ via a temperature T, so we need at least an equation for $T(r)$. Before doing that, let's see what we can learn from just these two equations; to proceed, we shall assume a simple form for $\rho(r)$, such as a constant density, and see what we can infer.

EXERCISE 2.5 — Let's suppose that ρ is constant throughout the star. In what follows, you should be able to express everything in terms of the star's mass M and radius R, along with physical constants such as G and k_{B}.

1. First, find ρ in terms of the total mass M and radius R.
2. Next, solve equation (2.6) to find $m(r)$ in terms of M and r/R.
3. Use this expression for $m(r)$ and your expression for ρ to integrate equation (2.7) and to find the pressure at the center, $P_c = P(r = 0)$.
4. Now that we have an expression for the central pressure in terms of M and R, let's try to understand what it means. Use your result from parts 1 and 3, as well as equation (2.5) to find the central temperature of the star, in terms of G, M, R, and the mean molecular weight of the gas μ. Evaluate T_c for $M = M_\odot$, $R = R_\odot$, and $\mu = 0.6$. Do you get a reasonable result?

2.4 *A closer look at hydrostatic equilibrium*

What would happen if the star fell out of equilibrium? Suppose we could turn off the pressure. If we did that, the star would implode. To understand how long this would take, let's calculate the amount of time a particle would need to free-fall from the surface to the center. We can get this from Kepler's law. Start with a circular orbit and deform it while keeping the center at one focus, as shown in Fig. 2.3.

The limit of these increasingly eccentric orbits is a fall into the center. The time is one-half of an orbital period, and the semi-major axis in this limiting case is $a = R/2$:

$$\tau_{\mathrm{ff}} = \frac{T}{2} = \frac{\pi}{\sqrt{GM}} \left(\frac{R}{2}\right)^{3/2}.$$

Notice that we have the combination $\sqrt{R^3/M}$. Let's convert this into an

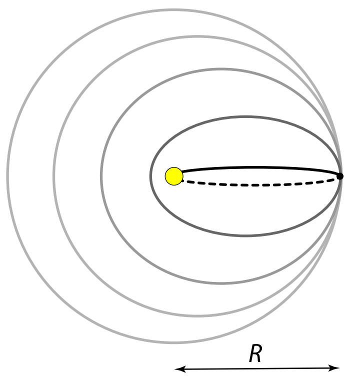


Figure 2.3: Deformation of an orbit until it becomes a fall to the center, denoted by the yellow dot.

expression involving the mean density $\bar{\rho}$:

$$\tau_{\rm ff} = \left(\frac{3}{32\pi}\right)^{1/2} \left(\frac{1}{G}\frac{4\pi R^3}{3M}\right)^{1/2} = \left(\frac{3}{32\pi}\right)^{1/2} \frac{1}{\sqrt{G\bar{\rho}}}. \qquad (2.8)$$

The time to collapse is proportional to $1/\sqrt{G\bar{\rho}}$ and depends on the average density of the star. We call $t_{\rm dyn} \equiv 1/\sqrt{G\bar{\rho}}$ the DYNAMICAL TIMESCALE of the star.

Let's avoid a collapse by turning the pressure back on. If part of the star is falling inward, the gas within the star will be compressed, the pressure will rise, and hydrostatic equilibrium will be restored. How quickly can the star respond? A change is pressure is communicated to the rest of the star by sound waves, which travel at a speed (see Box 2.1)

$$c_s = \left(\gamma\frac{P}{\rho}\right)^{1/2} = \left(\gamma\frac{k_{\rm B}T}{\mu m_{\rm u}}\right)^{1/2}. \qquad (2.9)$$

Here γ is the adiabatic index: for an ideal monatomic gas, $\gamma = 5/3$. How long would it take for a sound wave to go a distance R? Using the expression for the average temperature of a constant density sphere (cf. eq. [2.23]), we find

$$\tau_{\rm sc} = \frac{R}{c_s} = R\left(\frac{3R}{GM}\right)^{1/2} = \left(\frac{3}{2\sqrt{\pi}}\right)\frac{1}{\sqrt{G\bar{\rho}}}.$$

Both the sound-crossing time, $\tau_{\rm sc}$, and the free-fall time, $\tau_{\rm ff}$, are approximately equal to the dynamical timescale $1/\sqrt{G\bar{\rho}}$. This is another way of looking at hydrostatic equilibrium: the star is able to remain in balance because the time for pressure disturbances to propagate, $\tau_{\rm sc}$, is comparable to the time for large-scale motions of the fluid, $\tau_{\rm ff}$.

Box 2.1 The sound speed

Suppose we have a long tube filled with gas at pressure $P(x,t) = P_0$, density $\rho(x,t) = \rho_0$, and velocity $u(x,t) = U_0 = 0$. We then tap on one end of the tube; this causes a disturbance to propagate down the tube. Denote the cross-sectional area of the tube by A, and consider the volume $A\Delta x$ located between x and $x + \Delta x$; the mass in this small volume is $\rho A\,\Delta x$.

As a result of the disturbance, the pressure in the tube becomes $P(x,t) = P_0 + \sigma P_1(x,t)$. In this expression, σ is a bookkeeping parameter that we'll eventually set to unity. We will expand our equations and keep only terms that are linear in σ. The fluid will also acquire a velocity $u(x,t) = \sigma u_1(x,t)$. This compresses or rarifies the gas: $\rho(x,t) = \rho_0 + \sigma\rho_1(x,t)$. Because the pressure in the

Box 2.1 continued

tube is no longer uniform, our small mass will accelerate:

$$\begin{aligned} \rho A\,\Delta x \frac{\partial u}{\partial t} &= A\,\Delta x\,(\rho_0 + \sigma\rho_1)\frac{\partial(0 + \sigma u_1)}{\partial t} \\ &\approx \sigma\left[A\,\Delta x \rho_0 \frac{\partial u_1}{\partial t}\right] + \mathcal{O}(\sigma^2). \end{aligned}$$

The corresponding force on our mass is

$$A\left[P(x) - P(x+\Delta x)\right] \approx -\sigma A\left[P_1(x+\Delta x) - P_1(x)\right];$$

equating this to the expression—to order σ—for the acceleration, taking the limit $\Delta x \to 0$, and canceling common factors gives

$$\frac{\partial u_1}{\partial t} = -\frac{1}{\rho_0}\frac{\partial P_1}{\partial x}. \tag{2.10}$$

Because of the non-uniform velocity, the volume and hence density of our little mass will also change:

$$\frac{\partial V}{\partial t} = A\left[u(x+\Delta x) - u(x)\right] = \sigma A\,\Delta x\left[\frac{u_1(x+\Delta x) - u_1(x)}{\Delta x}\right]$$

or

$$\frac{1}{V}\frac{\partial V}{\partial t} = \sigma\frac{\partial u_1}{\partial x}. \tag{2.11}$$

This change in volume is related to the change in pressure. We are interested in fluctuations that are sufficiently quick that no heat is transferred into or out of our mass. This is an adiabatic process, for which $PV^\gamma = \text{const.}$. Here γ is called the adiabatic index; for an ideal gas, this is the ratio of specific heats, $\gamma = C_P/C_V$. (We shall discuss adiabatic processes again more thoroughly in chapter 6.)

As the pressure changes adiabatically from P_0 to σP_1, the volume changes as

$$\frac{\mathrm{d}V}{V} = \mathrm{d}\ln V = -\frac{1}{\gamma}\mathrm{d}\ln P.$$

Hence

$$\frac{\partial \ln V}{\partial t} = -\frac{1}{\gamma}\frac{\partial \ln(P_0 + \sigma P_1)}{\partial t} \approx -\frac{\sigma}{\gamma P_0}\frac{\partial \ln P_1}{\partial t} = \sigma\frac{\partial u_1}{\partial x}. \tag{2.12}$$

The last equality comes from equation (2.11).

We therefore have two equations for the perturbed velocity to order σ:

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial P_1}{\partial x} \\ \frac{\partial u_1}{\partial x} &= -\frac{1}{\gamma P_0}\frac{\partial P_1}{\partial t}; \end{aligned}$$

Box 2.1 continued

differentiating the top equation with respect to x and the bottom with respect to t, and equating the expressions for $\partial^2 u_1/\partial t \partial x$ gives

$$\frac{\partial^2 P_1}{\partial t^2} = \left(\frac{\gamma P_0}{\rho_0}\right) \frac{\partial^2 P_1}{\partial x^2}. \quad (2.13)$$

This is the equation for a wave: the solutions are $P_1(x,t) = P_1(x \pm c_s t)$, where the sound speed is $c_s = \sqrt{\gamma P/\rho}$.

For the sun, $\bar{\rho} = 1400\,\mathrm{kg\,m^{-3}} = 1.4\,\mathrm{g\,cm^{-3}}$; this is just a bit denser than you. The dynamical timescale for the sun is about one hour.

EXERCISE 2.6— The central temperature T_c is a measure of the average kinetic energy of a particle at the stellar center. Use the central temperature that you found for the constant density star in exercise 2.5 and estimate the time that such a particle would take to cross a distance R. How does this time compare to the orbital period of a satellite orbiting just outside the stellar surface?

2.5 Virial Equilibrium

With the assumption that $\rho = $ constant, we showed (exercise 2.5) that the central temperature and pressure depended on the total mass M, total radius R, and the gravitational constant G as

$$T_c = \frac{1}{2}\left\{\frac{GM}{R}\frac{\mu m_\mathrm{u}}{k_\mathrm{B}}\right\} \quad (2.14)$$

$$P_c = \frac{3}{8\pi}\left\{\frac{GM^2}{R^4}\right\}. \quad (2.15)$$

Our task now is to show that the scalings of T_c and P_c with M and R—the quantities in $\{\,\}$—hold in general for an a star in mechanical equilibrium.

To show this, we are going to employ a form of the *virial theorem*. Suppose we have a collection of N particles, all moving about and exerting forces on one another. If we let this system settle down into some kind of bound configuration, then we can add up the kinetic and potential energies of all the particles to get a total kinetic energy K and a total potential energy Ω. The virial theorem asserts that K is proportional to, and comparable in magnitude to, Ω; indeed if the potential between a pair of particles scales as r^{-1}, r being the distance between the particles, then $K = -\Omega/2$, as we'll now show.

Let us take the position and momentum of particle i to be $\boldsymbol{r}_i = (x_i, y_i, z_i)$ and $\boldsymbol{p}_i = (p_x, p_y, p_z)$. Then the total kinetic energy is

$$K = \frac{1}{2}\sum_{i=1}^{N} \boldsymbol{p}_i \cdot \frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t}$$

$$= \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N} \boldsymbol{p}_i \cdot \boldsymbol{r}_i\right) - \sum_{i=1}^{N} \boldsymbol{r}_i \cdot \frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t}\right]. \tag{2.16}$$

The quantity $G = \sum_i \boldsymbol{p}_i \cdot \boldsymbol{r}_i$ is called the "virial" of the system. By expressing the force $\boldsymbol{F}_i = \mathrm{d}\boldsymbol{p}_i/\mathrm{d}t$ on particle i as the gradient of a potential Ω, $\boldsymbol{F}_i = -\boldsymbol{\nabla}_i\Omega$, we can rewrite eq. (2.16) as

$$2K = \frac{\mathrm{d}G}{\mathrm{d}t} + \sum_{i=1}^{N} \boldsymbol{r}_i \cdot \boldsymbol{\nabla}_i\Omega. \tag{2.17}$$

So far, we've just shuffled and relabeled terms. The crucial step comes in taking the time-average of the kinetic energy, which we'll denote by $\langle\ \rangle$:

$$\langle f\rangle \equiv \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^{\tau} f(t)\,\mathrm{d}t.$$

Applying this to equation (2.17) gives

$$\begin{aligned}
2\langle K\rangle &= \left\langle \frac{\mathrm{d}G}{\mathrm{d}t}\right\rangle + \left\langle \sum_{i=1}^{N} \boldsymbol{r}_i \cdot \boldsymbol{\nabla}_i\Omega\right\rangle \\
&= \lim_{\tau\to\infty}\left[\frac{1}{\tau}\int_0^{\tau}\frac{\mathrm{d}G}{\mathrm{d}t}\,\mathrm{d}t\right] + \left\langle \sum_{i=1}^{N} \boldsymbol{r}_i \cdot \boldsymbol{\nabla}_i\Omega\right\rangle \\
&= \underbrace{\lim_{\tau\to\infty}\left[\frac{G(\tau)-G(0)}{\tau}\right]}_{\mathrm{I}} + \underbrace{\left\langle \sum_{i=1}^{N} \boldsymbol{r}_i \cdot \boldsymbol{\nabla}_i\Omega\right\rangle}_{\mathrm{II}}
\end{aligned}$$

Now, if the system is bound and in mechanical equilibrium, then the positions and momenta of all particles are finite: none of the particles can escape, and the system doesn't violently collapse so that momenta are diverging. Hence both $G(\tau)$ and $G(0)$ are finite numbers, so as $\tau \to \infty$, term I vanishes.

As for term II, we can show that if the potential between pairs of particles depends on $1/r$, where r is the distance between those particles, then term II is just $-\Omega$ (see Box 2.2). For now, I'll give a rough argument of why this is so: in a spherically symmetric system, then the potential just depends on the distance r from the origin; and since

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{1}{r},$$

the last term is just $-\Omega$ and our equation is

$$2\langle K\rangle + \langle\Omega\rangle = 0. \tag{2.18}$$

This is the virial theorem, applied to a r^{-1} potential.

Box 2.2 Working with vectors

In this sidebar we'll show that the second term in equation (2.17) is

$$\sum_{i=1}^{N} \boldsymbol{r}_i \cdot \boldsymbol{\nabla}_i \Omega = -\Omega. \tag{2.19}$$

First, we need an expression for Ω. Suppose we pick a pair of particles, i and k. The potential between this pair is

$$-\frac{Gm_i m_k}{r_{ik}} = -\frac{Gm_i m_k}{\sqrt{(\boldsymbol{r}_i - \boldsymbol{r}_k)^2}}.$$

Our total potential consists of a sum over the potentials between all $N(N-1)/2$ unique pairs of particles,

$$\Omega = -\frac{Gm_1 m_2}{\sqrt{(\boldsymbol{r}_1 - \boldsymbol{r}_2)^2}} - \ldots - \frac{Gm_i m_k}{\sqrt{(\boldsymbol{r}_i - \boldsymbol{r}_k)^2}} - \ldots$$

When we take the derivative in eq. (2.19), we apply $\boldsymbol{r}_i \cdot \boldsymbol{\nabla}_i$ to each term in the potential. For the term with the pair i, k, this will give

$$\begin{aligned}
&\sum_{i=1}^{N} \boldsymbol{r}_i \cdot \boldsymbol{\nabla}_i \left(-\frac{Gm_i m_k}{\sqrt{(\boldsymbol{r}_i - \boldsymbol{r}_k)^2}} \right) \\
&\quad = \quad -Gm_i m_k \left[\boldsymbol{r}_i \cdot \boldsymbol{\nabla}_i \left(\frac{1}{\sqrt{(\boldsymbol{r}_i - \boldsymbol{r}_k)^2}} \right) + \boldsymbol{r}_k \cdot \boldsymbol{\nabla}_k \left(\frac{1}{\sqrt{(\boldsymbol{r}_i - \boldsymbol{r}_k)^2}} \right) \right].
\end{aligned}$$

Since many of you aren't yet comfortable with vector expressions, we'll do this in detail for the x-component:

$$\begin{aligned}
&\left[\boldsymbol{r}_i \cdot \boldsymbol{\nabla}_i \left(\frac{1}{\sqrt{(\boldsymbol{r}_i - \boldsymbol{r}_k)^2}} \right) + \boldsymbol{r}_k \cdot \boldsymbol{\nabla}_k \left(\frac{1}{\sqrt{(\boldsymbol{r}_i - \boldsymbol{r}_k)^2}} \right) \right]_x \\
&\quad = x_i \frac{\partial}{\partial x_i} \left(\frac{1}{\sqrt{(\boldsymbol{r}_i - \boldsymbol{r}_k)^2}} \right) + x_k \frac{\partial}{\partial x_k} \left(\frac{1}{\sqrt{(\boldsymbol{r}_i - \boldsymbol{r}_k)^2}} \right) \\
&\quad = -\frac{x_i (x_i - x_k)}{(\boldsymbol{r}_i - \boldsymbol{r}_k)^{3/2}} + \frac{x_k (x_i - x_k)}{(\boldsymbol{r}_i - \boldsymbol{r}_k)^{3/2}} \\
&\quad = -\frac{(x_i - x_k)^2}{(\boldsymbol{r}_i - \boldsymbol{r}_k)^{3/2}}
\end{aligned}$$

The y- and z-components are similar, giving

$$\begin{aligned}
\sum_{i=1}^{N} \boldsymbol{r}_i \cdot \boldsymbol{\nabla}_i \left(-\frac{Gm_i m_k}{\sqrt{(\boldsymbol{r}_i - \boldsymbol{r}_k)^2}} \right) &= Gm_i m_k \frac{(\boldsymbol{r}_i - \boldsymbol{r}_k)^2}{(\boldsymbol{r}_i - \boldsymbol{r}_k)^{3/2}} \\
&= -\left(-\frac{Gm_i m_k}{\sqrt{(\boldsymbol{r}_i - \boldsymbol{r}_k)^2}} \right).
\end{aligned}$$

This can be done for every term in the sum, with the final result that

$$\sum_{i=1}^{N} \boldsymbol{r}_i \cdot \boldsymbol{\nabla}_i \Omega = -\Omega.$$

For an ideal monatomic gas in thermal equilibrium, the mean kinetic energy of a particle in the gas is $K = (3/2)k_\mathrm{B}T$, and we therefore may define an average temperature

$$2K = 3Nk_\mathrm{B}\bar{T} = -\Omega. \qquad (2.20)$$

The total number of particles is $N = M/(\mu m_\mathrm{u})$, and so

$$\bar{T} = -\frac{1}{3}\Omega\frac{\mu m_\mathrm{u}}{Mk_\mathrm{B}}. \qquad (2.21)$$

The total potential of the system depends on only three parameters: G, M, and R. The only way to make a quantity having dimensions of energy is for

$$\Omega \propto -\frac{GM^2}{R},$$

and so

$$\bar{T} \propto \frac{GM}{R}\frac{\mu m_\mathrm{u}}{k_\mathrm{B}}.$$

By using the ideal gas law, $\bar{P} = \bar{\rho}(k_\mathrm{B}/\mu m_\mathrm{u})\bar{T}$, we find

$$\bar{P} \propto \frac{GM^2}{R^4}.$$

As a concrete example, let's compute Ω for a constant density sphere. If we bring a small amount of mass $\mathrm{d}m$ from infinity onto a sphere of mass m and radius r, then the change in potential is

$$\mathrm{d}\Omega = -\frac{Gm}{r}\,\mathrm{d}m.$$

For a constant density, $r = R(m/M)^{1/3}$; upon substituting for r we have

$$\Omega_{\text{const. den.}} = -\int_0^M \frac{GM^{1/3}m^{2/3}}{R}\,\mathrm{d}m = -\frac{3}{5}\frac{GM^2}{R}. \qquad (2.22)$$

Using this in equation (2.21) gives us the mean temperature, and hence pressure, for a constant density sphere,

$$\bar{T} = \frac{1}{5}\frac{GM}{R}\frac{\mu m_\mathrm{u}}{k_\mathrm{B}}, \qquad (2.23)$$

$$\bar{P} = \frac{3}{20\pi}\frac{GM^2}{R^4}. \qquad (2.24)$$

These are comparable to the central values, eqn. (2.14) and (2.15).

	B2	B8	F0	F5	G5	M0	M7
$M/M_\odot$	9.8	3.8	1.6	1.3	0.92	0.51	0.12
$R/R_\odot$	5.6	3.0	1.5	1.3	0.92	0.60	0.18
$L/L_\odot$	5800.0	180.0	6.5	3.2	0.79	0.08	0.003

Table 2.2: Masses, radii, and luminosities for selected stellar types. The type—B2, B8, F0, and so forth—indicates what features are present in the star's spectrum and indicates the star's surface effective temperature.

EXERCISE 2.7 — We can infer a great deal from our simple virial scalings. Table 2.2 provides masses, radii, and luminosities, in units of $M_\odot$, $R_\odot$, and $L_\odot$, for stars from type B (hot blue stars) to type M (cool red stars). Using the constant density model, compute $\rho/\rho_\odot$, $T_c/T_{c,\odot}$, and $P_c/P_{c,\odot}$. You should find that each quantity depends only on $m = M/M_\odot$ and $r = R/R_\odot$. Describe your findings: do $P_c/P_{c,\odot}$, $\rho/\rho_\odot$, and $T_c/T_{c,\odot}$ vary in a similar fashion? If not, how do they change with stellar type?

2.6 *Contraction to the main sequence*

Stars are born when a cold, dense[5] cloud of gas and dust becomes unstable to gravitational collapse. The details of this process is a topic of current research; for our purposes, however, after a period of time a pre-main sequence star forms. This object is in hydrostatic balance, but with a radius much larger than its main-sequence value and a cool central temperature. What happens to this object?

[5] Dense is a relative term; here we mean ~ 100 *atoms* per cubic centimeter

The pre-main sequence star is in hydrostatic balance, so it doesn't collapse. But the interior, and hence the surface, is warm, so it radiates energy. The only source of energy is gravitational, so the pre-main sequence star *must* contract. How long would this take? For our sun, the total energy is

$$E_\odot = K + \Omega = \Omega/2 \approx -\frac{GM_\odot^2}{R_\odot};$$

the time to radiate this energy away is

$$t_{\mathrm{KH}} = \frac{|E_\odot|}{L_\odot} \approx \frac{GM_\odot^2}{R_\odot L_\odot} \approx 3 \times 10^7\,\mathrm{yr}. \tag{2.25}$$

This timescale is called the KELVIN-HELMHOLTZ TIMESCALE. Since $t_{\mathrm{KH}} \gg t_{\mathrm{dyn}} = (G\bar{\rho})^{-1/2}$ the star is, to an excellent approximation, in hydrostatic equilibrium throughout the whole contraction.

EXERCISE 2.8 — Using the constant density model (constant here means "constant throughout the star at any given time") of exercise 2.5 and the virial relations, give a qualitative sketch for how the pressure, density, temperature, radius, and total energy change with time as the protostar contracts.

EXERCISE 2.9 — Using the constant density model, derive an expression for the total energy (kinetic plus potential) as a function of central temperature. Plot this relation. What happens to the central temperature if additional heat is injected into the star?

EXERCISE 2.10— Let's examine the behavior of a thin layer at the surface of our constant density model. The pressure on the upper surface of our layer vanishes, and the pressure on the bottom surface of our layer is $P(R)$.

1. As a mathematical preliminary, suppose we have a function $f(x) = Ax^{\alpha}$ and that we expand about a point x_0 with $f_0 = Ax_0^{\alpha}$. Show that to lowest order in δx,
$$f(x_0 + \delta x) \approx f_0 \left(1 + \alpha \frac{\delta x}{x}\right).$$
2. Write $\mathrm{d}P/\mathrm{d}r$ as $\Delta P/\Delta r$, where Δr is the thickness of the layer, and then integrate the equation of hydrostatic balance (2.7) over the surface to show that
$$4\pi R^2 P(R) - \frac{GMm}{R^2} = 0, \tag{2.26}$$
where $m = 4\pi R^2 \rho \Delta r$ is the mass of the layer. For the rest of this exercise, we shall take m as fixed.
3. Now suppose of star expands by a small amount δR. Use the result of part 1 to find the new density ρ' in terms of the original density ρ and δR, to lowest order in $\delta R/R$.
4. If the contraction is adiabatic, then the new pressure obeys a relation $P'V'^{\gamma} = PV^{\gamma}$ (cf. Box 2.1). For our shell of mass m, show that this implies that $P'\rho'^{-\gamma} = P\rho^{-\gamma}$, and thus find P' in terms of P, $\delta R/R$, and γ, to lowest order in $\delta R/R$.
5. Insert the expressions for P' and $R' = R + \delta R$ into equation (2.26) and cancel any common factors; you should find that the pressure and gravitational forces no longer balance. Express the residual force in terms of GMm/R^2, γ and $\delta R/R$.
6. Equate this residual force with the acceleration of the shell, $m\ddot{\delta R}$, and show that the shell oscillates. For $\gamma = 5/3$ (appropriate for an ideal gas), find the period of oscillation in terms of $\rho = 3M/4\pi R^3$.

3
Edge of Darkness

We saw in chapter 2 that the equilibrium central temperature of a self-gravitating object—such as a star—with an ideal gas EOS depends *solely* on the mass, radius, and composition of that star. For the Sun, this temperature is $\approx$ 15 MK and is much higher than the surface effective temperature $T_{\mathrm{eff},\odot} = 5780\,\mathrm{K}$. The photons emitted from the Sun are therefore coming just from the cooler surface layers.

PHOTONS IN A PLASMA, SUCH AS IN THE INTERIOR OF THE SUN, TRANSPORT ENERGY. Were the sun transparent, these photons would immediately stream out, and the sun would release its stored energy in a fiery blast. This doesn't happen: a photon can only travel a short distance before being scattered or absorbed. The net effect is that radiation generated in the core must travel a tortuous path, rather like a pinball, before reaching the surface and escaping.

3.1 *Interaction of radiation and matter*

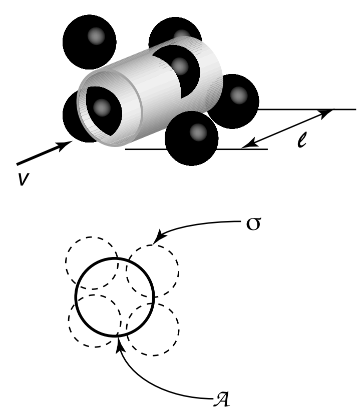


Figure 3.1: Schematic of a particle incident on a group of scattering or absorbing particles.

How far does a photon—or any particle, for that matter—travel, on average, in the interior of the sun? Imagine a particle traveling with speed v. Draw a cylinder, of length ℓ and cross-sectional area $\mathcal{A}$, around its path, as shown in Fig. 3.1. What the particle "sees" is that the cylinder is partly blocked by obstacles—other particles in its path.

What is the probability of our particle making it through the cylinder unscathed? The probability of the particle hitting an obstacle is the ratio

$$\mathcal{P} = \frac{\text{total area covered by obstacles}}{\text{area of cylinder}}$$

Denote the cross-sectional area of the other particles by σ. If the density of obstacles is n, then the number of obstacles in the cylinder is $n \times (\mathcal{A}\ell)$, and therefore the fraction of the area blocked by the obstacles is

We are taking ℓ and $\mathcal{A}$ sufficiently small that we don't have to worry about particles overlapping.

$$\mathcal{P} = \frac{n \times (\mathcal{A}\ell) \times \sigma}{\mathcal{A}} = n\sigma\ell. \tag{3.1}$$

The particle will suffer a collision when $\mathcal{P} \to 1$, or when

$$\ell = \frac{1}{n\sigma}. \tag{3.2}$$

We call ℓ the “mean free path”: it is the mean distance the particle travels freely before colliding.

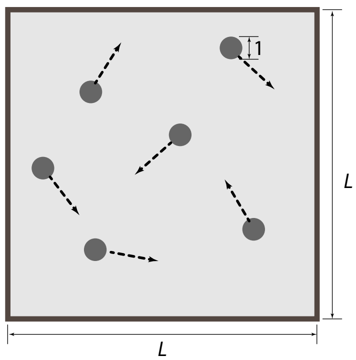


Figure 3.2: Schematic for Exercise 3.1

EXERCISE 3.1 — Suppose we have a flat, slippery surface on which hockey pucks are sliding around, as shown in Fig. 3.2. The pucks bounce off the walls as they slide around. Suppose there are N pucks, each with unit diameter, and the table is square with sides of length L. Estimate the mean free path of a puck.

Although we have motivated this derivation with a classical picture, the cross-section σ is just related to the probability of an interaction and can therefore be defined for quantum mechanical systems.

EXERCISE 3.2 — In the sun, free electrons scatter photons; the cross-section for this is

$$\sigma_{\mathrm{Th}} = \left(\frac{8\pi}{3}\right)\left(\frac{e^2}{m_e c^2}\right)^2 = 6.65 \times 10^{-29}\,\mathrm{m}^2.$$

What is the mean free path against this process for a photon at the average density of the solar interior?

As the ray of light traverses a small distance Δs through some matter, the probability of a photon being absorbed is $\mathcal{P} = n\sigma\Delta s$. Thus, out of every N photons, $\Delta N = N \times \mathcal{P} = N \times n\sigma\Delta s$ are absorbed. Since the intensity I_ν is proportional to the number of photons, the change in intensity is just

$$\Delta I_\nu = -n\sigma I_\nu \Delta s.$$

Taking the limit $\Delta s \to 0$, we obtain an equation for the absorption of light,

$$\left.\frac{\mathrm{d}I_\nu}{\mathrm{d}s}\right|_{\mathrm{absorption}} = -n\sigma I_\nu. \tag{3.3}$$

Rather than work with the microscopic cross-section, it is convenient to define the ABSORPTION OPACITY,

$$\kappa_\nu^{\mathrm{abs}} = \frac{n\sigma}{\rho},$$

so that $\mathrm{d}I_\nu/\mathrm{d}s = -\rho\kappa_\nu^{\mathrm{abs}} I_\nu$. The units of opacity are cm^2/g. We use a subscript ν to indicate that the opacity is a function of frequency. In terms of the opacity, the photon mean free path is $\ell = (\rho\kappa_\nu^{\mathrm{abs}})^{-1}$.

EXERCISE 3.3— A ray of light crosses a slab of absorbent material. Calculate the intensity I_ν as a function of distance traveled. Your expression should be in terms of ρ and $\kappa_\nu^{\mathrm{abs}}$. How far does the ray go before its intensity has dropped to $1/e$ of its original value?

IN ADDITION TO ABSORBING PHOTONS, THE MATTER CAN ALSO SPONTANEOUSLY EMIT THEM. Denote the power emitted per wavelength per volume per angle by ρj_ν. After traveling a distance Δs through matter with this EMISSIVITY, the ray will increase in intensity by $\rho j_\nu \Delta s$; or

$$\left.\frac{\mathrm{d}I_\nu}{\mathrm{d}s}\right|_{\mathrm{emission}} = \rho j_\nu. \tag{3.4}$$

EXERCISE 3.4— Suppose a ray traverses matter that both absorbs (opacity $\kappa_\nu^{\mathrm{abs}}$) and emits (emissivity j_ν), so that

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = \rho j_\nu - \rho\kappa_\nu^{\mathrm{abs}} I_\nu.$$

Solve for $I_\nu(s)$, and show that as $s \to \infty$, $I_\nu \to j_\nu/\kappa_\nu^{\mathrm{abs}}$.

FINALLY, THE MATTER CAN ALSO SCATTER LIGHT. This removes photons from a ray, similar to absorption, but it also adds them into a ray propagating in a different direction. If we assume that the direction into which the photon is scattered is random and isotropic (as is most often the case), then if the intensity in our ray is greater than the angle-average J_ν, scattering will cause a net reduction in intensity as more photons are scattered out of the ray than are scattered into it. Conversely, if $I_\nu < J_\nu$, then more photons will be scattered into the ray than out of it. Thus, the effect of scattering can be described via

$$\left.\frac{\mathrm{d}I_\nu}{\mathrm{d}s}\right|_{\mathrm{scattering}} = -\rho\kappa_\nu^{\mathrm{sca}} \left(I_\nu - J_\nu\right). \tag{3.5}$$

3.2 *The equation of radiative transfer*

Combining our expressions for absorption, emission, and scattering gives us the full expression for how the intensity changes along a ray,

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\rho\left(\kappa_\nu^{\mathrm{abs}} + \kappa_\nu^{\mathrm{sca}}\right) I_\nu + \rho j_\nu + \rho\kappa_\nu^{\mathrm{sca}} J_\nu. \tag{3.6}$$

This is a complicated INTEGRODIFFERENTIAL equation: it contains both the derivative $\mathrm{d}I_\nu/\mathrm{d}s$ of the intensity as well as its integral $J_\nu = (4\pi)^{-1}\int I_\nu\,\mathrm{d}\Omega$.

In general, eq. (3.6) must be solved numerically; but conditions in the deep interior of the star and near the surface allow us to make simplifying approximations and to obtain a solution that gives some insight into the physics. First, let's clean up the equation: divide through by $\rho\kappa_\nu \equiv \rho(\kappa_\nu^{\mathrm{abs}} + \kappa_\nu^{\mathrm{sca}})$,

$$\frac{1}{\rho\kappa_\nu}\frac{\partial I_\nu}{\partial s} = -I_\nu + \left[\frac{j_\nu + \kappa_\nu^{\mathrm{sca}} J_\nu}{\kappa_\nu}\right].$$

Next, define a new quantity, the OPTICAL DEPTH τ_ν via the equation

$$\frac{\partial \tau_\nu}{\partial s} = \rho\kappa_\nu = \rho(\kappa_\nu^{\mathrm{abs}} + \kappa_\nu^{\mathrm{sca}}),$$

which allows us to change variables, $\mathrm{d}I_\nu/\mathrm{d}s = (\mathrm{d}I_\nu/\mathrm{d}\tau_\nu)\cdot(\mathrm{d}\tau_\nu/\mathrm{d}s)$; and finally define the SOURCE FUNCTION S_ν as the term in $[\cdot]$. Doing all that gives us the simpler-looking equation,

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = -I_\nu + S_\nu.$$

This prettifying doesn't advance us any closer to the solution, but notice! The optical depth has a simple meaning:

$$\tau_\nu = \int_0^s \rho\kappa_\nu\,\mathrm{d}s = \int_0^s n\sigma_\nu\,\mathrm{d}s = \int_0^s \frac{\mathrm{d}s}{\ell}.$$

That is, the optical depth measures distance along the ray in units of mean free path. Said differently, if you have traveled one optical depth, then you have gone one mean free path. We also see from this equation that $I_\nu \to S_\nu$ as $\tau_\nu \to \infty$ (cf. exercise 3.4).

Thus, if your object has $\tau_\nu \ll 1$, then photons are hardly affected by the medium and the object is nearly transparent; if, on the other hand, $\tau_\nu \gg 1$, then photons cannot go through the object: it is opaque, and the emission is determined by the emissivity (via the source function S_ν) of the matter.

EXERCISE 3.5— For the electron scattering cross-section (Exercise 3.2), estimate the optical depth between the solar center and the solar photosphere.

SUPPOSE WE ARE IN A CAVITY IN WHICH THE RADIATION AND MATTER ARE IN A STEADY-STATE. The matter is not gaining or losing energy to the radiation. This requires balancing

$$(\text{energy emitted per unit volume}) = \rho\int j_\nu\,\mathrm{d}\nu\,\mathrm{d}\Omega$$

with

$$(\text{energy absorbed per unit volume}) = \rho\int \kappa_\nu^{\mathrm{abs}} I_\nu\,\mathrm{d}\nu\,\mathrm{d}\Omega,$$

so that
$$\int_0^\infty \left(j_\nu - \kappa_\nu^{\mathrm{abs}} J_\nu\right)\,\mathrm{d}\nu = 0. \tag{3.7}$$
We don't include scattering in this expression because scattering doesn't transfer energy between the radiation and the gas.

If, in addition, the matter and radiation are in thermal equilibrium, so that $J_\nu = B_\nu$, then eq. (3.7) implies that
$$\frac{j_\nu}{\kappa_\nu^{\mathrm{abs}}} = B_\nu(T). \tag{3.8}$$
Now j_ν and $\kappa_\nu^{\mathrm{abs}}$ are properties of the matter, and do not depend on the state of the radiation field. Hence, equation (3.8) must hold whenever the matter is in equilibrium, *regardless of the state of the radiation field*.

3.3 *Radiative diffusion*

We can now examine how heat transport works in the deep interior of a star. First, we need to adjust our coordinates. In equation (3.6), the coordinate s is distance along a ray; but we are considering many different rays. We shall therefore use radial distance r as a coordinate, and measure the optical depth it: $\mathrm{d}\tau_\nu = \rho\kappa_\nu\,\mathrm{d}r$. Since $\mathrm{d}r = \mu\,\mathrm{d}s$, where $\mu = \cos\theta$ is the cosine between ds and dr (Fig. 3.3), the equation of transfer becomes
$$\mu\frac{\mathrm{d}I_\nu}{\mathrm{d}r} = -\rho\kappa_\nu\left(I_\nu - S_\nu\right). \tag{3.9}$$

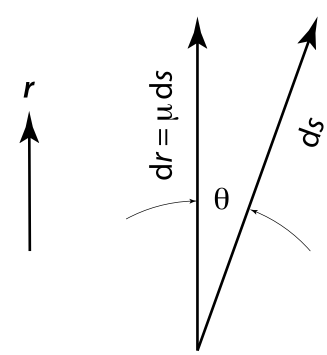


Figure 3.3: Schematic of the coordinate system used for solving the radiative transport equation.

LET'S EXAMINE THE TYPICAL SCALES OF TERMS IN THE RADIATIVE TRANSFER EQUATION, FOR CONDITIONS IN THE DEEP SOLAR INTERIOR. We'll start with eq. (3.6), and indicate some expected scales:
$$\underbrace{\mu\frac{\mathrm{d}I_\nu}{\mathrm{d}r}}_{\sim I_\nu/R_\odot} = -\underbrace{\rho\kappa_\nu I_\nu}_{\sim I_\nu/\ell} + \rho\kappa_\nu S_\nu.$$

If we are far from the surface of the star, then we should expect the intensity to change over lengthscales comparable to $R_\odot$. Of course, it won't be exactly this, but—as we'll show—the exact value doesn't matter so long as $|\mathrm{d}I_\nu/\mathrm{d}s|$ is in the ballpark of $I_\nu/R_\odot$. Notice the enormous disparity in scales:
$$\frac{|\mathrm{d}I_\nu/\mathrm{d}r|}{\rho\kappa_\nu I_\nu} \sim \frac{\ell}{R_\odot};$$
the left-hand side is smaller than the terms on the right by the ratio of the mean free path to the solar radius. This implies that conditions are nearly homogeneous. They are also isotropic, so that $I_\nu = J_\nu$. We expect that collisions are fast enough so that the matter is in thermal equilibrium and $j_\nu = \kappa_\nu^{\mathrm{abs}} B_\nu$. We also know from exercise 3.4 that $I_\nu \to j_\nu/\kappa_\nu^{\mathrm{abs}} = B_\nu$.

We can't have $I_\nu = B_\nu$ exactly, however, since in that case there is no net flux! We'll treat the intensity as being thermal plus a perturbation:

$$I_\nu = B_\nu + I_\nu^{(1)},$$

where the superscript "(1)" indicates that this is a small correction. Inserting this expansion into eq. (3.6) and keeping only the lowest-order terms on each side gives

$$I_\nu^{(1)} = -\mu \frac{\mathrm{d}B_\nu}{\mathrm{d}\tau_\nu}. \tag{3.10}$$

B_ν depends on the temperature T, so $\mathrm{d}B_\nu/\mathrm{d}\tau_\nu = \mathrm{d}B_\nu/\mathrm{d}T \cdot \mathrm{d}T/\mathrm{d}\tau_\nu$. To get the flux, multiply eq. (3.10) by μ and integrate over angles:

$$F_\nu = \int \mu I_\nu^{(1)}\,\mathrm{d}\Omega = -\int \mu^2 \frac{\mathrm{d}B_\nu}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}\tau_\nu}\,\mathrm{d}\Omega = -\frac{4\pi}{3}\frac{\mathrm{d}B_\nu}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}\tau_\nu}.$$

We can switch back to the radial coordinate:

$$F_\nu = -\frac{4\pi}{3}\left[\frac{1}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial T}\right]\frac{\mathrm{d}T}{\mathrm{d}r}. \tag{3.11}$$

The term in $[\cdot]$ controls which frequencies are most responsible for energy transport.

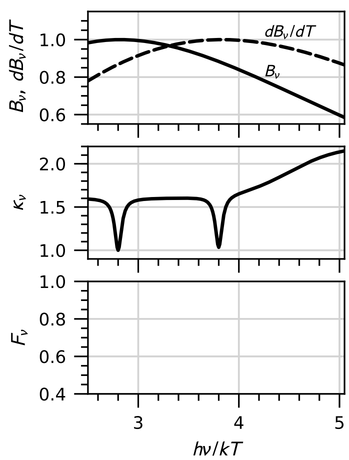


Figure 3.4: The specific flux for a hypothetical opacity

EXERCISE 3.6 — Let's examine the term $[\cdot]$ in eq. (3.11) more closely. Fig. 3.4 shows B_ν and $\mathrm{d}B_\nu/\mathrm{d}T$ (top panel) and a hypothetical κ_ν (middle). Sketch F_ν on the bottom panel. For which frequencies is it maximum?

To get the total flux, we integrate F_ν over all frequencies.

$$\begin{aligned} F = \int_0^\infty F_\nu\,\mathrm{d}\nu &= -\frac{4\pi}{3}\left[\int_0^\infty \frac{1}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial T}\right]\frac{\mathrm{d}T}{\mathrm{d}r} \\ &\equiv -\frac{4\pi}{3}\frac{1}{\rho\kappa_\mathrm{R}}\frac{\partial}{\partial T}\left[\int_0^\infty B_\nu\,\mathrm{d}\nu\right]\frac{\mathrm{d}T}{\mathrm{d}r}. \end{aligned}$$

Here we've defined the ROSSELAND MEAN of the opacity:

$$\frac{1}{\kappa_\mathrm{R}} = \left(\int_0^\infty \frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu\right)^{-1}\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu. \tag{3.12}$$

This is done to put the equation in more familiar terms. Since (cf. eq. [1.6]) $\int B_\nu\,\mathrm{d}\nu = \sigma_\mathrm{SB}T^4/\pi = caT^4/4\pi$, we can write the equation for the flux as

$$F = -\frac{1}{3}\frac{c}{\rho\kappa_\mathrm{R}}\frac{\mathrm{d}}{\mathrm{d}r}aT^4. \tag{3.13}$$

Equation (3.13) is known as the equation for radiative diffusion, for reasons that will become apparent in the next section.

If we multiply the flux by the surface area of a shell in the star we obtain the luminosity $L = 4\pi r^2 F$; we can therefore recast eq. (3.13) into an equation for the thermal gradient:

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3\rho\kappa_R}{4acT^3}\frac{L(r)}{4\pi r^2}. \tag{3.14}$$

EXERCISE 3.7 — Let's dissect eq. (3.13) to see how it sets the luminosity.

1. To keep the algebra simple, assume that F is constant throughout the star and that aT^4 is linear in r—that is, $aT^4 = aT_c^4(1 - r/R)$. Since F is constant, you can express it in terms of the luminosity at the surface L. Use this to transform eq. (3.13) into an expression for L in terms of R and T_c (along with ρ, κ_R, and c).
2. Write the luminosity as $L = E_\gamma/\tau$, where E_γ is the total radiative energy of the star, and τ some as-yet-undetermined *diffusion timescale*. Give an estimate of E_γ in terms of the mean temperature T and the radius R of the star.
3. Finally, assume that the photon mean free path $\ell = (\rho\kappa_R)^{-1}$ is constant. Substitute the results from parts 1 and 2 into equation (3.13). After simplifying, you should end up with a simple expression for τ in terms of c, R, and ℓ. For Thomson scattering, what is τ (express in years)?

3.4 *Diffusion*

In the presence of scattering or absorption, the photons crossing the face can travel one mean free path ℓ. Imagine a small cube with sides of length ℓ and filled with photons. The total radiant energy in the cube is ΔE. In a time $\Delta t = \ell/c$, all of the energy will leave this cube. The total luminosity is $\Delta E/\Delta t = c\Delta E/\ell$. If everything is isotropic, then the flux out of any one face is $1/6$ of the luminosity, divided by the area of that face:

$$F = \frac{1}{6\ell^2}\frac{c\Delta E}{\ell} = \frac{1}{6}cU,$$

where $U = E/\ell^3$ is the radiative energy density.

Now place two of these cubes against one another, with their common face located at position x. The energy density of the two cubes need not be the same; the energy density of the left cube is $U(x-\ell)$ and of the right cube is $U(x+\ell)$ (see Fig. 3.5). The net flux traveling in the x-direction through the common face is then

$$F = \frac{1}{6}cU(x-\ell) - \frac{1}{6}cU(x+\ell) \approx -\frac{1}{3}c\ell\frac{\mathrm{d}U}{\mathrm{d}x}.$$

This is an expression for a DIFFUSIVE FLUX. Although we gave a heuristic explanation, the formula is in general true:

$$\begin{aligned}(\text{flux of something}) \quad = \quad & -\frac{1}{3} \times (\text{speed of carriers}) \times (\text{MFP of carriers}) \\ & \times \nabla(\text{density of something}) \qquad (3.15)\end{aligned}$$

For radiation, the "something" is "radiative energy" and the carriers are photons.

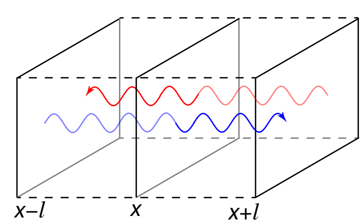


Figure 3.5: Illustration of net flux crossing a face between regions with slightly different energy densities.

EXERCISE 3.8— Compare this crude diffusion model,

$$F = -\frac{1}{3} c\ell \frac{\mathrm{d}U}{\mathrm{d}r},$$

with eq. (3.13). Using the results of exercise 3.6, give a succinct description for why the effective mean free path $\ell = 1/(\rho\kappa_{\mathrm{R}})$ is computed using a weighting function $\partial B_\nu / \partial T$.

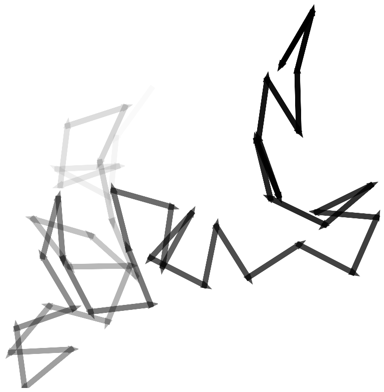


Figure 3.6: Schematic of a random walk of 50 steps.

To keep things simple, we'll imagine that after absorption the atom immediately emits an identical photon in a random direction.

AS AN ALTERNATE TAKE ON OUR ESTIMATION, LET'S MODEL THE TRANSPORT AS A PHOTON THAT IS RANDOMLY WALKING THROUGHOUT THE INTERIOR. The photon moves at speed c, but it can only go one mean free path ℓ before being absorbed or scattered, at which point it is sent off in a random direction. The path of the photon will therefore look something like that in Fig. 3.6.

We will just do our calculation for motion along a diameter, with the photon starting at the center. At each hop, the photon either goes left or right with equal probability. On average, the photon doesn't go anywhere; but after enough hops, there is some probability for the photon to reach the edge of the star and escape. Figure 3.7 shows the distribution of positions for walks of length $n = 10, 30, 100, 300$ steps, with each step having length 1.0. Suppose the edge of the star is at $x = \pm 10$ (red dotted lines). Although the average position is at $x = 0$, for $n \gtrsim 100$ steps, there is a reasonable probability of the photon escaping.

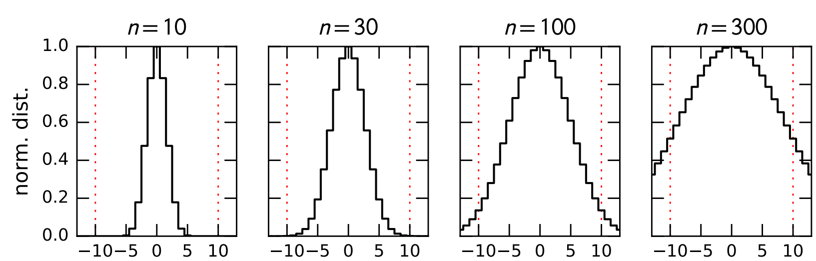


Figure 3.7: Distribution of positions after n steps in a random walk.

To make this into a workable model, let us first recall the basic features of a random walk. It is described by a binomial distribution: after n steps, the probability that m of them were to the right is

$$\mathcal{P}_n(m; p) = \frac{n!}{m!(n-m)!} p^m (1-p)^{n-m}. \tag{3.16}$$

Here p is the probability of any single step being to the right. The mean and root variance of m are

$$\langle m \rangle = np \tag{3.17}$$

$$\left[\left\langle (m - \langle m \rangle)^2 \right\rangle\right]^{1/2} = [np(1-p)]^{1/2}. \tag{3.18}$$

We can use these to estimate the diffusion timescale τ.

EXERCISE 3.9—

1. Show from equation (3.17) that the mean distance traveled by the photon after n steps is $\langle d\rangle = \ell(2np - n)$, so for $p = 1/2$, $\langle d\rangle = 0$.
2. If all the steps were in the same direction, how many steps would be needed to reach the edge, at a a distance R from the center? Assume all steps have the same length ℓ.
3. We want the distribution of steps (cf. Fig. 3.7) to be wide enough to reach the edge. Set the root variance—a measure of the width of the probability distribution—equal to the number of steps found in part 2 and use equation (3.18),
$$\left[\left\langle (m - \langle m\rangle)^2 \right\rangle\right]^{1/2} = \left[n_{\mathrm{edge}}p(1-p)\right]^{1/2},$$
to find n_{edge} in terms of R and ℓ.
4. What is the *total* distance traveled by the photon after n_{edge} steps? If the photon traveled at speed c, how long did it take? Compare your answer with that for part 3 of exercise 3.7.

3.5 The photosphere

We are now ready to investigate heat transport near the edge, where the optical depth $\tau_\nu \lesssim 1$ and photons begin to freely escape. We can no longer use the approximation of radiative diffusion, because conditions in the star are now changing over distances of a mean free path. Let's return to equation (3.6) for radiative transport:
$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\rho\left(\kappa_\nu^{\mathrm{abs}} + \kappa_\nu^{\mathrm{sca}}\right) I_\nu + \rho j_\nu + \rho\kappa_\nu^{\mathrm{sca}} J_\nu.$$

In general, this is difficult to solve: for some frequencies, the atmosphere will be nearly transparent, while for other frequencies it is quite opaque. Rather than develop the numerical machinery to solve the equation, we shall adopt a few simple approximations (indicated by highlighted bold text in the margins) that will allow us to obtain an approximate solution for the temperature of the stellar atmosphere.

Opacities are gray

First, we assume that the opacity is gray—that is, independent of frequency. This is unphysical, but the solutions for temperature and pressure will have the correct overall behavior. Because the opacity is gray, we shall drop the "ν" subscript in κ and τ.

EXERCISE 3.10— Does matter with a gray opacity in thermal equilibrium also have a gray emissivity j_ν?

We next define a coordinate system. Since we are in a thin layer near the edge of the star, we will adopt planar coordinates, with z being the altitude above some point. We'll pick $z = 0$ to be a point deep enough in the star that $I_\nu \approx B_\nu$. Then we define the optical depth as

$$\tau = \int_z^\infty \rho \left(\kappa^{\mathrm{abs}} + \kappa^{\mathrm{sca}}\right)\, \mathrm{d}z; \tag{3.19}$$

differentiating this expression gives

$$\frac{\mathrm{d}\tau}{\mathrm{d}z} = -\rho\left(\kappa^{\mathrm{abs}} + \kappa^{\mathrm{sca}}\right).$$

Note the "−": in these coordinates, as z gets larger, τ gets smaller.

We may rewrite the equation (2.1) of hydrostatic balance as

$$\begin{aligned} -\rho g = \frac{\mathrm{d}P}{\mathrm{d}z} &= \frac{\mathrm{d}P}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}z} = -\rho\kappa\frac{\mathrm{d}P}{\mathrm{d}\tau}, \\ \frac{\mathrm{d}P}{\mathrm{d}\tau} &= \frac{g}{\kappa}. \end{aligned} \tag{3.20}$$

Since we are in a thin layer, we can take the gravitational acceleration g as being approximately constant. By integrating hydrostatic equilibrium from where $\tau = 0, P = 0$ to where $\tau = 1$, we can get an approximate value of the photospheric pressure,

$$P_{\mathrm{ph}} = \int_0^{P_{\mathrm{ph}}} \mathrm{d}P = \int_0^1 \frac{g}{\kappa}\,\mathrm{d}\tau \approx \frac{g}{\kappa}.$$

The surface gravity sets the pressure at the ***photosphere****, the location where the optical depth is of order unity and where photons can escape from the star.*

EXERCISE 3.11— Suppose you observe a star that has a 10% larger mass and 10% larger radius than the Sun. All else being equal, how does the pressure at the photosphere of this star compare to that of the Sun?

Atmosphere is in steady-state LTE

For our second approximation, we assume that the matter is in LOCAL THERMAL EQUILIBRIUM (LTE). This means there is a well-defined temperature at each depth. Furthermore, the emissivity is related to the absorption opacity,

$$j_\nu = \kappa^{\mathrm{abs}} B_\nu.$$

Note that this does *not* imply anything about the radiation field. We can now take the radiative transfer equation (3.6) and substitue our definition of optical depth (eq. [3.19]) to obtain

$$\mu\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau} = I_\nu - S_\nu. \tag{3.21}$$

Here

$$S_\nu = \frac{j_\nu + \kappa^{\mathrm{sca}} J_\nu}{\kappa} = \frac{\kappa^{\mathrm{abs}} B_\nu + \kappa^{\mathrm{sca}} J_\nu}{\kappa}.$$

If, in addition, the matter is in steady-state, then the rate at which energy is absorbed from the radiation field, $\int \kappa^{\mathrm{abs}} I_\nu \,\mathrm{d}\nu\,\mathrm{d}\Omega$, must equal the rate at which energy is emitted, $\int j_\nu \,\mathrm{d}\nu\,\mathrm{d}\Omega$. Since we are in LTE,

$$\int \left(j_\nu - \kappa^{\mathrm{abs}} I_\nu\right)\,\mathrm{d}\nu\,\mathrm{d}\Omega = 4\pi\kappa^{\mathrm{abs}} \int \left(B_\nu - J_\nu\right)\,\mathrm{d}\nu = 0.$$

Since $J = \int J_\nu\,\mathrm{d}\nu = \int B_\nu\,\mathrm{d}\nu = B$, it follows that $S = \int S_\nu\,\mathrm{d}\nu = B$ as well:

For a gray atmosphere in steady-state, local thermal equilibrium, the integrated source function and mean intensity equal the Planck value:

$$S(\tau) = J(\tau) = B(\tau),$$

Note that this does *not* imply that $I_\nu = B_\nu$ or $J_\nu = B_\nu$.

We still have the problem that eq. (3.21) includes both the derivative and integral of I_ν. To get around this, we are going to expand I_ν in Legendre polynomials,

$$I_\nu(\tau,\mu) = I_{\nu,0}(\tau)\mathcal{P}_0(\mu) + I_{\nu,1}(\tau)\mathcal{P}_1(\mu) + I_{\nu,2}(\tau)\mathcal{P}_2(\mu) + \ldots$$

and then only include the first two terms, $\mathcal{P}_0(\mu) = 1, \mathcal{P}_1 = \mu$. Thus, I_ν is *linear* in μ: $I_\nu = I_{\nu,0}(\tau) + I_{\nu,1}(\tau)\mu$.

Intensity is linear in μ

In terms of this expansion, the angle-averaged specific intensity is

$$J_\nu(\tau) = \frac{1}{4\pi}\int I_\nu\,\mathrm{d}\mu\,\mathrm{d}\varphi = I_{\nu,0}(\tau),$$

and hence the specific energy density is $U_\nu = 4\pi/c \cdot J_\nu = 4\pi/c \cdot I_{\nu,0}$. The specific flux is

$$F_\nu(\tau) = \int \mu I_\nu\,\mathrm{d}\mu\,\mathrm{d}\varphi = \frac{4\pi}{3} I_{\nu,1}(\tau).$$

We can therefore write the intensity as

$$I_\nu(\tau) = J_\nu(\tau) + \frac{3\mu}{4\pi}F_\nu(\tau) = \frac{c}{4\pi}U_\nu(\tau) + \frac{3\mu}{4\pi}F_\nu(\tau). \tag{3.22}$$

Box 3.1 Expansion in Legendre polynomials

You may recall from electrostatics that we can decompose the field from a set of charges into a sum of moments: dipole, quadrupole, and so on. The basis functions for this are the Legendre polynomials $\mathcal{P}_n(\cos\theta)$, defined by the expansion

$$\frac{1}{\sqrt{1-2\mu z + z^2}} \equiv \sum_{n=0}^{\infty} \mathcal{P}_n(\mu) z^n,$$

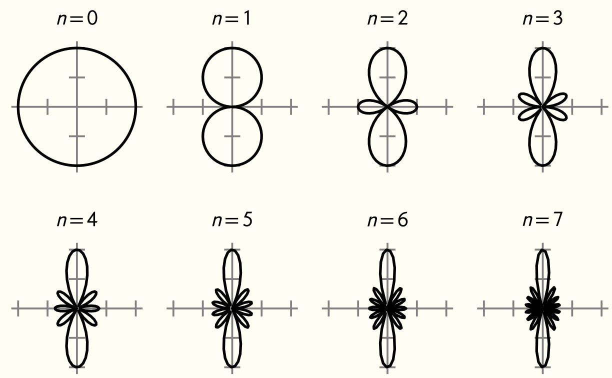
for $-1 < \mu < 1$, $|z| < 1$. The first four polynomials are

$$\mathcal{P}_0(\mu) = 1 \qquad \mathcal{P}_2(\mu) = \frac{1}{2}(3\mu^2 - 1)$$

Box 3.1 continued

$$\mathcal{P}_1(\mu) = \mu \qquad \mathcal{P}_3(\mu) = \frac{1}{2}(5\mu^3 - 3\mu),$$

and the first eight Legendre polynomials are plotted below.



As n increases, the angular variations become finer.

The Legendre polynomials are *orthogonal* in the following sense:

$$\int_{-1}^{1} \mathcal{P}_n(\mu)\mathcal{P}_m(\mu)\mathrm{d}\mu = \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1} & m = n \end{cases}. \qquad (3.23)$$

As a result of this orthogonality, we can decompose the radiative intensity into multipoles:

$$I = \sum_{n=0}^{\infty} I_n \mathcal{P}_n(\mu). \qquad (3.24)$$

EXERCISE 3.12— Use eq. (3.23) to show that $(4\pi)^{-1}\int I\,\mathrm{d}\Omega = I_0$ and $\int \mu I\,\mathrm{d}\Omega = (4\pi/3)I_1$, for $I = I_0 + I_1\mu$.

Insert the expansion (3.22) into the radiative transfer equation (3.21) and integrate over all angles and frequencies. Since τ is gray, we can pull the derivative out from the integral,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\int \mu I_\nu\,\mathrm{d}\nu\,\mathrm{d}\Omega &= \int I_\nu\,\mathrm{d}\nu\,\mathrm{d}\Omega - \frac{\kappa^{\mathrm{abs}}}{\kappa}\int B_\nu\,\mathrm{d}\nu\,\mathrm{d}\Omega - \frac{\kappa^{\mathrm{sca}}}{\kappa}\int J_\nu\,\mathrm{d}\nu\,\mathrm{d}\Omega \\
\frac{\mathrm{d}}{\mathrm{d}\tau}\int F_\nu\,\mathrm{d}\nu &= \frac{4\pi}{\kappa}\left[(\kappa^{\mathrm{abs}} + \kappa^{\mathrm{sca}})\int J_\nu\,\mathrm{d}\nu - \kappa^{\mathrm{abs}}\int B_\nu\,\mathrm{d}\nu - \kappa^{\mathrm{sca}}\int J_\nu\,\mathrm{d}\nu\right] \\
\frac{\mathrm{d}F}{\mathrm{d}\tau} &= 4\pi\frac{\kappa^{\mathrm{abs}}}{\kappa}\int (J_\nu - B_\nu) = 0.
\end{aligned}$$

Here we used $\kappa = \kappa^{\mathrm{abs}} + \kappa^{\mathrm{sca}}$ to simplify the right-hand side.

For a steady-state gray atmosphere in local thermal equilibrium, the total flux

$F = \int F_\nu \, \mathrm{d}\nu$ *is constant.*

The radiative energy is just passing through. Since the flux at $\tau = 0$, outside the star, is $F = \sigma_{\mathrm{SB}} T_{\mathrm{eff}}^4$, we can substitute that value in our expression for the intensity,

$$I(\mu, \tau) = \frac{c}{4\pi} U(\tau) + \frac{3\mu}{4\pi} \sigma_{\mathrm{SB}} T_{\mathrm{eff}}^4. \tag{3.25}$$

To solve for $U(\tau)$, multiply eq. (3.21) by μ and integrate over all angles and frequencies:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau} \int \mu^2 I_\nu \, \mathrm{d}\Omega \, \mathrm{d}\nu &= \int \mu I_\nu \, \mathrm{d}\Omega \, \mathrm{d}\nu - \int \mu S_\nu \, \mathrm{d}\Omega \, \mathrm{d}\nu \\ \frac{c}{4\pi} \frac{\mathrm{d}}{\mathrm{d}\tau} \int \mu^2 U \, \mathrm{d}\Omega + \frac{3}{4\pi} \sigma_{\mathrm{SB}} T_{\mathrm{eff}}^4 \int \mu^3 \, \mathrm{d}\Omega &= F - \int \mu S \, \mathrm{d}\Omega \\ \frac{c}{3} \frac{\mathrm{d}U}{\mathrm{d}\tau} &= \sigma_{\mathrm{SB}} T_{\mathrm{eff}}^4 \end{aligned} \tag{3.26}$$

In going from the first to the second line we have used eq. (3.25). In going from the second to the third line, the integrals of μ^3 and μS vanish because S is independent of angle and $\int_{-1}^{1} \mu \, \mathrm{d}\mu = \int_{-1}^{1} \mu^3 \, \mathrm{d}\mu = 0$.

Equation (3.26) is a first-order ODE, which upon integration yields

$$U(\tau) = \frac{3}{c} F(\tau + \tau_0), \tag{3.27}$$

where τ_0 is an integration constant. Our intensity is thus

$$I(\mu, \tau) = \frac{3}{4\pi} \sigma_{\mathrm{SB}} T_{\mathrm{eff}}^4 \left(\tau + \tau_0 + \mu \right).$$

To fix the integration constant τ_0, let's go to where $\tau = 0$. Here all of the radiation must be outward-bound. Hence if we integrate $\mu I(\mu, \tau = 0)$ over $0 \le \mu \le 1$, we should recover the flux:

$$\sigma_{\mathrm{SB}} T_{\mathrm{eff}}^4 = \int_0^{2\pi} \int_0^1 \mu I(\mu, \tau = 0) \, \mathrm{d}\mu \, \mathrm{d}\varphi = \frac{3}{4} \sigma_{\mathrm{SB}} T_{\mathrm{eff}}^4 \left(\tau_0 + \frac{2}{3} \right),$$

which fixes $\tau_0 = 2/3$.

To finish this, we note that $J = B$ since we are in steady-state local thermal equilibrium. The radiative energy density is thus $U = (4\pi/c) J = (4\pi/c) B = 4\sigma_{\mathrm{SB}}/c T^4$. Substituting this into eq. (3.27) then yields

$$T^4 = \frac{3}{4} T_{\mathrm{eff}}^4 \left(\tau + \frac{2}{3} \right). \tag{3.28}$$

This equation, along with eq. (3.20), determines the structure of the stellar atmosphere.

Box 3.2 Decomposition of intensity into moments

It is often useful to describe the intensity in terms of MOMENTS. A moment is simply an angle-weighted average of the radiative intensity, where the weight is a power of μ. For example, to take the zeroth-order moment, we multiply I_ν by $\mu^0 = 1$, integrate over all angles, and divide by 4π. This is just the average intensity $J_\nu = (4\pi)^{-1} \int I_\nu \, \mathrm{d}\Omega$. To take the first-order moment H_ν, we use a weight μ^1:

$$H_\nu = \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \mu I_\nu \, \mathrm{d}\mu \, \mathrm{d}\varphi.$$

To take the second-order moment K_ν, we use a weight μ^2:

$$K_\nu = \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \mu^2 I_\nu \, \mathrm{d}\mu \, \mathrm{d}\varphi.$$

The first three moments have physically interpretable meanings: the specific radiative energy density, flux, and pressure are $U_\nu = (4\pi/c)J_\nu$, $F_\nu = 4\pi H_\nu$, and $P_\nu = (4\pi/c)K_\nu$, respectively.

By taking moments of the radiative-transfer equation (3.21), we reduce the complicated integro-differential equation into a simpler ordinary differential equation. This comes at a cost, however; because the left-hand side contains $\mu\mathrm{d}/\mathrm{d}\tau$, the left hand side will have a higher-order moment than the right-hand side. By multiplying eq. (3.21) by successively higher powers of μ and integrating, we will generate an infinite series of ODE's for successively higher moments of I_ν. The trick is to adopt a CLOSURE RELATION that truncates this series. The classic scheme, due to Eddington, is to take $K = J/3$. The Eddington closure scheme is equivalent to expanding the radiative intensity to terms linear in μ.

EXERCISE 3.13— Show that if we approximate the intensity as $I_\nu(\mu, \tau) = I_{\nu,0}(\tau) + \mu I_{\nu,1}(\tau)$ (cf. eq. [3.22]), then $K_\nu = J_\nu/3$ identically.

EXERCISE 3.14— Deep in the star, we expect the radiation to be nearly isotropic, while it becomes outward-bound as $\tau \to 0$. Let's investigate this. We'll measure the anisotropy of the radiation field using the first two moments of the intensity (Box 3.2).

1. Demonstrate that $H_\nu/J_\nu = 0$ if the radiation is isotropic.
2. Next, suppose the radiation is completely anisotropic: all the photons are headed along precisely the same direction $\mu = 1$. We describe this mathematically as
$$I_\nu(\mu) = a_\nu \delta(\mu - 1), \tag{3.29}$$
where $\delta(x)$ is the DIRAC DELTA FUNCTION (see Box 3.3). Show that $H_\nu/J_\nu = 1$ for this case.
3. Now compute $H(\tau)/J(\tau)$ for our gray atmosphere.What is the degree of anisotropy at $\tau = 0$? at $\tau = 2/3$? at $\tau = 10$?

Box 3.3 The Dirac delta function

The Dirac delta function $\delta(x)$ has the following properties: $\delta(x) = 0, \forall x \neq 0$; and
$$\int_{-\varepsilon}^{\varepsilon} \delta(x)\,\mathrm{d}x = 1.$$
From this definition, one can show that
$$\int f(x)\delta(x - a)\,\mathrm{d}x = f(a),$$
where the integral is over any domain containing $x = a$.

4
Rainbow in the Dark

Now that we've discussed radiative transport in the star, we'll explore how the emergent spectrum of a star serves as a diagnostic of ambient conditions in the photosphere; we'll then discuss how nuclear reactions generate the luminosity in chapter 5.

4.1 Overview

If light from the Sun is passed through a grating (a piece of glass with finely etched lines), the light is dispersed in wavelength and creates a spectrum, such as the highly detailed one show in Fig. 4.1.

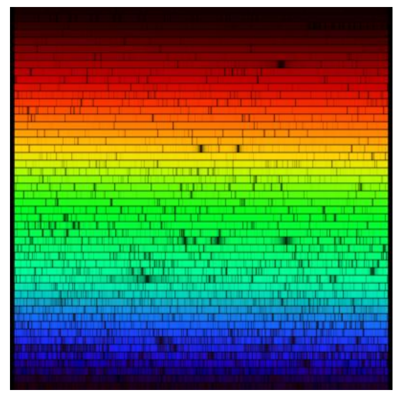


Figure 4.1: Visible spectrum of the Sun. Frequency increases along a row from left to right, and by rows from top to bottom. *Credit: N.A.Sharp, NOAO/NSO/Kitt Peak FTS/AURA/NSF.*

Superposed on the slow variation from red to violet are dark ABSORPTION LINES. The ions, atoms, and molecules in the solar atmosphere absorb light at specific frequencies and create these lines.

Beginning in the late 1800's, astronomers began classifying stars by the observed absorption lines in the spectra. At this time, Edward Pickering and Williamina Fleming of the Harvard College Observatory began amassing a vast catalog of stellar spectra. They classified these spectra according to the strength of observed hydrogen Balmer lines (the first four are Hα: 657 nm; Hβ: 486 nm; Hγ: 434 nm; Hδ: 410 nm). Stars, such as Vega, with the strongest Balmer lines were classified as type "A", those with the next strongest were type "B", and so forth. Annie Jump Cannon, who would later succeed Fleming as curator of astronomical photography at the observatory, simplified and reorganized the scheme, and added decimal subdivisions (0 . . . 9) for each type[1]. When stellar color is taken into account, the ordering of stars, from blue to red, is "OBAFGKM". In the 1990's the "L" and "T" classes were added[2] for cool stars and brown dwarfs (stellar-like objects that do not reach central temperature sufficient for fusion of hydrogen into helium). With the introduction of the "Y" stellar type[3], this classification was further extended to even cooler objects having $T_{\mathrm{eff}} \lesssim 500\,\mathrm{K}$.

Hertzsprung and Russell independently noticed that most stars tended to lie along a band, termed the MAIN SEQUENCE, in a plot of

[1] For example, the Sun's type is G2

[2] J. D. Kirkpatrick, I. N. Reid, J. Liebert, et al. Dwarfs Cooler than "M": The Definition of Spectral Type "L" Using Discoveries from the 2 Micron All-Sky Survey (2MASS). *ApJ*, 519:802–833, July 1999

[3] Michael C. Cushing, J. Davy Kirkpatrick, Christopher R. Gelino, et al. The Discovery of Y Dwarfs using Data from the Wide-field Infrared Survey Explorer (WISE). *ApJ*, 743:50, December 2011. DOI: 10.1088/0004-637X/743/1/50

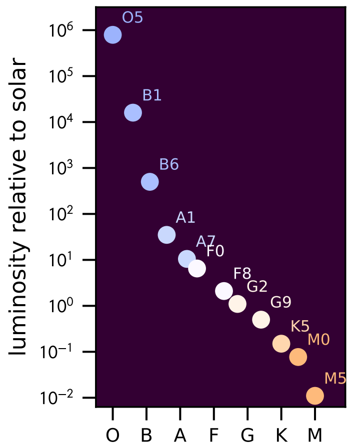


Figure 4.2: Hertzsprung-Russell diagram showing standard main-sequence stars. Colors are approximate translations of the spectra.

absolute magnitude (or luminosity) against stellar type (now known as a HERTZSPRUNG-RUSSELL DIAGRAM). Figure 4.2 shows some standard main-sequence stars, along with their stellar type and approximate color.

In an influential PhD thesis, Cecilia Payne-Gaposhkin[4] applied the Boltzmann and Saha equations to show that different stellar spectra were consistent with changes in temperature, rather than composition, of the stellar photosphere. The sequence of stellar types is therefore a temperature sequence, with "O" stars being the hottest.

[4] C. H. Payne. *Stellar Atmospheres; a Contribution to the Observational Study of High Temperature in the Reversing Layers of Stars.* PhD thesis, RADCLIFFE COLLEGE., 1925

4.2 The hydrogen atom

To understand why the Balmer lines are strongest in a certain range of temperatures, we first need to review the workings of a hydrogen atom.

The electrons bound to an atom or molecule can only occupy states having a discrete set of energies. For example, the electron in a hydrogen atom only has energies

$$E_n = -13.6\,\mathrm{eV} \times \frac{1}{n^2}, \tag{4.1}$$

where $n > 0$ is an integer known as the PRINCIPAL QUANTUM NUMBER. These energies are negative, relative to a free electron. For example, the ground state ($n = 1$) has energy $-E_{\mathrm{Ry}} = -13.6\,\mathrm{eV}$, meaning that 13.6 eV is required to remove an electron in its ground state from the atom.

Because the electrons in an atom can only have certain energies, the atom can only absorb or emit light at specific wavelengths, such that the energy of the photon matches the difference in energy between two levels. For example, a hydrogen atom in its ground state can absorb a photon of energy

$$E_{1\to 2} = -E_{\mathrm{Ry}}\left(\frac{1}{2^2} - \frac{1}{1^2}\right) = 10.2\,\mathrm{eV}$$

corresponding to the energy required to excite the electron from level $n = 1$ to $n = 2$. The wavelengths that can be absorbed by a hydrogen atom at rest can be found by substituting $E = hc/\lambda$ into equation (4.1):

$$\lambda_{m\to n} = \lambda_0 \left(\frac{1}{m^2} - \frac{1}{n^2}\right)^{-1}, \tag{4.2}$$

where $\lambda_0 = 91.2\,\mathrm{nm}$ and $n > m$. The transitions from the lowest levels are named after their discoverers: Lyman for $1 \to n$, Balmer for $2 \to n$, Paschen for $3 \to n$. A greek letter is used to denote the higher state: for example Lyman α (abbr. Lyα) means $1 \to 2$, with $\lambda_{\mathrm{Ly}\alpha} = 121.6\,\mathrm{nm}$. Note that $\lambda_{m\to n} > \lambda_0$; photons with wavelengths $\lambda < 91.2\,\mathrm{nm}$ have sufficient energy to knock the electron out of the atom, thereby producing a hydrogen ion and a free electron. The first line transition in the Balmer series is $2 \to 3$, and is designated Hα: $\lambda_{\mathrm{H}\alpha} = 656.3\,\mathrm{nm}$. The first 20 lines for each of the Lyman, Balmer, and Paschen series are shown in Fig. 4.3; note the $3 \to 4$ transition is outside the plot range. The Balmer lines lie in the visible range.

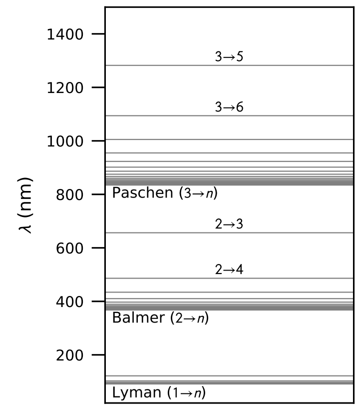


Figure 4.3: Spectral lines of neutral hydrogen

4.3 The Boltzmann Equation

In order to produce a Balmer absorption line, we must have some hydrogen atoms in the photosphere with electrons in the energy level $n = 2$. The more atoms in a state $n = 2$, the more absorption and the stronger the line. To find the number of atoms with energy level $n = 2$, we make use of a fundamental result, due to Boltzmann, from statistical (thermal) physics; namely, that if our sample of atoms is in thermal equilibrium, then the ratio of the number of atoms with energy E_i to the number of atoms with energy E_j is

$$\frac{N_i}{N_j} = \frac{g_i}{g_j} \exp\left(-\frac{E_i - E_j}{k_\mathrm{B} T}\right). \tag{4.3}$$

Here the number g_n gives the number[5] of quantum mechanical states having energy $E_n = -E_\mathrm{Ry}/n^2$. For an energy level n, there are n^2 possible states, each having a different angular momentum. For each of these n^2 states, both the electron and proton may each have 2 possible spins. The total number of states for energy E_n is therefore $g_n = 2 \times 2 \times n^2$.

[5] g_n is known as the *degeneracy* of a given level n

Suppose we wish to know the fraction of atoms in a given state i: that is, we wish to know

$$x_i = \frac{N_i}{N_1 + N_2 + \ldots + N_i + \ldots}?$$

Using equation (4.3), we can express x_i as

$$\begin{aligned} x_i &= \frac{g_i e^{-E_i/k_\mathrm{B}T}}{g_1 e^{-E_1/k_\mathrm{B}T} + g_2 e^{-E_2/k_\mathrm{B}T} + \ldots + g_i e^{-E_i/k_\mathrm{B}T} + \ldots} \\ &\equiv \frac{g_i e^{-E_i/k_\mathrm{B}T}}{\mathcal{Q}}. \end{aligned} \tag{4.4}$$

The quantity

$$\mathcal{Q} = \sum_n g_n \exp\left(-\frac{E_n}{k_\mathrm{B} T}\right) \tag{4.5}$$

is called the *partition function*: loosely speaking, it indicates the number of ways the sample of atoms can be partitioned among the different energy levels.

EXERCISE 4.1 — Assuming that the first term $g_1 e^{-E_1/k_\mathrm{B}T}$ dominates the sum in the partition function (see Box 4.1), plot the fraction of neutral hydrogen in its $n = 2$ state as a function of temperature, for $5\,000\ \mathrm{K} < T < 20\,000\ \mathrm{K}$.

Box 4.1 The partition function for neutral hydrogen

The partition function for neutral hydrogen, eq. (4.5), has some interesting features. Substituting $g_n = 4n^2$ and $E_n = -E_{\mathrm{Ry}}/n^2$ and factoring out common terms gives

$$\mathcal{Q} = 4e^{\beta E_{\mathrm{Ry}}} \sum_n n^2 e^{-\beta E_{\mathrm{Ry}}(1-1/n^2)},$$

with $\beta = (k_{\mathrm{B}}T)^{-1}$. The sum diverges, since for $n \gg 1$ the individual terms approach $n^2 e^{-\beta E_{\mathrm{Ry}}}$. In practice, this isn't a problem, as there is an upper limit on n set by ambient conditions. For example, the mean distance of the electron from the nucleus is $\approx a_{\mathrm{B}}n^2$, where $a_{\mathrm{B}} = 5.29 \times 10^{-11}$ cm is the Bohr radius. As a result, each atom takes up a volume $\approx a_{\mathrm{B}}^3 n^6$; if we want the atoms to not overlap, then the volume per atom, $V/N = 1/\xi$, must be larger than this by some factor. Suppose we set that the volume of an atom must be less than half of that available in our gas; then

$$\xi = \frac{N}{V} < \frac{N}{N \cdot 2a_{\mathrm{B}}^3 n^6}.$$

Thus the maximum level is $n < (2a_{\mathrm{B}}^3 \xi)^{-1/6}$. For a typical A-star photospheric density $\xi \sim 10^{15}\ \mathrm{cm}^{-3}$, the energy level cutoff is $n \approx 35$. In practice the cutoff will be even lower because of collisions.

The precise maximum value of n is not that important for most applications. The reason is that the terms in the partition function increase only slowly. As an example, the terms and cumulative sum in the partition function at a temperature $T = 10^4$ K are as follows.

n	$n^2 e^{\beta E_{\mathrm{Ry}}(1-1/n^2)}$	$4\sum_{i=1}^{n} i^2 e^{-\beta E_{\mathrm{Ry}}(1-1/i^2)}$
1	1.00e+00	4.0000
2	2.88e-05	4.0001
3	7.23e-06	4.0001
	⋮	
26	9.62e-05	4.0038
	⋮	
52	3.78e-04	4.0274
	⋮	
268	9.99e-03	7.5901

As we can see from the cumulative sum (rightmost column), the partition function is insensitive to the precise value of the cutoff until n is quite large; indeed, for many applications it is reasonably accurate to just use the first term: $Q \approx 4e^{\beta E_{\mathrm{Ry}}}$.

4.4 *Ionization: The Saha equation*

As the temperature in the gas rises, there are more photons with sufficient energy to eject electrons from an atom. In addition, collisions between atoms also become sufficiently energetic to ionize the atom. In astronomical nomenclature, the ionization state is denoted by a small Roman numeral: Fe I denotes neutral iron, Fe II denotes singly-ionized iron (charge $+1$), Fe III denotes doubly-ionized iron (charge $+2$), and so on. In thermal equilibrium, the rate at which atoms are ionized must equal the rate at which ions and electrons recombine: for example, in a gas consisting of hydrogen atoms, hydrogen ions (i.e., protons), and electrons the reaction

$$\mathrm{H\,II} + e \longleftrightarrow \mathrm{H\,I}$$

is in equilibrium. We'd like to extend equation (4.3) to find the ratio of two ionization states N_{i+1}/N_i. Although deriving this equation, termed the SAHA EQUATION[6], is beyond the scope of the course, what we shall do is take the equation apart and try to understand how it works. The Saha equation for the ratio of the populations of two ionization states [7] N_{i+1} and N_i is

$$\frac{N_{i+1}}{N_i} = \left[\frac{2}{n_e}\left(\frac{m_e k_\mathrm{B} T}{2\pi\hbar^2}\right)^{3/2}\right]\frac{\mathcal{Q}_{i+1}}{\mathcal{Q}_i}. \tag{4.6}$$

In this equation, n_e denotes the electron density—the number of free electrons per unit volume—and m_e is the electron mass. The terms $\mathcal{Q}_{i+1}$ and $\mathcal{Q}_i$ are the partition functions for the two states, *both measured with respect to the same zero-point for energy*.

Let's start with the term Q_{i+1}/Q_i. If both partition functions are dominated by the ground state term[8] then

$$\begin{aligned}\frac{Q_{i+1}}{Q_i} &= \frac{g_{i+1,1}}{g_{i,1}} e^{-\beta(E_{i+1,1}-E_{i,1})} \\ &= \frac{g_{i+1,1}}{g_{i,1}} e^{-\beta E_\mathrm{ion}}\end{aligned}$$

Here $E_\mathrm{ion} = E_{i+1,1} - E_{i,1}$ is the energy needed to remove an electron from an ion in state i and we use the common shorthand $\beta = (k_\mathrm{B}T)^{-1}$. Thus Q_{i+1}/Q_i resembles equation (4.3).

The term in $[\cdot]$ in eq. (4.6) arises because we also need to allow for the number of free electron states. When the atom is ionized, each electron quickly acquires an average kinetic energy $(3/2)k_\mathrm{B}T$. There are many different states having this energy: the electron can be in different locations and moving in different directions, for example.

You might think that there would be an infinitude of possible electron states. Quantum mechanics, however, sets limitations on this number. First, we have the Pauli exclusion principle: no two electrons can be in the same location with the same momentum and same spin. What

[6] Derived by Meghnad Saha in 1920

[7] In this context, N_{i+1}/N_i refers to ratios such as $N_{\mathrm{Fe\,II}}/N_{\mathrm{Fe\,I}}$. Each population N can be divided into sub-populations based on the different electron energy levels. For example, $N_{\mathrm{H\,I}} = N_{\mathrm{H\,I},1} + N_{\mathrm{H\,I},2} + \ldots$, where $N_{\mathrm{H\,I},2}$ are the number of hydrogen in ionization state I with an electron in the second energy level.

[8] see Box 4.1

do we mean by same location and momentum? Recall the Heisenberg uncertainty principle: the electrons x-position and x-momentum are spread about a range of values Δx and Δp_x, and these uncertainties are related via

$$\Delta x\,\Delta p_x \gtrsim h.$$

Thus, if we imagine dividing our volume into little boxes of volume

$$\Delta V = \Delta x\,\Delta y\,\Delta z \approx \frac{h^3}{\Delta p_x\,\Delta p_y\,\Delta p_z},$$

each box can hold two electrons.[9] Suppose we have a volume V; how many boxes are there? The number of available boxes is

$$\frac{V}{\Delta V} \approx \frac{V\,\Delta p_x\,\Delta p_y\,\Delta p_z}{h^3}.$$

[9] Because electrons have spin 1/2, we can put two electrons into the same position and momentum state if their spins are oppositely directed.

To estimate the size of $\Delta p_x\,\Delta p_y\,\Delta p_z$, let's take $\Delta p_x \sim p_x$ and similarly for Δp_y and Δp_z; further, if everything is isotropic then $p_x \approx p_y \approx p_z$ on average, so $\Delta p_x\,\Delta p_y\,\Delta p_z \sim p_x^3$. Now the kinetic energy of the electron is $p^2/2m_e$, and $p^2 = p_x^2 + p_y^2 + p_z^2 \approx 3p_x^2$. Hence the kinetic energy is $(3/2)p_x^2/m_e$; in thermal equilibrium, however, the kinetic energy has an average value of $(3/2)k_\mathrm{B}T$. The value of p_x^2 is therefore

$$p_x^2 \approx m_e k_\mathrm{B} T,$$

and the number of boxes is

$$\frac{V}{\Delta V} \sim V\frac{p_x^3}{h^3} \sim V\frac{(m_e k_\mathrm{B} T)^{3/2}}{h^3}.$$

If our volume V contains N_e electrons, then the number of states electron is

$$\frac{2V}{N_e \Delta V} \sim \frac{2V}{N_e}\frac{(m_e k_\mathrm{B} T)^{3/2}}{h^3}.$$

The factor of 2 appears because each box can hold 2 electrons. Recognizing that $N_e/V = n_e$, we see that this number of states per free electrons corresponds to the factor in [ ] in equation (4.6). When the numerical calculation is done correctly, the additional factor of 2π arises.

The number of states per free electron plays an important role in setting the temperature at which a species ionizes. You might expect, since a term $e^{-E_\mathrm{ion}/k_\mathrm{B}T}$ appears in the ratio N_{i+1}/N_i, that a species would ionize at a temperature $E_\mathrm{ion}/k_\mathrm{B}$. In fact the ionization temperature is much lower. To see how this works, define

$$\zeta = \ln\left[\frac{1}{n_e}\left(\frac{m_e k_\mathrm{B} T}{2\pi\hbar^2}\right)^{3/2}\right].$$

We can then write eq. (4.6)—with the approximation that the partition functions are dominated by the ground state—as

$$\frac{N_{i+1}}{N_i} = \frac{2g_{i+1,1}}{g_{i,1}} \exp\left(\zeta - \beta E_\mathrm{ion}\right).$$

Now the factor $g_{i+1,1}/g_{i,1}$ is of order unity. Hence, when the gas ionizes and $N_{i+1,1} \approx N_{i,1}$, we must have that $\zeta \approx \beta E_{\text{ion}}$; put differently, the ionization temperature will not be $E_{\text{ion}}/k_{\text{B}}$ but rather $E_{\text{ion}}/k_{\text{B}}\zeta$. Under conditions in the photosphere of an A star ($T \approx 10^4$ K, $n \sim 10^{15}\,\text{cm}^{-3}$), $\zeta \approx 15$.

In more intuitive terms, when an electron is ejected from an atom, it has an enormously large number $\sim e^{15}$ number of different states available. To rejoin with an ion requires being in the right place at the right time with the right energy. The large number of available states makes this unlikely, so the electron must wander lonely through a vast and desolate phase space until at long last it reunites with an ion. In a sense, the large number of available states per electron makes ionization easier than recombination; as a result the temperature at which ionization occurs is considerably lower than $E_{\text{ion}}/k_{\text{B}}$.

EXERCISE 4.2— Let n_{I} be the density of H I and n_{II} be the density of H II. Denote the fraction of neutral hydrogen as $x = n_{\text{I}}/(n_{\text{I}} + n_{\text{II}})$, so that $1 - x = n_{\text{II}}/(n_{\text{I}} + n_{\text{II}})$ is the fraction of ionized hydrogen. Take $n_{\text{I}} + n_{\text{II}} = 10^{15}\,\text{cm}^{-3}$, and assume that all free electrons come from the ionization of hydrogen, so that $n_e = n_{\text{II}}$. Plot x as a function of temperature for $7\,500\,\text{K} \le T \le 15\,000\,\text{K}$, and find the temperature at which $x = 1/2$. Then multiply x by the fraction n_2/n_1, as set by the Boltzmann equation, to find the fraction of hydrogen in the $n = 2$ level.

As shown in exercise 4.2, the Balmer lines, which correspond to transitions $2 \to 3, 2 \to 4, \ldots$, are most prominent in A stars. These stars have $T_{\text{eff}} = (7\,500\text{–}9\,500)$ K. At lower temperatures, the population of hydrogen atoms in the level $n = 2$ decreases as $e^{-E_2/k_{\text{B}}T}$ and the lines become weak. At higher temperatures, the number of neutral hydrogen atoms decreases; most of the hydrogen is ionized, and the Balmer lines again become weaker.

These arguments apply to other species present in the stellar photosphere. Figure 4.4 displays spectra for selected stellar types at optical wavelengths. In the hottest stars (type O: $T_{\text{eff}} > 30\,000$ K), hydrogen is mostly ionized and the lines are from He II and multiply-ionized metals. As the temperature cools into the B and A series, the hydrogen lines increase in strength. Going from F into G ($T_{\text{eff}} = (5\,000\text{–}6\,000)$ K, the hydrogen lines decrease, while lines from singly-ionized and neutral metals such as Ca II, Ca I, and Fe I become strong. At still lower temperatures in the K and M ($T_{\text{eff}} < 3\,500$ K) types, absorption from molecules such as TiO becomes prominant. An example is the broad trough seen in the K spectrum near $\lambda = 500$ nm.

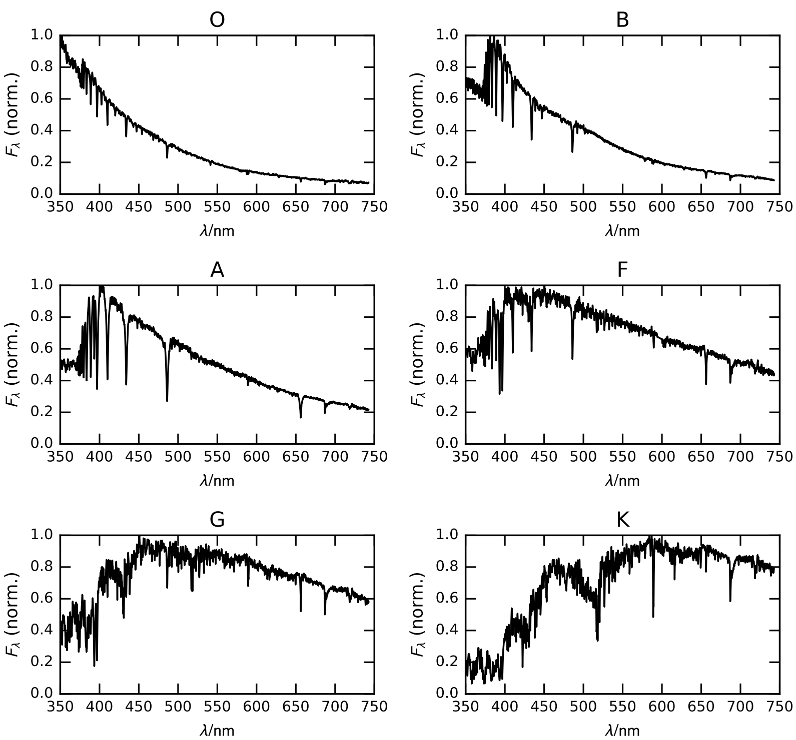


Figure 4.4: Spectra from main-sequence stars of spectral types O–K. Data from Jacoby et al. [1984].

4.5 Pressure broadening of lines

We've now demonstrated how stars may be classified by the absorption lines in their spectra, and how this classification gives us the photosphere effective temperature. We can also obtain information about the pressure at the photosphere, and hence the surface gravity of the star, by looking at the shape of the absorption lines. A zoomed-in view of the Hγ line ($2 \to 5$ transition in H I) from a main-sequence A1 star is show in Fig. 4.5. The line is spread over a few nanometers, compared against a central value of 434 nm.

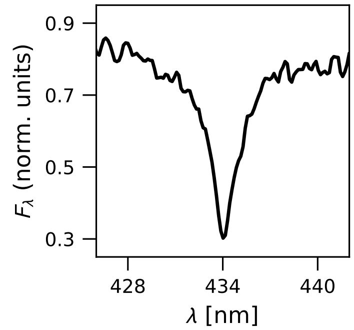


Figure 4.5: Hγ absorption line observed from the main-sequence A1 star HD16608. Spectrum from Jacoby et al. [1984].

TO UNDERSTAND WHAT SETS THE SHAPE, AND WIDTH, OF THE ABSORPTION LINE, we need to model our atomic transition. Consider an electronic transition in an atom between two energy levels, E_m and E_n. The natural frequency of this transition is $\nu_0 = |E_n - E_m|/h$. Light incident on the atom with frequency $\nu \neq \nu_0$ drives the electron at frequency ν.

Since the transition between two states has a definite frequency associated with it, let's start with a simple harmonic oscillator, which is described by an equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega_0^2 x = 0.$$

Here $\omega_0 = 2\pi\nu_0$. Light is described as an electromagnetic wave, so classically the electron feels a force $eE\cos(\omega t)$, where $\omega = 2\pi\nu$. An accelerating electron radiates, which damps the acceleration of the electron. The damping can be modeled as a force that is proportional to the velocity, $-m\Gamma \mathrm{d}x/\mathrm{d}t$. Classically, the transition in an atom can therefore be modeled as an electromagnetic oscillator with damping and driving terms,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + m\Gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = \frac{eE}{m}\cos(\omega t).$$

This has a well known solution (see Box 4.2). The amplitude of oscillation is proportional to the energy removed from the incident light, which is proportional to the cross-section. The classical cross-section for absorption of radiant energy by an electromagnetic oscillator is thus

$$\sigma = \left(\frac{\pi e^2}{m_e c}\right)\left\{\frac{\Gamma/4\pi}{(\nu_0 - \nu)^2 + (\Gamma/4\pi)^2}\right\}. \tag{4.7}$$

The function

$$\mathcal{L}(\nu;\Gamma) = \frac{1}{\pi}\frac{\Gamma/4\pi}{(\nu_0 - \nu)^2 + (\Gamma/4\pi)^2}$$

is known as a LORENTZIAN. In contrast to a Gaussian, a Lorentzian is characterized by broad "wings" (Fig. 4.6) away from the central frequency

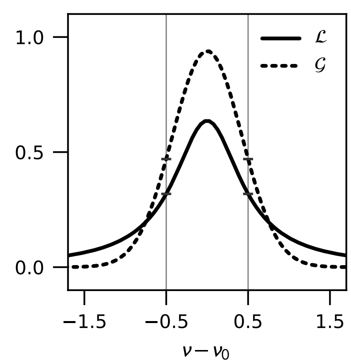


Figure 4.6: Comparison of a Lorentzian ($\mathcal{L}$, solid line) and a Gaussian ($\mathcal{G}$, dotted line), both with FWHM $= 1$. The area under each curve is unity.

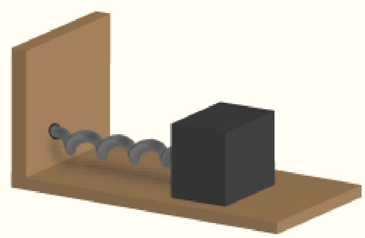
ω_0. The actual value of the cross-section must be calculated using quantum mechanics. The overall shape of the cross-section is still in the form of equation (4.7), however, so the opacity is just

$$\rho\kappa_\nu = n_{\mathrm{ion},m} \left(\frac{\pi e^2}{m_e c}\right) f_{mn} \left\{ \frac{\Gamma/4\pi}{(\nu_0 - \nu)^2 + (\Gamma/4\pi)^2} \right\}. \tag{4.8}$$

In this equation, f_{mn} is a number, called the *oscillator strength*, that results from the calculation of the transition probability from state m to state n, and $n_{\mathrm{ion},m}$ is the density of atoms in state m. The key point is that f_{mn} depends only on the details of the transition: the energies, spins, and parities of the atomic states. It does not depend on environmental parameters such as temperature and pressure. As a result, f_{mn} can be measured or computed once and then tabulated.

Box 4.2 The driven damped oscillator

Let's begin with a simple system: a mass m attached to a spring with force $F = -kx$.



If we put the origin of our coordinate system where the mass is at rest with the spring relaxed, then the equation of motion of the mass is

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{k}{m} x = 0. \tag{4.9}$$

You have solved this equation before: the most general solution is

$$x(t) = x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t) \tag{4.10}$$

with $\omega_0^2 = k/m$ and with x_0 and v_0 being the initial position and velocity of the mass. The angular frequency ω_0 is related to the period of oscillation T as $\omega_0 = 2\pi/T = 2\pi\nu$.

NOW LET'S PUSH ON OUR MASS WITH AN OSCILLATING FORCE, $F\cos(\omega t)$ WITH $\omega \neq \omega_0$. A real world example would be holding a vibrating tuning fork near another fork tuned to a different frequency. The equation of motion is now

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega_0^2 x = \frac{F}{m}\cos(\omega t). \tag{4.11}$$

Box 4.2 continued

You can verify by substitution that a general solution is

$$x(t) = \frac{F/m}{(\omega_0^2 - \omega^2)} \cos(\omega t) + A\cos(\omega_0 t) + B\sin(\omega_0 t).$$

Let's start with our harmonic oscillator at rest ($v_0 = \mathrm{d}x/\mathrm{d}t|_{t=0} = 0$) and at $x|_{t=0} = 0$. With these conditions, we can determine the constants A and B; the solution is

$$x(t) = \frac{F/m}{(\omega_0^2 - \omega^2)} \left[\cos(\omega t) - \cos(\omega_0 t)\right].$$

Let's recast this by defining $\Delta = \omega_0 - \omega$ and $\omega_m = (\omega_0 + \omega)/2$. Then

$$\begin{aligned}
\omega_0^2 - \omega^2 &= (\omega_0 - \omega)(\omega_0 + \omega) = 2\Delta\omega_m, \\
\cos(\omega_0 t) &= \cos\left(\omega_m t + \Delta t/2\right), \\
\cos(\omega t) &= \cos\left(\omega_m t - \Delta t/2\right);
\end{aligned}$$

using the cosine addition rules and combining terms, we can write the solution as

$$x(t) = \left[\frac{F/m}{\Delta\omega_m} \sin(\Delta t/2)\right] \sin(\omega_m t). \tag{4.12}$$

This illustrates the phenomena of beats: the oscillation consists of a carrier signal at frequency ω_m with the amplitude modulated at the slower frequency $\Delta/2$. Notice that the amplitude increases as $\Delta \to 0$, i.e., $\omega \to \omega_0$.

Now let's make our model even more realistic. We add a frictional force that is proportional to velocity, $F_{\text{friction}} = -m\Gamma \mathrm{d}x/\mathrm{d}t$. Our complete equation of motion is then

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \Gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = \frac{F}{m}\cos(\omega t). \tag{4.13}$$

The solution to this is straightforward to find, although the algebra is tedious (trust me on this). The general solution for initial conditions $x|_{t=0} = x_0$ and $\mathrm{d}x/\mathrm{d}t|_{t=0} = v_0$ is

$$\begin{aligned}
x(t) = {} & \frac{F(\omega_0^2 - \omega^2)/m}{(\omega_0^2 - \omega^2)^2 + \Gamma^2\omega^2} \cos(\omega t) \\
&+ \frac{\Gamma\omega F/m}{(\omega_0^2 - \omega^2)^2 + \Gamma^2\omega^2} \sin(\omega t) \\
&+ \left[x_0 - \frac{F(\omega_0^2 - \omega^2)/m}{(\omega_0^2 - \omega^2)^2 + \Gamma^2\omega^2}\right] e^{-\Gamma t/2} \cos(\omega_\Gamma t) \\
&+ \left[\frac{v_0}{\omega_\Gamma} - \frac{\Gamma\omega F/m}{(\omega_0^2 - \omega^2)^2 + \Gamma^2\omega^2} \frac{\omega}{\omega_\Gamma}\right] e^{-\Gamma t/2} \sin(\omega_\Gamma t),
\end{aligned} \tag{4.14}$$

Box 4.2 continued

with

$$\omega_\Gamma = \omega_0 \left(1 - \frac{\Gamma^2}{4\omega_0^2}\right)^{1/2}.$$

Let's simplify this a bit. First, the last two terms decay as $e^{-\Gamma t/2}$: these are transients set by the initial conditions. After a time $t \gg 2/\Gamma$ only the first two terms, which oscillate at the driving frequency ω, will remain.

We can simplify these first two terms even further: if we write

$$\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}, \quad \sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i},$$

we can combine them and obtain

$$\begin{aligned} x(t) &= \frac{F}{2m}\left[\frac{1}{(\omega_0^2 - \omega^2) + i\Gamma\omega}\right] e^{i\omega t} \\ &\quad + \frac{F}{2m}\left[\frac{1}{(\omega_0^2 - \omega^2) - i\Gamma\omega}\right] e^{-i\omega t} \\ &= \Re\left\{\frac{F}{m}\left[\frac{1}{(\omega_0^2 - \omega^2) + i\Gamma\omega}\right] e^{i\omega t}\right\} \end{aligned} \tag{4.15}$$

We use the symbol "$\Re$" to denote taking the real part of a complex quantity. The oscillation is thus described as the real part of a complex quantity $Ae^{i\omega t}$, with

$$A = \frac{F}{m}\left[\frac{1}{(\omega_0^2 - \omega^2) + i\Gamma\omega}\right]$$

being the (complex) amplitude.

For $\omega \approx \omega_0$, we approximate $(\omega_0^2 - \omega^2) \approx 2\omega_0(\omega_0 - \omega)$ and take the square of the amplitude to find,

$$\begin{aligned} |A|^2 &= \left(\frac{F}{2m\omega_0}\right)^2 \frac{1}{(\omega_0 - \omega)^2 + (\Gamma/2)^2} \\ &= \frac{\pi}{2\Gamma}\left(\frac{F}{m\omega_0}\right)^2 \left\{\frac{1}{\pi}\frac{\Gamma/2}{(\omega_0 - \omega)^2 + (\Gamma/2)^2}\right\} \end{aligned} \tag{4.16}$$

We rewrote the amplitude in the second line so that the term in $\{\cdot\}$ is normalized. The amplitude is a Lorentzian function of the driving frequency ω.

In a stellar atmosphere, the width Γ is set by collisions. For example, when an electron passes close by our atom, the electric field shifts the energy levels of the atom[10]. The greater the collision rate, the larger the width. If we have two stars of the same photospheric temperature (so that both stars have the same lines), then a way to increase the collision rate is to increase the pressure. Recall, however, that in the stellar atmosphere $P = (g/\kappa)\tau$; as a result, stars with a higher surface gravity will have broader lines. The inset in Figure 4.7 illustrates the broadening of the Balmer Hγ line ($2 \to 5$) in the spectrum of a main-sequence A1 star compared with that of a supergiant A1 star.

[10] This is an application of the *Stark* effect that you learn about in quantum mechanics.

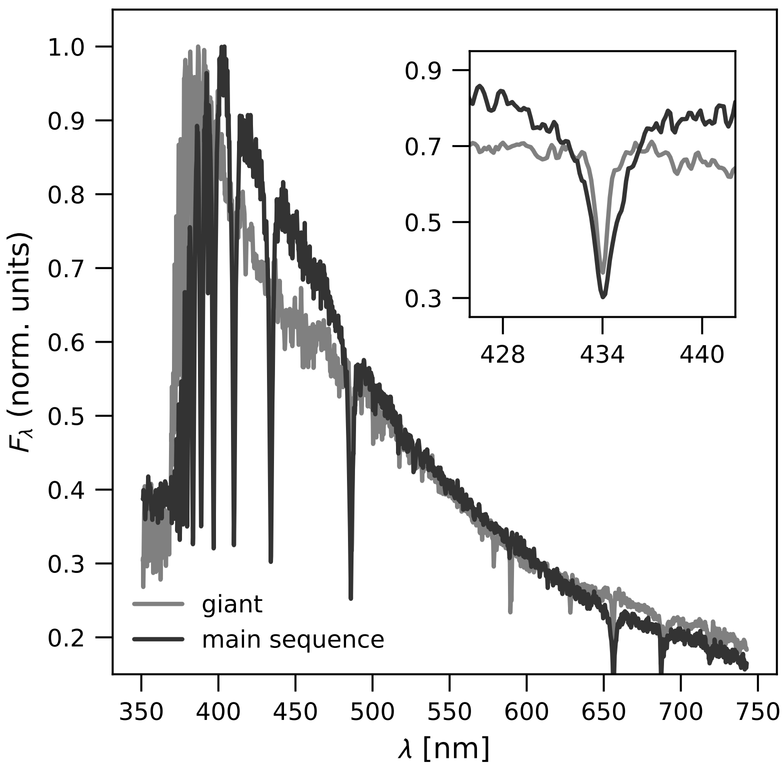


Figure 4.7: Spectra of two A1 stars, HD 16608 (a main sequence star) and SAO 12149 (a supergiant star). Spectra are from Jacoby et al. [1984].

5
Burn

To recap, we have established a description for the basic features of a self-gravitating fluid:

1. For a set mass and radius, hydrostatic equilibrium (balance of pressure and gravity) is established on the time needed for a sound wave to cross the star. Once this equilibrium is established, the central pressure, density, and temperature are established.

2. The gradient in temperature from center to surface drives a luminosity, which is controlled by the opacity of material in the stellar interior.

3. The ambient pressure and temperature near the stellar photosphere (where $\tau \sim 1$) are set by the surface gravity and opacity.

In this chapter we now discuss how the luminosity is generated by nuclear reactions in the core of a star, and the conditions needed to generate that luminosity.

5.1 The nucleus

Experimentally, nuclei are on the order of femtometers[1] in size. Like an atom, the nucleus also has excited states; typical energies for these are on the order of MeV[2]. It therefore makes sense to use fm and MeV as our units of length and energy. In these units, the combination

$$\hbar c = 197\,\mathrm{MeV\,fm}$$

to three significant figures. In quantum field theory, the strength of the electromagnetic interaction is characterized by the dimensionless FINE STRUCTURE CONSTANT

$$\alpha_{\mathrm{F}} = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = \frac{1}{137},$$

[1] $1\,\mathrm{fm} = 10^{-15}\,\mathrm{m}$. This unit is sometimes called a FERMI.

[2] $1\,\mathrm{MeV} = 10^6\,\mathrm{eV}$; an ELECTRON VOLT (eV) is the energy acquired by an electron being accelerated through a potential difference of 1 volt.

again to three significant figures. From these two quantities, we find the electron (or proton) charge in these units,

$$\frac{e^2}{4\pi\varepsilon_0} = \alpha_F \hbar c = 1.44\,\mathrm{MeV\,fm}.$$

Put another way, the Coulomb potential energy between two protons separated by 1 fm is 1.44 MeV.

The strong nuclear force differs from electromagnetism and gravity in several ways. First, the strong nuclear force is short-range: the interaction vanishes for distances $\gtrsim 2\,\mathrm{fm}$. It is weakly attractive for distances $1\,\mathrm{fm} \lesssim r \lesssim 2\,\mathrm{fm}$ and becomes strongly repulsive at distances $\ll 1\,\mathrm{fm}$. The potential between the neutron and proton in a deuterium (^{2}H) nucleus (called a deuteron) therefore looks something like that sketched in Fig. 5.1. The deuteron's ground state (black dotted line) is at $E_d = -2.2\,\mathrm{MeV}$, so the nucleus is weakly bound ($|E_d| \ll |V|$, where V is the depth of the potential well).

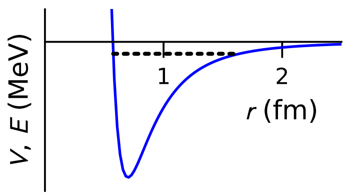


Figure 5.1: Schematic of the nuclear potential for a deuteron (^{2}H). The binding energy of the deuteron is shown as a black dotted line.

In our units of MeV and fm, some relevant masses are

$$\begin{aligned} m_n &= 939.6\,\mathrm{MeV}/c^2 \\ m_p &= 938.3\,\mathrm{MeV}/c^2 \\ m_u &= 931.5\,\mathrm{MeV}/c^2 \\ m_e &= 0.5110\,\mathrm{MeV}/c^2 \end{aligned}$$

EXERCISE 5.1— We can estimate the depth of the well in Fig. 5.1. Since this is a two-body problem, transfer to center-of-mass coordinates and solve for a single particle with a reduced mass $m_p m_n/(m_p + m_n) \approx m_n/2$. Use the uncertainty principle, with Δx being the width of the well, to get an estimate of $p \sim \Delta p$ and from this estimate the kinetic energy of the particle. Finally, use the small value of the binding energy (sum of potential and kinetic energies) to estimate the depth of the potential well.

Also unlike electromagnetism and gravity, the strong nuclear force does not obey superposition: we cannot write the energy of the nucleus as a sum over the potential between all pairs of nucleons. Further, the strong nuclear force is not a central force, meaning that it depends on more than just the distance between any two nucleons. The atomic nucleus is thus much more complicated to describe than the electronic structure of the atom.

DESPITE THESE COMPLICATIONS, WE CAN CONSTRUCT A PHENOMENOLOGICAL FORMULA FOR THE NUCLEAR MASS THAT IS REASONABLY ACCURATE. Let us write the mass of a nucleus with A nucleons—Z protons and $N = A - Z$ neutrons—as

$$M(Z,N) = Zm_p + Nm_n - B(Z,N)/c^2,$$

where $B(Z,N)$ is the BINDING ENERGY—the amount of energy that must be supplied to the nucleus in order to break it into its constituent protons and neutrons. Because the nuclear force is weakly attractive for separations $1\,\mathrm{fm} \lesssim r \lesssim 2\,\mathrm{fm}$ and repulsive at shorter distances (Fig. 5.1), there is a characteristic spacing between nucleons that is a bit larger than 1 fm. In a large nucleus, we therefore expect the nucleons

An alternative formulation uses the MASS EXCESS, defined via

$$M(Z,N) = Am_u + \Delta(Z,N)/c^2$$

with $\Delta(^{12}\mathrm{C}) \equiv 0$. Here $M(Z,N)$ is the *atomic mass*, including electrons.

to have a roughly constant density, so that the volume of the nucleus is proportional to A; experimentally, the radius of the nucleus is roughly[3]

$$r_A = (1.1 \text{ to } 1.8)\,\text{fm} \times A^{1/3}.$$

[3] The value of the radius depends on how it is measured; scattering with various light particles (protons, neutrons, alpha, electrons) agree, however, that $r_A \propto A^{1/3}$.

Notice that because the nucleon-nucleon potential is short-ranged, nucleons in a large nucleus only interact with their nearest neighbors. Indeed the nucleon-nucleon interaction is similar in form to the potential between molecules in a fluid, such as a water drop. This motivates developing a simple formula that gives a decent approximation for the binding energy. For the first term, we estimate the binding energy of a large nucleus as just the (constant) binding energy of a single nucleon multiplied by the number of nucleons. Experimentally, it is found that for large nuclei this is the case: the binding energy per nucleon is roughly constant. We say that the nuclear interaction SATURATES, so that $B(Z,N) \propto A = (Z+N)$.

EXERCISE 5.2 — To see how the nuclear force differs from the long-range Coulomb and gravitational forces, suppose instead that the nuclear force acted like a super-gravity: that is, the potential is $\propto 1/r$. Use the results from our constant-density model of a star (eq. [2.22]) to derive how the binding energy would scale with A in this case.

It is energetically favorable to have equal numbers of neutrons and protons. We therefore define an asymmetry parameter $\eta \equiv (N-Z)/(N+Z) = 1 - 2Z/A$, so that $-1 \leq \eta \leq 1$. The nuclear contribution to the binding energy is maximized for $\eta = 0$ (equal numbers of protons and neutrons). Because the nuclear force does not distinguish between neutrons and protons, the binding energy is quadratic in η, so that B doesn't depend on the sign of η. Thus our first approximation for the binding energy is $B \approx (a_V - a_A\eta^2)A$. Here a_V and a_A are as-yet-undetermined coefficients.

In a fluid drop there is a correction for the surface tension. Heuristically, we imagine that nuclei in the surface have fewer neighbors and are therefore not as bound. We therefore subtract from our formula a term proportional to the surface area, $\propto r_A^2 \propto A^{2/3}$. The next iteration of our liquid-drop approximation is thus $B \approx (a_V - a_A\eta^2)A - a_S A^{2/3}$.

Finally, the protons in the nucleus are charged and therefore repel one another. This Coulomb repulsion also reduces the binding energy. We therefore subtract a term $\propto Z^2/r_A \propto Z^2/A^{1/3}$ from our mass formula to obtain

$$B = \left(a_V - a_A\eta^2\right)A - a_S A^{2/3} - a_C \frac{Z^2}{A^{1/3}}. \tag{5.1}$$

This is a version of the SEMI-EMPIRICAL MASS FORMULA, also known as the BETHE-WEIZSÄCKER MASS FORMULA. The coefficients a_V, a_A, a_S, a_C are found by fitting the formula to measured nuclear masses (Table 5.1).

Table 5.1: Coefficients for the fit to nuclear masses, (5.1), in units of MeV.

a_V	a_A	a_S	a_C
15.5	22.7	16.6	0.71

This fit should have another term to account for the pairing of neutrons and protons, so that the binding energy is increased for even Z and N. We omit that term here for simplicity.

EXERCISE 5.3— For a given nuclear mass number A, derive an expression for the charge number $Z_\star(A)$ that maximizes the binding energy (eq. [5.1] with coefficients from Table 5.1).

1. Plot the ratio $Z_\star/A$ for $4 \le A \le 128$. Give a physical explanation for the behavior of $Z_\star/A$.
2. Plot the binding energy per nucleon B/A as a function of $Z_\star$ and A, for $4 \le A \le 128$.
3. Find the atomic number Z and atomic mass A of the nucleus with the maximum B/A.

5.2 *Nuclear reactions*

From mass-energy conservation, the heat evolved during a nuclear reaction equals the change in mass of the reacting system. For example, in the reaction

$$^3\mathrm{He} + {}^3\mathrm{He} \to {}^4\mathrm{He} + \mathrm{p} + \mathrm{p},$$

the binding energy of ^{3}He is 7.718 MeV and that of ^{4}He is 28.296 MeV; the heat evolved by this reaction is therefore

$$\begin{aligned}
&2\left[2m_p + m_n - B(^3\mathrm{He})\right] - \left[2m_p + 2m_n - B(^4\mathrm{He})\right] - 2m_p \\
&\quad= B(^4\mathrm{He}) - 2B(^3\mathrm{He}) \\
&\quad= 28.296\,\mathrm{MeV} - 15.437\,\mathrm{MeV} \\
&\quad= 12.859\,\mathrm{MeV}.
\end{aligned}$$

EXERCISE 5.4— Fusion of hydrogen into helium entails converting 4 hydrogen atoms (including the 4 electrons) into 1 helium atom (2 protons, 2 neutrons, 2 electrons) with $B = 28.296$ MeV. What is the heat evolved *per hydrogen atom*? Assume that the sun has been shining with its current luminosity over its life. What mass of hydrogen atoms would need to undergo fusion to supply this energy? How large is this mass relative to the total mass of the sun?

YOU MIGHT THINK THAT BECAUSE THE NUCLEAR INTERACTION IS SHORT-RANGE, THE CROSS-SECTION IS SOMETHING LIKE πr_n^2, WHERE $r_n \approx$ (1 TO 2) fm. Things are a bit more subtle, however, and in this section we shall explore how the reaction rate works. First, the "size" of a particle is in general proportional to the "size" of the wavefunction. From the uncertainty principle,

$$\pi\,\Delta x^2 \approx \pi\left(\frac{\hbar}{\Delta p}\right)^2 = \pi\frac{\hbar^2}{2mE}.$$

where we've taken $\Delta p \sim p$. Notice that if we multiply and divide by c^2, then we can estimate the area of the wavepacket as

$$\frac{(\hbar c)^2}{m_p c^2}\frac{1}{E} \sim 4\times 10^4\,\mathrm{fm}^2 \times \left(\frac{\mathrm{keV}}{E}\right) = 400\,\mathrm{b}\left(\frac{\mathrm{keV}}{E}\right).$$

Here we've introduced a convenient unit for cross-sections, the BARN[4] (b), with $1\,\mathrm{b} = 10^{-28}\,\mathrm{m}^2 = 100\,\mathrm{fm}^2$.

[4] as in hitting the broad side of

The key point is that the size of the wave packet is $\propto 1/E$, which is in general true. This geometrical size of the wave packet is then multiplied by the probability of the nucleons forming a bound state, so we write the nuclear portion of the cross-section as

$$\sigma_{\text{nuclear}}(E) = \frac{S(E)}{E}.$$

The function $S(E)$ contains the details of the nuclear interaction; in general $S(E)$ must be measured experimentally.

The final part of the cross-section concerns the Coulomb potential. Because protons repel one another, at large separations the nuclei interact only via the Coulomb potential. Consider the case of two nuclei with masses[5] $A_1 m_\mathrm{u}$ and $A_2 m_\mathrm{u}$. Transform to the center-of-mass frame; the problem then reduces to that of one particle, mass $Am_\mathrm{u} = A_1A_2/(A_1+A_2)\times m_\mathrm{u}$, moving in a potential, which at large separations is purely Coulomb,

[5] When doing kinematics, we shall make the approximation $m \approx Am_\mathrm{u}$.

$$\frac{Z_1Z_2e^2}{4\pi\varepsilon_0 r} = \frac{Z_1Z_2\alpha_\mathrm{F}\hbar c}{r} = 1.44\,\mathrm{MeV}\times Z_1Z_2\left(\frac{1\,\mathrm{fm}}{r}\right).$$

Fig. 5.2 has a schematic of the potential. While at short distances the nuclear interaction forms a deep potential well, outside the nucleus the Coulomb potential dominates.

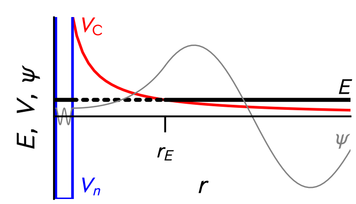


Figure 5.2: Tunneling through the Coulomb potential barrier. Not to scale.

EXERCISE 5.5— For the sun, typical center-of-mass energies are $E \sim 1\,\mathrm{keV}$ (horizontal black line in Fig. 5.2). Suppose we have two protons heading towards one another with this kinetic energy. What is their distance of closest approach?

As shown in Exercise 5.5, the turning radius r_E at typical stellar energies is much larger than the nuclear radius. Classically the particle can't penetrate the region $r_n < r < r_E$ where $E < V$ (dotted black line, Fig. 5.2); under classical physics, there would be no nuclear reactions at typical stellar temperatures because two particles would never find themselves close enough to be bound by the nuclear force.

The world is quantum, however, and the uncertainty in a particle's position means there is a small probability for the nucleons to be close enough for the nuclear force to come into play. In the classically forbidden region $r_n < r < r_E$, the particle wavefunction (thin gray line, Fig. 5.2) decreases exponentially, and the probability to reach $r \sim 1\,\mathrm{fm}$ is

$$\mathcal{P} \approx \exp\left[-2\pi^2 \frac{r_E}{\lambda}\right]$$

where $\lambda = h/p$, p being the momentum of the particle. It is convenient to rewrite the argument of the exponential in terms of the particle's energy,

$$\frac{2\pi^2 r_E}{\lambda} = 2\pi^2 \left(\frac{Z_1 Z_2 e^2}{E}\right)\left(\frac{p}{h}\right) = \left[\pi \frac{Z_1 Z_2 e^2 \sqrt{2m}}{\hbar}\right]\left(\frac{1}{E}\right)^{1/2},$$

so that the probability of "tunneling" through the Coulomb barrier is

$$\mathcal{P} \approx \exp\left[-\left(\frac{E_\mathrm{G}}{E}\right)^{1/2}\right], \qquad (5.2)$$

with

$$E_\mathrm{G} \equiv \text{"Gamow Energy"} = \left[\frac{2\pi^2 Z_1^2 Z_2^2 e^4 m}{\hbar^2}\right] = Z_1^2 Z_2^2 A \times 979\,\mathrm{keV}.$$

Our reaction cross-section is therefore the nuclear cross-section multiplied by the probability of tunneling,

$$\sigma(E) = \frac{S(E)}{E} \exp\left[-\left(\frac{E_\mathrm{G}}{E}\right)^{1/2}\right]. \qquad (5.3)$$

For many reactions $S(E)$ is nearly constant over the range of typical energies in a stellar plasma; this is helpful, as the reaction cross-section can be measured in the lab at higher energies and then extrapolated to the much lower stellar energies.

TO GET THE REACTION RATE FROM THE CROSS-SECTION, recall that the mean-free path of a particle is $\ell = (n\sigma)^{-1}$, where n is the density of targets. For definiteness, let us consider a particle of type 1. The mean-free path of this particle against reactions with particles of type 2 is therefore $\ell = (n_2\sigma)^{-1}$. If the particles are traveling with relative speed $v = |\boldsymbol{v}_1 - \boldsymbol{v}_2|$, then the mean time between collisions is ℓ/v. Thus in a large ensemble of particles, the mean rate of reactions is

$$r_{12} = \frac{n_1 v}{\ell} = n_2 n_1 \langle \sigma v \rangle.$$

Here $\langle \sigma v \rangle$ is the mean value of σv for all pairs of particles in the plasma. For reactions between particles of the same type, we replace $n_1 n_2$ with $n^2/2$; the factor of $1/2$ is to avoid double-counting.

A detailed calculation of the thermally averaged cross-section $\langle \sigma v \rangle$ is presented in Box 5.1; here we'll just give a brief physical explanation for its value. There are two competing terms. First, the cross-section has an exponential term $\exp[-(E_G/E)^{1/2}]$ that increases rapidly with energy: more energetic particles have a much higher probability of tunneling through the Coulomb barrier. On the other hand, in thermal equilibrium the number of particles with energy E decreases as $\exp(-E/k_B T)$. As a result, reactions predominately occur in a narrow window of energies about a sort of geometric mean between E_G and $k_B T$:

$$E_{pk} = \frac{E_G^{1/3}(k_B T)^{2/3}}{4^{1/3}}.$$

The reaction rate is suppressed for $E \ll E_{pk}$ because the probability of penetrating the Coulomb barrier is so small; the reaction rate is suppressed for $E \gg E_{pk}$ because there simply aren't enough particles with the relevant center-of-mass energy.

Box 5.1 The thermally averaged cross-section

Since the cross-section depends on energy, the rate at which any given particle of type 1, traveling with velocity $\boldsymbol{v}_1$, will react with particles of type 2 having velocities $\boldsymbol{v}_2$ in a range d^3v_2 is

$$n_2\sigma|\boldsymbol{v}_1 - \boldsymbol{v}_2| \left(\frac{m_2}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{m_2 v_2^2}{2k_B T}\right) d^3v_2.$$

The extra terms are because the particles have a Maxwell-Boltzmann distribution of velocities. To get the total rate per unit volume, we then have to multiply by the number of particles of type 1 having velocities $\boldsymbol{v}_1$ in a range d^3v_1 and integrate over $d^3v_1\, d^3v_2$:

$$\begin{aligned} r_{12} = n_1 n_2 &\left[\frac{m_1 m_2}{(2\pi k_B T)^2}\right]^{3/2} \\ &\times \int \sigma(E) v \exp\left(-\frac{m_1 v_1^2}{2k_B T} - \frac{m_2 v_2^2}{2k_B T}\right) d^3v_1\, d^3v_2. \end{aligned} \tag{5.4}$$

Now E and v are the relative energies and velocity in the center-of-mass frame. We can change variable using the relations

$$\begin{aligned} \boldsymbol{v}_1 &= \boldsymbol{V} - \frac{m_2}{m_1 + m_2}\boldsymbol{v} \\ \boldsymbol{v}_2 &= \boldsymbol{V} + \frac{m_1}{m_1 + m_2}\boldsymbol{v}. \end{aligned}$$

where V is the center-of-mass velocity. It is straightforward to show that $dv_{1,x}\, dv_{2,x} = dV_x dv_x$, and likewise for the y, z directions. Furthermore, $m_1 v_1^2 + m_2 v_2^2 = (m_1 + m_2)V^2 + mv^2$, and multiplying

Box 5.1 continued

and dividing the integral in equation (5.4) by $m_1 + m_2$ allows us to write

$$r_{12} = n_1 n_2 \left(\frac{m_1+m_2}{2k_B T}\right)^{3/2} \left(\frac{m}{2k_B T}\right)^{3/2} \times \int d^3V \int d^3v\, \sigma(E) v \exp\left[-\frac{mv^2}{2k_B T}\right] \exp\left[-\frac{(m_1+m_2)V^2}{2k_B T}\right].$$

The integral over d^3V can be factored out and is normalized to unity. Hence we have for the reaction rate between a pair of particles 1 and 2,

$$\begin{aligned} r_{12} &= n_1 n_2 \left\{ \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_0^\infty \sigma(E) v \exp\left(-\frac{mv^2}{2k_B T}\right) 4\pi v^2\, dv \right\}. \\ &\equiv n_1 n_2 \langle \sigma v \rangle. \end{aligned} \tag{5.5}$$

The term in $\{\}$ is the averaging over the joint distribution of the cross-section times the velocity, and is usually denoted as $\langle \sigma v \rangle$. Note that if particles 1 and 2 were identical, then we would need to divide r_{12} by 2.

Changing variables to $E = mv^2/2$ in equation (5.5) and inserting the formula for the cross-section, equation (5.3), gives

$$\langle \sigma v \rangle = \left(\frac{8}{\pi m}\right)^{1/2} \left(\frac{1}{k_B T}\right)^{3/2} \int_0^\infty S(E) \exp\left[-\left(\frac{E_G}{E}\right)^{1/2} - \frac{E}{k_B T}\right] dE. \tag{5.6}$$

Now, we've assumed that $S(E)$ varies slowly; but look at the argument of the exponential. This is a competition between a rapidly rising term $\exp[-(E_G/E)^{1/2}]$ and a rapidly falling term $\exp(-E/k_B T)$. As a result, the exponential will have a strong peak, and we can expand the integrand in a Taylor series about the maximum. Let

$$f(E) = -\left(\frac{E_G}{E}\right)^{1/2} - \frac{E}{k_B T}.$$

Then we can write

$$\int_0^\infty S(E) \exp\left[-\left(\frac{E_G}{E}\right)^{1/2} - \frac{E}{k_B T}\right] dE \approx \int_0^\infty S(E_{pk}) \exp\left[f(E_{pk}) + \frac{1}{2} \left.\frac{d^2 f}{dE^2}\right|_{E=E_{pk}} (E - E_{pk})^2\right] dE.$$

Here E_{pk} is found by solving $(df/dE)|_{E=E_{pk}} = 0$. By expanding the argument of the exponential, we have approximated the

Box 5.1 continued

integrand by a Gaussian,

$$\exp\left[-\frac{(E-E_{\text{pk}})^2}{2\varsigma^2}\right]$$

where

$$\frac{1}{\varsigma^2} = -\left.\frac{\mathrm{d}^2 f}{\mathrm{d}E^2}\right|_{E=E_{\text{pk}}}.$$

This trick of approximating a steeply peaked function as a Gaussian is known as the METHOD OF STEEPEST DESCENT.

Solving for E_{pk}, we get

$$E_{\text{pk}} = \frac{E_{\text{G}}^{1/3}(k_{\text{B}}T)^{2/3}}{2^{2/3}},$$

and

$$\exp\left[f(E_{\text{pk}})\right] = \exp\left[-3\left(\frac{E_{\text{G}}}{4k_{\text{B}}T}\right)^{1/3}\right].$$

Further,

$$\frac{1}{2}\left.\frac{\mathrm{d}^2 f}{\mathrm{d}E^2}\right|_{E=E_{\text{pk}}} = -\frac{3}{2(2E_{\text{G}})^{1/3}(k_{\text{B}}T)^{5/3}} = -\frac{3}{4E_{\text{pk}}k_{\text{B}}T}.$$

Defining a variable $\Delta = 4(E_{\text{pk}}k_{\text{B}}T/3)^{1/2}$, our integral becomes

$$\langle\sigma v\rangle = \left(\frac{8}{\pi m}\right)^{1/2}\left(\frac{1}{k_{\text{B}}T}\right)^{3/2} S(E_{\text{pk}}) \times \exp\left[-3\left(\frac{E_{\text{G}}}{4k_{\text{B}}T}\right)^{1/3}\right]\int_0^\infty \exp\left[-\frac{(E-E_{\text{pk}})^2}{(\Delta/2)^2}\right]\mathrm{d}E. \quad (5.7)$$

Another simplification can be made because both the Gaussian and the original integrand go to zero as $E \to 0$. As a result, we can extend the lower bound of our integral (eq. [5.7]) to $-\infty$, which allows us to evaluate the integral analytically and obtain

$$\begin{aligned}\langle\sigma v\rangle &\approx \left(\frac{8}{m}\right)^{1/2}\left(\frac{1}{k_{\text{B}}T}\right)^{3/2} S(E_{\text{pk}})\exp\left[-3\left(\frac{E_{\text{G}}}{4k_{\text{B}}T}\right)^{1/3}\right]\frac{\Delta}{2} \\ &= \frac{2^{13/6}}{\sqrt{3m}}\frac{E_{\text{G}}^{1/6}S(E_{\text{pk}})}{(k_{\text{B}}T)^{2/3}}\exp\left[-3\left(\frac{E_{\text{G}}}{4k_{\text{B}}T}\right)^{1/3}\right]. \quad (5.8)\end{aligned}$$

Using this approximation the rate can be evaluated, with $\langle\sigma v\rangle$ being given by eq. (5.8). The rate has the temperature dependence

$$r \propto T^{-2/3}\exp\left[-3\left(\frac{E_{\text{G}}}{4k_{\text{B}}T}\right)^{1/3}\right];$$

since $E_G \propto Z_1^2 Z_2^2 A$, at any given temperature lighter nuclei typically have much faster reaction rates. Also note that at stellar energies, reaction rates are incredibly sensitive to temperature. To quantify this, approximate the rate at a given temperature as a power-law, $r(T) \propto T^n$. Then the exponent is

$$n(T) = \frac{\partial \ln r}{\partial \ln T} = -\frac{2}{3} + \left(\frac{E_G}{4k_B T}\right)^{1/3}, \tag{5.9}$$

as you can verify for yourself (Exercise 5.6). Table 5.2 lists E_G, E_{pk}, and n for some common reactions. In the table, the peak reaction energy E_{pk} and exponent $n(T)$ are evaluated at $T = 10^7$ K ($k_B T = 0.86$ keV). Note the large value of $n(T)$ at stellar temperatures—this is a consequence of the largeness of $E_G/k_B T$.

Table 5.2: Parameters for non-resonant reactions

	$\mathrm{p}+\mathrm{p}$	$\mathrm{p}+{}^3\mathrm{He}$	${}^3\mathrm{He}+{}^3\mathrm{He}$	$\mathrm{p}+{}^7\mathrm{Li}$	$\mathrm{p}+{}^{12}\mathrm{C}$
E_G (MeV)	0.489	2.94	23.5	7.70	32.5
$E_{pk}\vert_{T=10^7\,\mathrm{K}}$ (keV)	4.5	8.2	16.3	11.3	18.2
$n(T = 10^7\,\mathrm{K})$	4.6	8.8	18.3	12.4	20.5

EXERCISE 5.6— Suppose we wish to approximate a function $f(x)$ at a point x_0 with a power-law, $p(x; A, n) = Ax^n$. Impose the condition $p(x_0; A, n) = f(x_0)$ and $\mathrm{d}p/\mathrm{d}x|_{x=x_0} = \mathrm{d}f/\mathrm{d}x|_{x=x_0}$ to find the parameters A and n, and show that

$$n = \frac{\mathrm{d}\ln f}{\mathrm{d}\ln x}.$$

Apply this to the reaction rate, eq. (5.8), and thus derive eq. (5.9).

5.3 *Stellar nuclear reactions*

Hydrogen burning via pp reactions: the weak nuclear interaction

In the previous section, we established that lighter nuclei, because of their lower Coulomb repulsion, will tend to fuse at lower temperatures. Thus we expect that the first reaction that can occur is p + p and therein lies a problem: there is no bound state of ^{2}He. The only possible way for two protons to fuse is for one of the protons to transmute into a neutron, giving the reaction

$$\mathrm{p} + \mathrm{p} \to {}^2\mathrm{H} + e^+ + \nu_e. \tag{5.10}$$

This reaction is possible because there are two nuclear forces: the strong and the weak. The strong is what binds nuclei together; the weak mediates the conversion of a neutron into a proton (and vice versa). Two leptons are also involved (either emitted or absorbed) in this type of weak reaction: an electron (or its anti-particle, the positron) and an electron neutrino (or anti-neutrino). Three conservation laws determine which particles are involved:

More precisely, what the strong force mainly does is bind quarks into neutrons or protons.

1. the number of nucleons is conserved;

2. the charge is conserved; and

3. the number of leptons is conserved.

With regard to item 3, electrons (e^-) and electron neutrinos (ν_e) have lepton number $+1$ while positrons (e^+) and anti-electron neutrinos ($\bar{\nu}_e$) have lepton number -1. Neutrinos, as the name implied, do not carry charge.

Applying these rules to the reaction (5.10), the number of nucleons on both sides of this reaction is the same, so rule 1 is satisfied. The positron on the right hand side balances charge to satisfy rule 2. Finally, the emission of an electron neutrino ensures that the lepton number on the right-hand side is zero to satisfy rule 3.

The weak cross section goes roughly as $\sigma_{\text{weak}} \sim 10^{-20}\,\text{b}\,(E/\text{keV})$, so that

$$\frac{\sigma_{\text{weak}}}{\sigma_{\text{nuc}}} \sim 10^{-23}\left(\frac{E}{\text{keV}}\right).$$

As a result the characteristic temperature for reaction (5.10) to occur is $\approx 1.5 \times 10^7$ K, much higher than the temperature at which $\mathrm{p} + {}^2\mathrm{H}$ occurs; at this temperature, the lifetime of a proton to forming deuterium via capture of another proton is about 6 Gyr. Once a deuterium nucleus is formed, it is immediately destroyed via ${}^2\mathrm{H} + p \to {}^3\mathrm{He}$. The nucleus ${}^4\mathrm{Li}$ is unbound with a lifetime of 10^{-22} s; the nucleus ${}^6\mathrm{Be}$ is likewise unbound ($\tau \sim 5 \times 10^{-21}$ s). As a result, the next reaction that can occur is ${}^3\mathrm{He} + {}^3\mathrm{He} \to 2\mathrm{p} + {}^4\mathrm{He}$. Despite having a much greater Gamow energy than $\mathrm{p} + \mathrm{p}$, this reaction still is much faster than $\mathrm{p} + \mathrm{p}$ owing to the small weak cross-section.

Because the weak cross section is so small, the first reaction that occurs in a contracting pre-main sequence star is ${}^2\mathrm{H} + \mathrm{p} \to {}^3\mathrm{He}$; in fact, this reaction can occur in objects as small as $\approx 12\,M_{\text{Jupiter}}$. The small primordial abundance of deuterium, however, prevents this reaction from doing anything more than slowing contraction slightly.

In addition to capturing another ${}^3\mathrm{He}$, it is also possible for ${}^3\mathrm{He}$ to react with ${}^4\mathrm{He}$ and trigger the reactions

In eq. (5.11) and (5.12), τ refers to the half-life for the nucleus on the left.

$$\begin{aligned}
{}^3\mathrm{He} + {}^4\mathrm{He} &\to {}^7\mathrm{Be} + \gamma \\
{}^7\mathrm{Be} + e^- &\to {}^7\mathrm{Li} + \nu_e \qquad (\tau = 53\,\mathrm{d}) \\
{}^7\mathrm{Li} + \mathrm{p} &\to 2{}^4\mathrm{He} + \gamma;
\end{aligned} \tag{5.11}$$

furthermore, at slightly higher temperatures ${}^7\mathrm{Be}$ can capture a proton instead of an electron to yield

$$\begin{aligned}
{}^7\mathrm{Be} + \mathrm{p} &\to {}^8\mathrm{B} + \gamma \\
{}^8\mathrm{B} &\to {}^8\mathrm{Be} + e^+ + \nu_e \qquad (\tau = 770\,\mathrm{ms}) \\
{}^8\mathrm{Be} &\to 2{}^4\mathrm{He} \qquad (\tau = 10^{-16}\,\mathrm{s}).
\end{aligned} \tag{5.12}$$

The end result of these chains is the conversion of hydrogen to helium, although the amount of energy carried away by neutrinos differs from one chain to the next.

Hydrogen burning via the CNO cycle

As we saw in the previous section, the smallness of the p+p cross-section means that proton captures onto heavier nuclei can occur at similar, or even faster rates, than p + p despite the larger Coulomb barrier. At $T_6 = 10$, proton captures onto ^{12}C have a comparable cross-section to p + p; at $T_6 = 20$, proton captures onto ^{16}O have a comparable cross-section. Thus at temperatures slightly greater than that in the solar center, the following catalytic cycle becomes possible:

$$\begin{aligned} {}^{12}\mathrm{C} + \mathrm{p} &\to {}^{13}\mathrm{N} \\ {}^{13}\mathrm{N} &\to {}^{13}\mathrm{C} + e^{+} + \nu_e \\ {}^{13}\mathrm{C} + \mathrm{p} &\to {}^{14}\mathrm{N} \\ {}^{14}\mathrm{N} + \mathrm{p} &\to {}^{15}\mathrm{O} \\ {}^{15}\mathrm{O} &\to {}^{15}\mathrm{N} + e^{+} + \nu_e \\ {}^{15}\mathrm{N} + \mathrm{p} &\to {}^{12}\mathrm{C} + {}^{4}\mathrm{He} \end{aligned}$$

The net result of this cycle is the ingestion of 4 protons and release of 1 ^{4}He nucleus. The reaction $^{14}\mathrm{N} + \mathrm{p} \to {}^{15}\mathrm{O}$ is by far the slowest step in the cycle; as a result, all of the CNO elements are quickly converted into ^{14}N in the stellar core, and this reaction controls the rate of heating. At $T = 2 \times 10^7$ K, $\partial \ln \epsilon_{\mathrm{CNO}} / \partial \ln T = 18$; in contrast the p + p reaction has a temperature exponent of only 4.5.

5.4 *The luminosity equation*

Suppose we have a shell of mass $\Delta m = 4\pi r^2 \rho \Delta r$ lying between surfaces r and $r + \Delta r$ (Fig. 5.3). Nuclear reactions in the shell heat it at a rate $\Delta m \times \epsilon$, where ϵ is the heating rate per unit mass. In addition, heat enters the shell from the bottom at a rate $L(r)$ and leaves from the top at a rate $-L(r + \Delta r)$. If the shell is neither gaining or losing heat, then all these terms must balance:

$$4\pi r^2 \rho \epsilon \, \Delta r + L(r) - L(r + \Delta r) = 0.$$

Taking the limit $\Delta r \to 0$ produces our fourth equation of stellar structure,

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \rho \epsilon.$$

At the center, $L(r)_{r \to 0} \to 0$, while at the surface $L(r)_{r \to R} \to 4\pi R^2 \sigma_{\mathrm{SB}} T_{\mathrm{eff}}^4$.

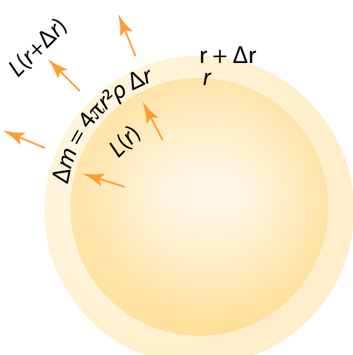


Figure 5.3: Heat balance in a shell Δm.

6
Star

We now have almost all of the physics necessary to describe the structure of a star. We only need two additional items: we must consider whether the fluid is at rest or whether there is circulation, and we must discuss how the equation of state deviates from that of a classical ideal gas at high densities. These changes in the equation of state are important for low-mass stars and set the minimum stellar mass.

6.1 Convection

We've established that in the interior of the star a temperature gradient,

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3\rho\kappa_R}{4acT^3}\frac{L(r)}{4\pi r^2},$$

arises to transport heat outward (cf. eq. [3.14]). This gradient becomes steeper as we increases either the flux $L/4\pi r^2$ or the mean opacity κ_R. There is a limit, however, to the magnitude of $|\mathrm{d}T/\mathrm{d}r|$: if the gradient is too steep, the warm fluid becomes buoyant relative to the cooler fluid above it and begins to rise. You are familiar with this phenomenon: picture a hot summer day. As the ground absorbs sunlight, it warms the air just above the ground. The warm air rises and forms updrafts. You have perhaps seen hawks circling as they are carried aloft by these updrafts. This circulation of fluid induced by a temperature gradient is known as CONVECTION.

You can do a home demonstration of convection. Brew tea, and pour the hot tea into a saucepan that is on an unlit burner. Use a straw to inject a layer of cold milk under the warm tea in the saucepan. The temperature difference between the tea and milk will inhibit their mixing. Light the burner, and watch for the development of convection—you will know it when you see it (Fig. 6.1).

Convection can also occur in stars, in regions of high flux and/or high opacity. During convection, the fluid velocities in question are typically quite subsonic, so hydrostatic equilibrium abides. But the fluid motions do make an enormous difference to heat transport! Warm fluid is carried

Figure 6.1: Onset of convection in a tea-milk mixture.

upward and cool fluid sinks. The net result is that heat is transported upward much faster than it would have been if only diffusion had been operating. This upward transport of heat modifies the temperature gradient. In this chapter, we shall derive the condition for the onset of convection, and the value of the temperature gradient in the presence of subsonic, efficient convection.

The onset of convection

To understand when convection starts, it helps to recall why a parcel of warm air rises. Recall Archimedes' law:

> *The buoyant force on an object, either wholly or partially immersed in a fluid under a constant gravitational acceleration, equals the weight of the fluid it displaces.*

What does this mean? A boat of mass m displaces (pushes aside) a volume v of water (density ρ_{w} when floating. The weight of this displaced water, $\rho_{\mathrm{w}} v g$, must equal the weight of the boat mg, so that $v = m/\rho_{\mathrm{w}}$.

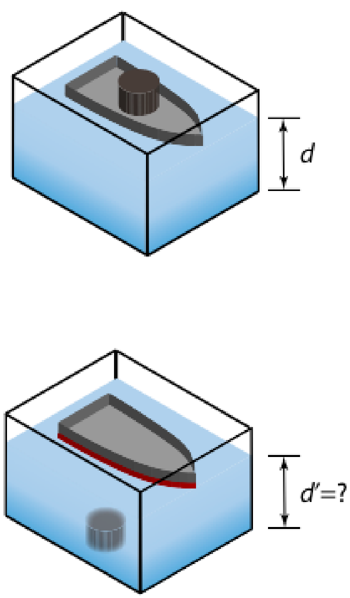


Figure 6.2: A boat with a weight in a tank.

EXERCISE 6.1— Suppose we have a toy boat carrying a weight and floating in a tank as shown in the top panel of Fig. 6.2. The depth of the water in the tank is d. The weight is then removed from the boat and allowed to sink to the bottom of the tank (bottom panel, Fig. 6.2). Does the depth of water in the tank increase, decrease, or stay the same? Explain your reasoning.

We can use Archimedes' law—which is just an application of hydrostatic equilibrium—to determine whether a fluid in planar geometry and hydrostatic equilibrium,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho g, \tag{6.1}$$

and with a temperature gradient is unstable to convection. Imagine moving a blob of fluid upwards from r to $r + h$. We raise the blob slowly enough that it is in hydrostatic equilibrium with its new surroundings,[1] $P_b(r + h) = P(r + h)$. We do move the blob quickly enough, however,

[1] We'll use the subscript b to denote properties of the blob; quantities without a subscript refer to the background fluid.

that it doesn't exchange heat with its surroundings and therefore doesn't remain in *thermal* equilibrium with its new environment.

As a result of this lack of heat exchange, the upward motion of the blob is ADIABATIC. To understand what this means, recall the first law of thermodynamics, which relates the change in internal energy $\mathrm{d}U$ and in volume $\mathrm{d}V$ to the heat transferred $\mathrm{d}Q = T\,\mathrm{d}S$:

$$\mathrm{d}Q = T\,\mathrm{d}S = \mathrm{d}U + P\,\mathrm{d}V, \tag{6.2}$$

where P is the pressure, T the temperature, and S the entropy. During an adiabatic process, $\mathrm{d}Q = T\,\mathrm{d}S = 0$. The entropy of the blob is therefore constant, $S_b(r+h) = S_b(r) = S(r)$, and is therefore not equal, in general, to the entropy of the surrounding gas at $r+h$: $S_b(r+h) \neq S(r+h)$. The pressure in the blob, however, is the same as in the surrounding gas: $P_b(r+h) = P(r+h)$.

Recall that pressure equilibrium in the blob is established over the time a sound wave takes to cross the blob. Thus, moving the blob slowly enough to maintain pressure equilibrium means that the motion is quite subsonic. Moving the blob quickly enough to prevent heat transport means (cf. exercise 3.9) that the blob is much larger than a mean free path so the time for photons to random walk across the blob is longer than time taken to raise the blob.

As in our discussion of the equation of state (cf. eq. [2.2]), it isn't really convenient to write things in terms of volume. To put eq. (6.2) into a more convenient form, divide both sides by the mass of the blob m_b:

$$\begin{aligned}
\mathrm{d}\left(\frac{Q}{m}\right) = T\,\mathrm{d}\left(\frac{S}{m}\right) &= \mathrm{d}\left(\frac{U}{m}\right) + P\,\mathrm{d}\left(\frac{V}{m}\right) \\
T\,\mathrm{d}s &= \mathrm{d}u + P\,\mathrm{d}\left(\frac{1}{\rho}\right) \\
T\,\mathrm{d}s &= \mathrm{d}u - \frac{P}{\rho^2}\mathrm{d}\rho
\end{aligned} \tag{6.3}$$

Here we denote the *entropy per mass* and the *energy per mass* by s and u respectively; and we identify the volume per mass with $1/\rho$, where ρ is the mass density. Equation (6.3) is the first law of thermodynamics as written for fluid dynamics.

After the blob has moved from r to $r+h$, it has expanded so that its density is

$$\rho_b(r+h) = \rho[P_b(r+h), s_b(r+h)] = \rho[P(r+h), s(r)].$$

Here we've written the density as a function of pressure and entropy: $\rho(P,s)$. Now we can apply Archimedes' law: if the density of the blob is greater than that of the surrounding fluid, then the buoyant force will be less than the weight of the blob; as a consequence, the blob will sink back to its original location. The fluid is thus stable. In contrast, if the density of the blob is less than that of the surrounding fluid, then the buoyant force is greater than the weight of the blob; a result, the fluid is unstable, as a small perturbation leads to the acceleration of the blob upwards. Figure 6.3 has a schematic of this criterion.

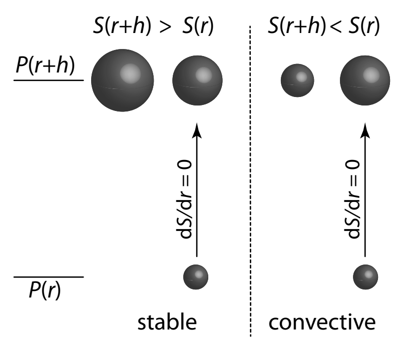


Figure 6.3: Illustration of criteria for convective instability. On the left, raising a blob a distance h adiabatically and in pressure balance with its surrounding results in a higher density $V_b < V$, or $\rho_b > \rho$. This is stable: the blob will sink back. On the right, the blob is less dense and hence buoyant: it will continue to rise.

Thus, for the fluid to be stable, we require that the density of the displaced blob be greater than that of the surrounding fluid:

$$\begin{aligned}
\rho_b(r+h) &> \rho(r+h) \\
\rho[P(r+h), s(r)] &> \rho[P(r+h), s(r+h)].
\end{aligned} \tag{6.4}$$

If condition (6.4) is satisfied, then the blob will be restored to its original location after a perturbation, and the system is stable. If condition (6.4) is not satisfied, then the blob will continue to rise following a perturbation; the system is thus unstable.

Since h is an infinitesimal displacement, we can expand the right-hand side of eq. (6.4):

$$\rho[P(r+h), s(r+h)] \approx \rho[P(r+h), s(r)] + \left(\frac{\partial \rho}{\partial s}\right)_P \frac{\mathrm{d}s}{\mathrm{d}r}\, h.$$

Here the notation $(\partial\rho/\partial s)_P$ means taking the derivative of ρ with respect to s while holding P fixed. The condition for stability is therefore, after canceling common factors,

$$\left(\frac{\partial \rho}{\partial s}\right)_P \frac{\mathrm{d}s}{\mathrm{d}r} < 0. \tag{6.5}$$

We've dropped h from the left-hand side since it is positive. We can put eq. (6.5) into a more useful form by changing variables from entropy ρ to temperature T via

$$\left(\frac{\partial \rho}{\partial T}\right)_P = \left(\frac{\partial \rho}{\partial s}\right)_P \left(\frac{\partial s}{\partial T}\right)_P = \left(\frac{\partial \rho}{\partial s}\right)_P \frac{C_P}{T},$$

where we used the specific heat at constant pressure, $C_P \equiv T(\mathrm{d}s/\mathrm{d}T)_P$. Inserting this expression into eq. (6.5) gives

$$\frac{T}{C_P}\left(\frac{\partial \rho}{\partial T}\right)_P \frac{\mathrm{d}s}{\mathrm{d}r} < 0,$$

Now, $(\partial\rho/\partial T)_P$ is negative (gas expands on being heated), while C_P is positive; hence eq. (6.5) will be satisfied wherever

$$\frac{\mathrm{d}s}{\mathrm{d}r} > 0. \tag{6.6}$$

In a convectively stable star, the entropy must increase with radius.

If this condition is not satisfied, if $\mathrm{d}s/\mathrm{d}r < 0$, then convection occurs: high-entropy material is buoyant and moves outward, while lower-entropy material sinks and moves inward. Eventually the rising fluid will mix with the surrounding material; when it does, its entropy will be added to the surrounding material, thereby raising its entropy. As a result of this mixing, the entropy gradient will be driven toward the marginally stable configuration $\mathrm{d}s/\mathrm{d}r = 0$.

The adiabatic thermal gradient

Condition (6.6) for convective stability is not directly useful, since our equations of stellar structure do not directly involve the entropy. We'd instead like to have the criterion for the onset of convection be expressed in terms of pressure and temperature, since those quantities appear in our stellar structure equations. To obtain such an equation, let's return to the first law expressed in terms of mass-specific quantities (eq. [6.3]):

$$\mathrm{d}q = T\,\mathrm{d}s = \mathrm{d}u - \frac{P}{\rho^2}\,\mathrm{d}\rho.$$

We can express the energy $u = u(\rho, T)$ as a function of temperature T and density ρ. Then taking the differential gives

$$\mathrm{d}u = \left(\frac{\partial u}{\partial T}\right)_\rho \mathrm{d}T + \left(\frac{\partial u}{\partial \rho}\right)_T \mathrm{d}\rho,$$

and thus the first law can be written as

$$\mathrm{d}q = T\mathrm{d}s = \left(\frac{\partial u}{\partial T}\right)_\rho \mathrm{d}T + \left[\left(\frac{\partial u}{\partial \rho}\right)_T - \frac{P}{\rho^2}\right]\mathrm{d}\rho.$$

Hence the heat needed to raise the temperature of one kilogram of fluid while holding density fixed is

$$C_\rho \equiv T\left(\frac{\partial s}{\partial T}\right)_\rho = \left(\frac{\partial u}{\partial T}\right)_\rho. \tag{6.7}$$

For an ideal gas, $u = u(T)$ and C_ρ is approximately constant; hence we may integrate equation (6.7) to obtain $u = C_\rho T + \text{const}$.

In Eq. (6.3), the last term is $-(P/\rho)\,(\mathrm{d}\rho/\rho) = -(P/\rho)\,\mathrm{d}\ln\rho$. This motivates the following trick: take the logarithm of the equation of state, $\ln(P) = \ln(\rho) + \ln(T) + \ln(k_\mathrm{B}/\mu m_\mathrm{u})$, and then take the differential to obtain

$$\frac{\mathrm{d}P}{P} = \frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}T}{T}.$$

Now eliminate $\mathrm{d}\rho/\rho$ in the equation

$$T\,\mathrm{d}s = C_\rho \mathrm{d}T - \frac{P}{\rho}\frac{\mathrm{d}\rho}{\rho}$$

to obtain an expression for the heat transferred as a function of temperature and pressure,

$$T\,\mathrm{d}s = \left[C_\rho + \frac{P}{\rho T}\right]\mathrm{d}T - \frac{1}{\rho}\mathrm{d}P = \left[C_\rho + \frac{k_\mathrm{B}}{\mu m_\mathrm{u}}\right]\mathrm{d}T - \frac{1}{\rho}\mathrm{d}P. \tag{6.8}$$

The heat needed to raise the temperature of a mass of fluid while holding pressure fixed is therefore

$$C_P \equiv T\left(\frac{\partial s}{\partial T}\right)_P = C_\rho + \frac{k_\mathrm{B}}{\mu m_\mathrm{u}}. \tag{6.9}$$

For a plasma of ions and electrons, $C_\rho = (3/2)k_\mathrm{B}/(\mu m_\mathrm{u})$ and hence $C_P = (5/2)k_\mathrm{B}/(\mu m_\mathrm{u})$. The ratio of specific heats is

$$\gamma = \frac{C_P}{C_\rho} = \frac{5/2}{3/2} = \frac{5}{3}. \tag{6.10}$$

This value of γ is for an ideal gas and does not hold universally.

DURING ADIABATIC MOTION, THERE IS NO HEAT EXCHANGE: hence, the entropy is constant and we can write eq. (6.8) as

$$T\mathrm{d}s = \mathrm{d}q = 0 = C_P\mathrm{d}T - \frac{1}{\rho}\mathrm{d}P. \tag{6.11}$$

We replace $1/\rho$ using the ideal gas equation of state to obtain

$$\begin{aligned} C_P\mathrm{d}T &= \frac{k_\mathrm{B}}{\mu m_\mathrm{u}}\frac{T}{P}\mathrm{d}P \\ \frac{\mathrm{d}T}{T} &= \frac{C_P - C_\rho}{C_P}\frac{\mathrm{d}P}{P} \\ &= \frac{\gamma - 1}{\gamma}\frac{\mathrm{d}P}{P}. \end{aligned} \tag{6.12}$$

Integrating both sides of the equation gives

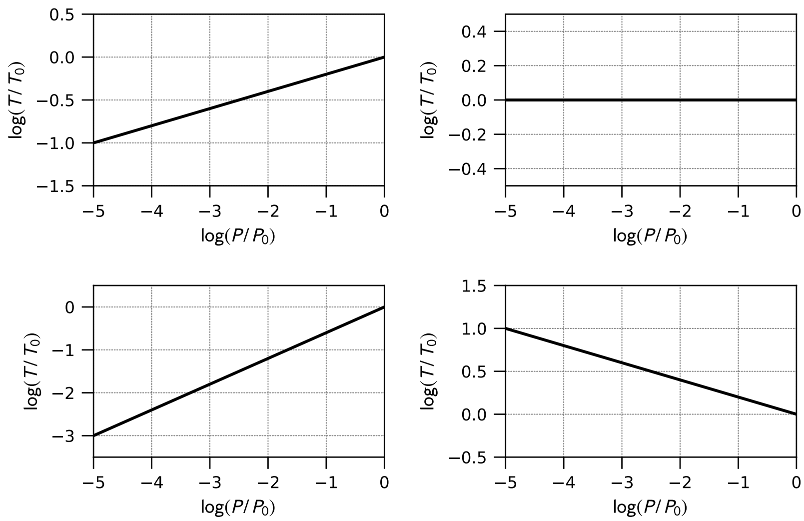
$$T = T_0\left(\frac{P}{P_0}\right)^{(\gamma-1)/\gamma}, \tag{6.13}$$

where T_0 and P_0 are the temperature and pressure at the beginning of the adiabatic process. Equation (6.13) tells us how the temperature changes with pressure along an adiabat for an ideal gas[2]. Using the ideal gas equation of state we can convert eq. (6.13) into a relation between temperature and density or between density and pressure along an adiabat.

[2] We used a similar relation in determining the sound speed (Box 2.1).

EXERCISE 6.2 — Use equations (6.13) and (2.5) to derive a relation between temperature and density, and a relation between density and pressure, along an adiabat.

EXERCISE 6.3 — The figure shows some hypothetical runs of temperature with respect to pressure in a gas in hydrostatic equilibrium. Indicate which of these situations is convectively unstable, and explain why. Draw on that plot the pressure-temperature relation that would ensue once convection sets in.



6.2 *Convection in stars*

When convection is absent, the temperature gradient in the star is (eq. [3.14])

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3\rho\kappa}{4acT^3}\frac{L(r)}{4\pi r^2}.$$

Here κ is the opacity and $L(r)$ is the luminosity at radius r: $L/4\pi r^2$ is the flux. If this thermal gradient, $|\mathrm{d}T/\mathrm{d}r|$, becomes too large, however, the fluid becomes unstable: warm fluid begins to rise while cold fluid sinks. Over a wide range of stellar conditions this mixing drives the entropy gradient in the convectively unstable region to $\mathrm{d}s/\mathrm{d}r = 0$. The sun has a convective region just below its photosphere, Fig. 6.4.

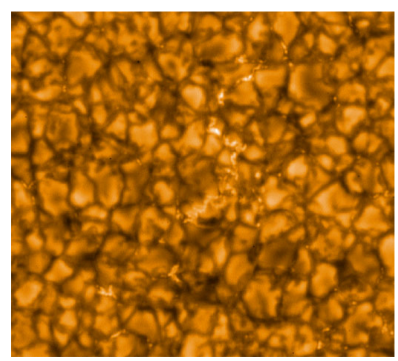


Figure 6.4: Solar convection cells, imaged with the Hinode Solar Optical Telescope. Image credit: Hinode JAXA/NASA/PPARC.

We can recast Equation (6.12) as

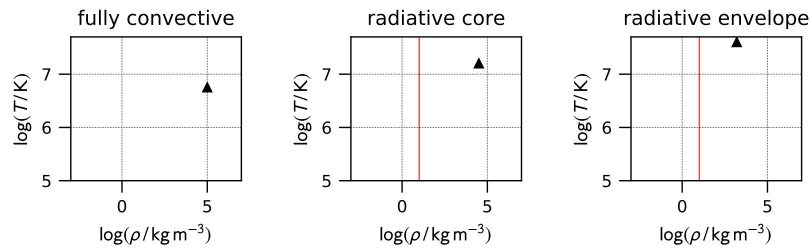
$$\frac{P}{T}\left(\frac{\partial T}{\partial P}\right)_s = \left(\frac{\partial \ln T}{\partial \ln P}\right)_s = \frac{\gamma - 1}{\gamma}. \qquad (6.14)$$

Hence, in a convective region,

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}r} &= \frac{T}{P}\left(\frac{\partial \ln T}{\partial \ln P}\right)_s \frac{\mathrm{d}P}{\mathrm{d}r} \\ &= \frac{\gamma - 1}{\gamma}\frac{T}{P}\frac{\mathrm{d}P}{\mathrm{d}r}. \qquad (6.15)\end{aligned}$$

The last form is specific to the case of an ideal gas.

EXERCISE 6.4 — The figure below indicates the central density and temperature (*triangle*) for 3 hypothetical stars: (*left*) a star that is fully convective; (*center*) a star with a radiative (i.e., stable against convection) core (densities greater than $10\,\mathrm{kg\,m^{-3}}$) and a convective envelope; (*right*) a star with a convective core and a radiative envelope. For each star, sketch a plausible run of temperature with density within the star. In the center and right panels, the boundary between radiative and convective regions is marked with a vertical solid line.



We can now collect the equations describing the structure of a star in steady-state. Previously, we established the relations for the enclosed mass,

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho, \tag{6.16}$$

and the pressure,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho \frac{Gm}{r^2}. \tag{6.17}$$

To these we add the equations for the temperature,

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{L}{4\pi r^2}\frac{3\rho\kappa}{4acT^3} \quad \text{where radiative; and} \tag{6.18}$$

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{T}{P}\left(\frac{\partial \ln T}{\partial \ln P}\right)_S \frac{\mathrm{d}P}{\mathrm{d}r} \quad \text{where convective.} \tag{6.19}$$

We finally add the equation for the luminosity,

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \rho \epsilon. \tag{6.20}$$

Equations (6.16)–(6.20), or equivalently (6.21)–(6.25), are supplemented by an equation of state $P = P(\rho, T, \{X\})$, opacity $\kappa = \kappa(\rho, T, \{X\})$, and heating rate $\epsilon = \epsilon(\rho, T, \{X\})$. Here $\{X\}$ refers to the abundances of the various isotopes. Note that we've omitted equations for the change in composition ($\mathrm{d}X/\mathrm{d}t$) due to nuclear burning. We've also omitted terms containing $\mathrm{d}r/\mathrm{d}t$, which describe expansion or contraction, from these equations.

Box 6.1 The equations of stellar structure in Lagrangian form

In general, the equations (6.16), (6.17), (6.18)-(6.19), and (6.20) must be solved numerically. In practice, the radius r is not the most convenient variable to use as a coordinate. In one dimension, the mass in each shell remains distinct, so the enclosed mass

$$m(r) = \int_0^r 4\pi r^2 \rho \, \mathrm{d}r$$

makes a useful coordinate. Using the enclosed mass as a coordinate is called a LAGRANGIAN description of the star. Upon changing variables from r to m, the structure equations become

$$\frac{\mathrm{d}r}{\mathrm{d}m} = \frac{1}{4\pi r^2 \rho} \tag{6.21}$$

$$\frac{\mathrm{d}P}{\mathrm{d}m} = \frac{\mathrm{d}P}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4} \tag{6.22}$$

$$\frac{\mathrm{d}T}{\mathrm{d}m} = -\frac{3\kappa L}{64\pi^2 r^4 acT^3} \quad \text{where radiative} \tag{6.23}$$

$$\frac{\mathrm{d}T}{\mathrm{d}m} = -\frac{T}{P}\left(\frac{\partial \ln T}{\partial \ln P}\right)_S \frac{Gm}{4\pi r^4} \quad \text{where convective} \tag{6.24}$$

$$\frac{\mathrm{d}L}{\mathrm{d}m} = \epsilon. \tag{6.25}$$

6.3 *Contraction to the main sequence*

Stars are formed when clouds of gas and dust fall out of pressure balance and become unstable to gravitational collapse. Often, the cloud fragments into a myriad of small collapsing regions, such as in the Soul Nebula pictured in Fig. 6.5. In the center of these dense knots, a core comes into hydrostatic equilibrium and grows in mass as matter continues to infall. Much of this process is obscured from view by the surrounding clouds of gas and dust.

As the nebula thins out, the star continues to contract slowly on a Kelvin-Helmholtz timescale, eq. (2.25), as the core is still too cool for nuclear reactions to power the luminosity from the surface (remember, the luminosity is set by the mass of the star and its opacity). As the central temperature rises, the nuclear reaction rate increases rapidly until the heat released by reactions balances that emitted from the surface. At that point the star is on the ZERO-AGE MAIN SEQUENCE (ZAMS). Of course, not all collapsing stellar-like objects reach the ZAMS—objects that are too low in mass will not ignite hydrogen fusion, while objects that are too high in mass tend to be unstable and eject mass. We'll explore these limits in the next few sections.

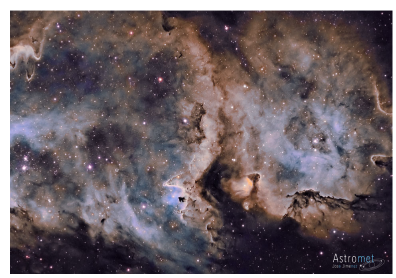


Figure 6.5: Image of the Soul Nebula (IC 1848) in the constellation Cassiopeia. Credit: José Jiménez Priego (Astromet).

EXERCISE 6.5— This exercise revisits problem 2.8. In that exercise you modeled how the density and temperature changed as a pre-main-sequence star contracted. Table 6.1 gives central densities and temperatures of stars at the onset of hydrogen fusion (known as the ZERO-AGE MAIN SEQUENCE). These temperatures and densities are plotted below and labeled by stellar mass. Assume an ideal-gas equation of state and use the virial relations for the temperature and central density to plot the tracks in this plane each star followed during its contraction.

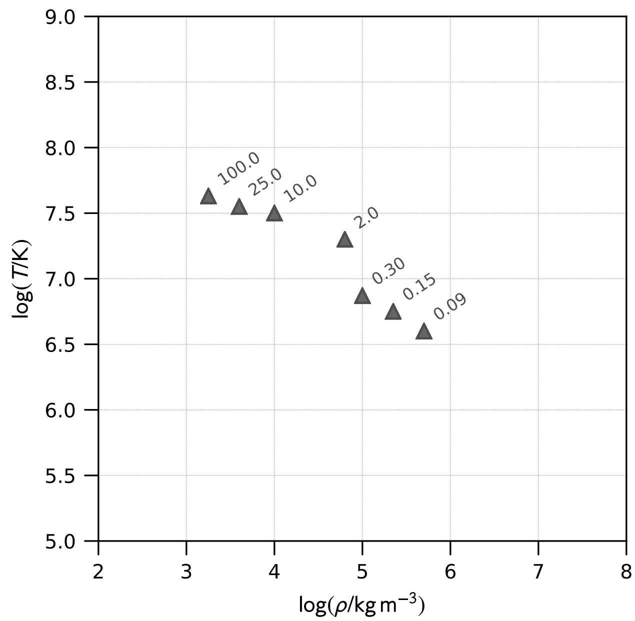


Table 6.1: Selected central densities and temperatures of zero-age main-sequence stars, computed with the `MESA` stellar evolution code [Paxton et al., 2011].

$M/M_\odot$	$\log(\rho_c/\mathrm{kg\,m^{-3}})$	$\log(T_c/K)$
0.09	5.70	6.60
0.15	5.35	6.75
0.30	5.00	6.87
2.0	4.80	7.30
10.0	4.00	7.50
25.0	3.60	7.55
100.0	3.25	7.63

You will use this plot for exercises 6.7 and 6.10 as well.

Degeneracy

As a star contracts, the particles within it are packed ever closer together. As we saw from our discussion of ionization, quantum mechanics enters the description of particle behavior when the separation between particles is of the order of the uncertainty in their positions. Said differently, our classical description breaks down when the particle density exceeds roughly

$$\frac{1\ \text{particle}}{(\Delta x)^3} = \left(\frac{\Delta p}{h}\right)^3 \sim \left(\frac{m k_\mathrm{B} T}{h^2}\right)^{3/2}. \tag{6.26}$$

Another way to put this is that quantum effects become important when there is roughly 1 particle in a normalized phase space volume $\mathrm{d}^3x\,\mathrm{d}^3p/h^3$.

Suppose we have two identical particles in a quantum state. Since the particles are identical, if we exchange them the wavefunction can only change by a phase factor[3] $e^{i\delta}$. If we exchange the particles again, we are back to our original state; as a result, $e^{2i\delta} = 1$, and therefore $\delta = 0$ or π. Hence upon the exchange of particles, the wavefunction either is unchanged ($\delta = 0$) or it changes sign ($e^{i\pi} = -1$).

[3] See Box 6.2

> *There are two types of wavefunctions in this world: those that change sign under exchange; and those that don't.*

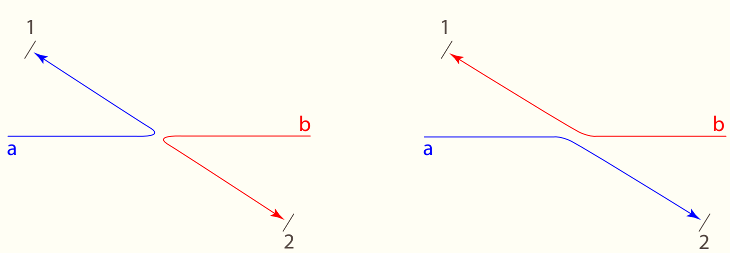
Particles that don't change sign under exchange are called BOSONS and have integer spin. Photons (spin $= 1$) are bosons. Particles that change sign under exchange are called FERMIONS and have half-integer spin. Electrons, neutrinos, protons, and neutrons (spin $= 1/2$) are all fermions.

A consequence of the fermion wavefunction changing sign when any two particles are exchanged is that the wavefunction vanishes if any two particles are in the same state—that is, they have the same position, momentum, and spin. For spin-half particles like electrons, this means we can put at most two such electrons in the same position and momentum state; we do this by having their spins antiparallel.

Box 6.2 Identical particles

To understand how the interchange of identical particles works in more detail, let's start by recalling some features of quantum mechanics. This discussion is based on Feynman et al. [1989]. We denote a particle's state as $|a\rangle$, where a is just a label. For example, a could be "electron with such-and-such momentum". The probability of finding the electron in some other state $|\varphi\rangle$ is given by $|\langle\varphi|\, a\rangle|^2$, where $\langle\varphi|\, a\rangle$ is a complex number known as the probability amplitude formed via an inner product of $|\varphi\rangle$ and $|a\rangle$.

Now suppose we have two particles, a and b, and we scatter them so that one particle ends up in detector 1 and the other ends up in detector 2. There are two ways this can go, as shown here.



Classically, we would argue that the probability of getting either particle in detector 1 is just

$$\mathcal{P}(\text{a or b in 1}) = \mathcal{P}(\text{a in 1}) + \mathcal{P}(\text{b in 1}). \tag{6.27}$$

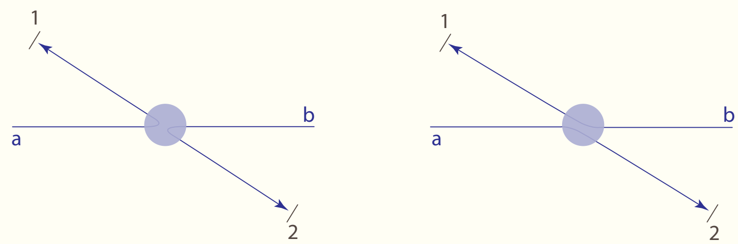
Box 6.2 continued

If particles a and b are different—e.g., one is a ^{12}C nucleus and the other is an ^{16}O nucleus—then this holds in quantum mechanics as well. Quantum mechanically, we write

$$\mathcal{P}(\text{a or b in 1}) = |\langle 1|\, a\rangle \langle 2|\, b\rangle|^2 + |\langle 2|\, a\rangle \langle 1|\, b\rangle|^2. \qquad (6.28)$$

If the particles are identical, however—for example, if a and b are two electrons with identical spin—then this picture is wrong.

Because of the uncertainty principle, we cannot follow the trajectories of a and b with infinite precision to see which is which; instead, the situation is more analogous to the depiction shown here.



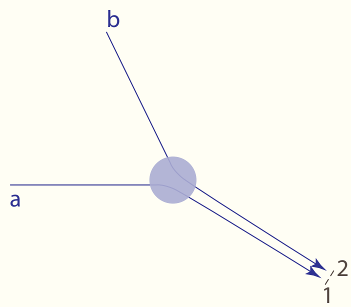
There are now two indistinguishable ways of arriving at the final state—in this case, an electron in detector 1 and an electron in detector 2. According to quantum mechanics, we must therefore sum the amplitudes for getting to the final state, *before taking the square*. That is, the probability for this one particle to end up in detector 1 and the other to end up in detector 2 is

$$\begin{aligned}
\mathcal{P}(\text{a or b in 1}) &= |\langle 1|\, a\rangle \langle 2|\, b\rangle + \langle 2|\, a\rangle \langle 1|\, b\rangle|^2 \\
&= |\langle 1|\, a\rangle \langle 2|\, b\rangle|^2 + |\langle 2|\, a\rangle \langle 1|\, b\rangle|^2 \\
&\quad + \left[\langle 1|\, a\rangle^* \langle 2|\, b\rangle^* \langle 2|\, a\rangle \langle 1|\, b\rangle \right.\\
&\quad \left. + \langle 2|\, a\rangle^* \langle 1|\, b\rangle^* \langle 1|\, a\rangle \langle 2|\, b\rangle\right] \\
&= \mathcal{P}(\text{a in 1}) + \mathcal{P}(\text{b in 1}) \\
&\quad + \left[\langle 1|\, a\rangle^* \langle 2|\, b\rangle^* \langle 2|\, a\rangle \langle 1|\, b\rangle \right.\\
&\quad \left. + \langle 2|\, a\rangle^* \langle 1|\, b\rangle^* \langle 1|\, a\rangle \langle 2|\, b\rangle\right]. \qquad (6.29)
\end{aligned}$$

The probability of scattering an electron into detector 1 is the classical value plus the *additional* interference term in $[\cdot]$.

TO SEE THE EFFECT OF THIS INTERFERENCE TERM ON THE THERMAL PROPERTIES OF THE SYSTEM, let's imagine putting two particles into the same small volume. To do this, we imagine the detectors 1 and 2 sliding together until they overlap, as shown here.

Box 6.2 continued



Since detectors 1 and 2 are approaching one another, we must have

$$|\langle 1|\, a\rangle\, \langle 2|\, b\rangle\,|^2 = |\,\langle 2|\, a\rangle\, \langle 1|\, b\rangle\,|^2. \tag{6.30}$$

This does not imply, however, that $\langle 1|\, a\rangle\, \langle 2|\, b\rangle = \langle 2|\, a\rangle\, \langle 1|\, b\rangle$: the amplitudes could differ by a phase factor, so that interchanging the particles would yield

$$\langle 2|\, a\rangle\, \langle 1|\, b\rangle = e^{i\delta}\, \langle 1|\, a\rangle\, \langle 2|\, b\rangle\,.$$

If we interchange the particles, and then interchange them again, we get

$$\langle 1|\, a\rangle\, \langle 2|\, b\rangle = e^{2i\delta}\, \langle 1|\, a\rangle\, \langle 2|\, b\rangle\,;$$

since swapping the particles twice just gets up back to the original situation, we must have that $e^{2i\delta} = 1$ and therefore $e^{i\delta} = \pm 1$.

If there is no change of sign, i.e., $\langle 2|\, a\rangle\, \langle 1|\, b\rangle = \langle 1|\, a\rangle\, \langle 2|\, b\rangle$, then from equation (6.29) we have

$$\mathcal{P}(\text{a or b in 1}) = 2|\,\langle 1|\, a\rangle\, \langle 2|\, b\rangle\,|^2 + 2|\,\langle 2|\, a\rangle\, \langle 1|\, b\rangle\,|^2. \tag{6.31}$$

This is *twice* the classical value: the probability of the particles entering the same state is enhanced.

In contrast, if the sign changes under exchange, i.e., if $\langle 2|\, a\rangle\, \langle 1|\, b\rangle = -\,\langle 1|\, a\rangle\, \langle 2|\, b\rangle$, then equation (6.29) implies that

$$\begin{aligned} \mathcal{P}(\text{a or b in 1}) &= |\,\langle 1|\, a\rangle\, \langle 2|\, b\rangle\,|^2 + |\,\langle 2|\, a\rangle\, \langle 1|\, b\rangle\,|^2 \\ &\quad -|\,\langle 1|\, a\rangle\, \langle 2|\, b\rangle\,|^2 - |\,\langle 2|\, a\rangle\, \langle 1|\, b\rangle\,|^2 \\ &= 0. \end{aligned} \tag{6.32}$$

We cannot have 2 identical particles with the same momentum, position, and spin if their wavefunction changes sign when the particles are exchanged.

Particles with integer spin (i.e., their angular momentum is an integer multiple of $\hbar$) have wavefunctions that do not change sign

Box 6.2 continued

under exchange; these particles are said to obey BOSE-EINSTEIN STATISTICS and are called BOSONS. Particles with half-integer spin have wavefunctions that do change sign under exchange; these particles are said to obey FERMI-DIRAC STATISTICS and are called FERMIONS. Photons are bosons; electrons, protons, neutrons, and neutrinos are fermions.

TO ACCOUNT FOR FERMI-DIRAC STATISTICS WITHIN THE EQUATION OF STATE, we imagine a small volume containing N electrons. Motivated by eq. (6.26), we divide the phase space into cells,

$$\frac{\mathrm{d}^3x\,\mathrm{d}^3p}{h^3},$$

and into each cell we place 2 electrons with opposing spins. We always add the electrons to the lowest open energy level, and repeat the process until we have added all N electrons. This procedure is represented by the equation

$$N = \frac{2}{h^3}\int_V \mathrm{d}^3x \int_0^{E_\mathrm{F}} \mathrm{d}^3p \qquad (6.33)$$

In this equation E_F, the *Fermi energy*, is the energy of the last electron added and is the largest filled energy level.

If our volume is isotropic, then we can change variables: first, to spherical momentum coordinates, $\mathrm{d}^3p = 4\pi p^2\,\mathrm{d}p$; second, from $\mathrm{d}p$ to $\mathrm{d}\epsilon$. Since $p = \sqrt{2m\epsilon}$, where ϵ is the energy of a single electron,

$$\mathrm{d}p = \sqrt{\frac{m}{2\epsilon}}\,\mathrm{d}\epsilon;$$

upon changing variables and integrating over ϵ from 0 to E_F we obtain

$$N = \frac{8\pi}{h^3}V\int_0^{E_\mathrm{F}} \sqrt{2}m^{3/2}\epsilon^{1/2}\,\mathrm{d}\epsilon = \frac{8\pi}{3h^3}V(2m)^{3/2}E_\mathrm{F}^{3/2}.$$

Solving for the Fermi energy gives

$$E_\mathrm{F} = \frac{h^2}{2m}\left(\frac{3}{8\pi}\frac{N}{V}\right)^{2/3}. \qquad (6.34)$$

What is the total energy of our system? We again integrate over phase space, with each electron multiplied by its energy ϵ:

$$E = \frac{8\pi}{h^3}V\int_0^{E_\mathrm{F}} \sqrt{2}m^{3/2}\epsilon^{3/2}\,\mathrm{d}\epsilon = \frac{8\pi}{5h^3}V(2m)^{3/2}E_\mathrm{F}^{5/2}. \qquad (6.35)$$

Using eq. (6.34) to substitute for E_F in eq. (6.35), we can find the energy per unit volume,

$$\frac{E}{V} = \frac{3}{5}\left(\frac{3}{8\pi}\right)^{2/3}\frac{h^2}{2m}n^{5/3} = \frac{3}{5}nE_\mathrm{F},$$

where $n = N/V$ is the density of electrons.

For a non-relativistic gas the pressure is $P = (2/3)(E/V)$. Hence the pressure of our electron gas is

$$P = \frac{2}{3}\frac{E}{V} = \frac{2}{5}nE_{\mathrm{F}} = \frac{2}{5}\left(\frac{3}{8\pi}\right)^{2/3}\frac{h^2}{2m}n^{5/3}. \tag{6.36}$$

Notice that the pressure is independent of the temperature.

Electrons, being more than 1000 times lighter than nuclei, become degenerate first. Suppose our composition consists of species with charge Z_i and mass number A_i. Then the number of electrons per unit volume[4] is

[4] assuming complete ionization

$$n_e = \sum_i n_i Z_i = \frac{\rho}{m_{\mathrm{u}}}\sum_i X_i \frac{Z_i}{A_i}.$$

By analogy with the mean molecular weight, we define an electron mean weight

$$\mu_e \equiv \left(\sum_i X_i \frac{Z_i}{A_i}\right)^{-1} \tag{6.37}$$

so that $n_e = \rho/(m_{\mathrm{u}}\mu_e)$.

EXERCISE 6.6 — Use equation (6.37) in eq. (6.36) to express the pressure as a function of mass density ρ. The use the virial scalings for $P(M,R)$ and $\rho(M,R)$ to obtain a relation $R(M)$ for a degenerate object.

As you found in exercise 6.6, when the star becomes degenerate, there is a unique radius for a given mass and composition. This is in contrast to the non-degenerate case, for which a star of a given mass can have a wide range of possible radii depending on the internal temperature.

Consider a contracting pre-main-sequence star. Initially, the star has a low density and the equation of state is that of an ideal non-degenerate gas. According to the virial theorem, as the radius decreases, both the central temperature and density increase. The radius decreases because the star is radiating away energy, and a star with an ideal, non-degenerate equation of state has a total energy that depends on its radius.

At some density, the equation of state will become degenerate. At this point, contraction comes to a halt. The star continues to radiate energy, but instead of contracting, the star simply cools while remaining at constant radius. If the contracting pre-main-sequence star is to become a main-sequence star, then, it must reach temperatures sufficient for hydrogen fusion to occur *before* becoming degenerate.

EXERCISE 6.7 — The equation of state becomes degenerate roughly where $k_\mathrm{B}T = E_\mathrm{F}$, with E_F begin given by eq. (6.34). From this and eq. (6.37), assuming a H-He composition with $X_\mathrm{H} = 0.7$ and $X_\mathrm{He} = 0.3$, derive a relation between $\log(T)$ and $\log(\rho)$. Plot this relation on the phase diagram in exercise 6.5, and on the plot indicate which side of the relation is degenerate. Given that contraction halts when the equation of state becomes degenerate, what does this plot imply for the minimum mass required to initiate hydrogen fusion?

AS SHOWN IN EXERCISE 6.7, THERE IS A MINIMUM MASS NEEDED TO INITIATE HYDROGEN FUSION. Contracting star-like objects of lower mass are known as BROWN DWARFS. Although dim, they are observable with spectral types "L", "T" or "Y"[5].

[5] J. D. Kirkpatrick, I. N. Reid, J. Liebert, et al. Dwarfs Cooler than "M": The Definition of Spectral Type "L" Using Discoveries from the 2 Micron All-Sky Survey (2MASS). *ApJ*, 519:802–833, July 1999; and Michael C. Cushing, J. Davy Kirkpatrick, Christopher R. Gelino, et al. The Discovery of Y Dwarfs using Data from the Wide-field Infrared Survey Explorer (WISE). *ApJ*, 743:50, December 2011. DOI: 10.1088/0004-637X/743/1/50

EXERCISE 6.8 — You might notice that the degenerate mass-radius relation you found in exercise 6.6 can't hold for very light objects (or very heavy ones, for that matter). Earth, for example has a much larger mass than Mars, and also has a larger radius, contrary to what the degenerate relation predicts. What happens is that at low pressures, the Coulomb force comes into play—the atomic and molecular bonds that add variety to life. These bonds set the size and spacing of atoms, and therefore fix the density of matter. Let's model this. The typical size of an atom is the Bohr radius,

$$a_\mathrm{B} = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} = 5.29 \times 10^{-11}\,\mathrm{m}.$$

1. Let take our mass density as being one average nuclear mass per volume a_B^3. We'll again use our solar composition, $X_\mathrm{H} = 0.7, X_\mathrm{He} = 0.3$. What is the value of this density? Is it plausible?
2. If matter is at this density, what is $R(M)$?
3. Roughly for what mass object, if any, does this $R(M)$ relation intersect the relation for degenerate matter? This sets the mass at which degeneracy becomes important for a cold object. Compare this mass with objects in the solar system.

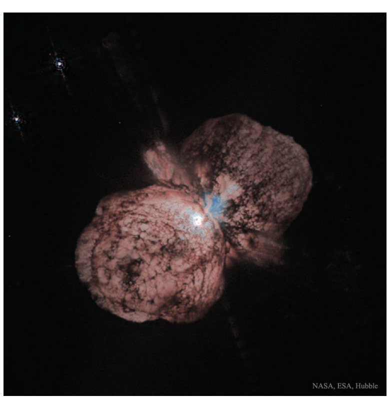


Figure 6.6: Image of the massive star Eta Carinae. Credit: J. Morse (Arizona State U.), K. Davidson (U. Minnesota) et al., WFPC2, HST, NASA.

Radiation pressure

Radiation in thermal equilibrium exerts a pressure (eq. 1.10): $P_\mathrm{rad} = aT^4/3$. Because of this strong dependence on temperature, radiation pressure becomes an increasingly large fraction of the total pressure for massive stars. Stars that are radiation-pressure dominated tend to be unstable: they have strong winds and violent fits of mass ejection (see the image of Eta Carinae, Fig. 6.6). As a result, they lose copious amounts of mass while on the main sequence. This effectively sets a rough upper limit on the mass of a star.

EXERCISE 6.9 — Use the virial relations for density and temperature to estimate how the ratio $P_{\mathrm{rad}}/P_{\mathrm{gas}}$ depends on the mass of the star.

EXERCISE 6.10 — The equation of state becomes dominated by radiation roughly where $P(\text{ideal gas}) \approx P(\text{radiation})$. Derive from this criterion a relation between $\log(T)$ and $\log(\rho)$, and plot this relation on the figure for exercise 6.5. Indicate which side of this relation is radiation-pressure dominated. What do your findings in this exercise imply for the mass range of main-sequence stars?

6.4 *Life on the main-sequence*

With the initiation of hydrogen fusion, the star settles into thermal and mechanical equilibrium, with its structure described by the solution of equations[6] (6.16)–(6.20), along with the equation of state and prescriptions for the opacity κ and heating rate ϵ.

[6] Or, in Lagrangian form, (6.21)–(6.25).

The reason for star's stability on the main sequence is a consequence of the relation, derived in exercise 2.9, between the star's total energy and temperature. If the reaction rate were to increase and deposit more energy into the star, then since the total energy is $\propto -GM^2/R$, the star would expand. This expansion would cause the central temperature to decrease, thereby reducing the reaction rate.

The star is not in complete equilibrium, however, as hydrogen in the core is gradually being converted to helium. The timescale over which the composition changes is much longer than the dynamical timescale (sets hydrostatic equilibrium), the radiative diffusion timescale (sets thermal gradient), and the Kelvin-Helmholtz timescale (sets core temperature via growth or contraction of stellar radii). The gradual build-up of a helium-rich core does not, therefore, affect the stability of the star, but it does lead to a slow brightening of the star over its main sequence lifetime. For our sun, the gradual enrichment of the core in helium causes a slow increase in luminosity of $\approx 10\%$ for each billion years.

Although this slow increase in luminosity is not a drastic change, it has significant implications for life on Earth. The expected warming is sufficient to make Earth uninhabitable within about a billion years from now.

EXERCISE 6.11 — You computed in exercise 5.4 the energy released from the conversion of 4 hydrogen atoms into helium. Express this number in terms of the energy released per mass of hydrogen burned; this number should be in units of J/kg. Now assume that the Sun's luminosity comes from the fusion of hydrogen into helium in the innermost 10% of the Sun's mass. For a composition that is 70% hydrogen by mass, how long would it take to deplete the hydrogen in the solar core? This sets the main-sequence lifetime of the sun.

The cool outer layers of low-mass stars have large opacities: for example many elements are not ionized, so there are many potential lines for absorption. As a result, stars with $M \lesssim M_{\odot}$ have convective regions in their outer parts. The fraction of the star that is convective is larger for low-mass, cool stars; and stars with $M \lesssim 0.3\,M_\odot$ are fully convective, so that the whole interior lies along an adiabat. For more massive, hotter, stars, the opacities are lower, and as a result, the outer convective region vanishes for stars with $M \gtrsim M_\odot$.

EXERCISE 6.12— We can estimate how the luminosity depends on stellar mass for stars that have a mostly radiative structure. Start with equation (6.18) for the temperature gradient and approximate $\mathrm{d}T/\mathrm{d}r \approx T_c/R$, $\rho \approx \bar{\rho}$, $L/4\pi r^2 \approx L/4\pi R^2$, and $T \approx T_c$. Take the opacity κ to be constant, use the virial estimate for the central temperature T_c and express the mean density $\bar{\rho}$ in terms of stellar mass M and radius R. After some algebra, you should find that the luminosity L depends on M to some power. Compare this scaling against the data in Table 2.2. Obtain an expression for the stellar lifetime as a function of mass, and calibrate it to the Sun's main-sequence lifetime, $\tau_\odot \sim 10\,\mathrm{Gyr}$.

STARS MORE MASSIVE THAN THE SUN HAVE SUFFICIENTLY HIGH CORE TEMPERATURES FOR HYDROGEN TO BE CONSUMED VIA THE CNO CYCLE. The strong temperature dependence of the CNO burning has two effects on the structure of the star. First, it makes the central temperature nearly constant over a wide range of stellar masses for $M > 1\,M_\odot$—a small rise in temperature is sufficient to raise the heat production ϵ to match the rise in luminosity. A nearly constant central temperature implies, via the virial theorem, that $R \propto M$ on the upper main sequence. The second consequence is that nearly all of the star's luminosity is generated in a small region about the stellar center. The flux, $L/4\pi r^2$, in this small region is enormous, and this makes the core of the star convective. The convection can mix hydrogen fuel into the core, which makes the lifetime somewhat longer than the estimate from exercise 6.12. A summary of the structure of main sequence stars is contained in Table 6.2.

Table 6.2: Characteristics of main-sequence stars

	$M \lesssim M_\odot$	$M \gtrsim M_\odot$
$4\,{}^1\mathrm{H} \to {}^4\mathrm{He}$	pp	CNO
core is	radiative	convective
envelope is	convective	radiative

7
End of the Line

The depletion of hydrogen in the core heralds the end of the star's placid main-sequence life. We shall give an overview of the changes that ensue before discussing in more detail the events marking the end of the star's life. Fusion of helium requires a temperature $\gtrsim 10^8$ K, substantially higher than that required for the fusion of hydrogen. As a consequence, when the hydrogen is used up, helium burning cannot immediately begin and the core contracts. The main difference from the pre-main-sequence contraction is that hydrogen is fusing into helium in a shell surrounding the core. This shell burning causes drastic changes to the star's structure, surface temperature, and luminosity.

Once the core becomes sufficiently hot, helium fuses into carbon, and the core again reaches a state of thermal and mechanical equilibrium. When the helium is depleted the core must again contract. As with pre-main sequence stars, the critical question is whether the core becomes degenerate before a particular reaction can ignite. For stars with main-sequence masses $\lesssim (8\text{–}10)\,M_\odot$, the core becomes degenerate before the onset of ^{12}C fusion, which requires temperatures $\approx 8 \times 10^8$ K. Indeed, for stars around a solar mass, the fusion of ^{4}He occurs under moderately degenerate conditions.[1] As a result, the cores of low-mass stars end up composed of carbon and oxygen (or perhaps oxygen and neon) and supported by degenerate electrons; such objects are known as WHITE DWARFS.

For stars with masses $\gtrsim (8\text{–}10)\,M_\odot$, reactions in the core will successively make heavier and heavier isotopes until reaching ^{56}Fe. At this point the matter reaches its maximum binding energy[2]. A degenerate core forms and grows in mass due to reactions in shells surrounding the core. There is a maximum mass, known as the CHANDRASEKHAR MASS, that can be supported by electron degeneracy pressure. When the core exceeds this mass, it violently implodes. The implosion halts when matter reaches nuclear density and the repulsive strong nuclear force provides pressure support. In this implosion, most of the electrons and protons combine, $e^- + \mathrm{p} \to \mathrm{n} + \nu_e$. The core is then composed mostly

[1] Stars with masses $\lesssim 0.5\,M_\odot$ will become degenerate before reaching temperatures sufficient for helium to fuse; the main-sequence lifetime of such stars is much greater than the age of the universe, so making a helium white dwarf requires some kind of mass loss, such as in a binary.

[2] cf. exercise 5.3

of neutrons[3] and is known as a NEUTRON STAR. The resulting torrent of neutrinos injects energy into the outer layers of the star; in many cases this is sufficient to eject the outer layers of the star and produce a SUPERNOVA. If the envelope is not ejected, matter will accumulate onto the neutron star. The maximum mass that can be supported by the nuclear force is uncertain, but is somewhere between $(2\text{–}3)\,M_\odot$; when this maximum mass is exceeded, the neutron star collapses into a black hole.

[3] At densities substantially above that of an atomic nucleus other constituents, such as hyperons, may appear.

Having given a brief summary of post-main sequence evolution, we shall now explore the various evolutionary tracks in slightly more detail.

7.1 *Low-mass stars*

Ascent of the red-giant branch

With the depletion of hydrogen in the core, the core contracts. During this contraction, hydrogen fusion continues in a shell surrounding the core. The shell hydrogen fusion produces helium, which adds to the core mass. As the core contracts its temperature rises. The rising temperature and pressure at the base of the hydrogen-burning shell causes the reactions in the shell to go at an ever-increasing rate. The resulting increase in luminosity inflates the envelope, now fully convective, to large radii and hence to a low surface temperature. The star becomes a red giant. The high luminosity, combined with the low surface gravity of the distended envelope, drives a strong wind[4] so that the star loses a substantial amount of mass during the giant phase.

[4] Calculations of the rate of mass loss are still crude, but there are some observational constraints.

Helium burning: the horizontal branch

There are no stable isotopes with mass number $A = 5$ or $A = 8$, which makes the fusion of ^{4}He somewhat tricky. Although unstable, the isotope ^{8}Be is relatively long-lived (10^{-16} s) compared to a nuclear timescale[5]. As a result, when the core temperature reaches $\approx 10^8$ K, the reaction

[5] Roughly the time for a pion to cross a nucleus, $\sim 10^{-22}$ s.

$$^4\text{He} + {}^4\text{He} \longleftrightarrow {}^8\text{Be}$$

builds up a minute abundance of ^{8}Be. This abundance is sufficient for the reaction

$$^8\text{Be} + {}^4\text{He} \longleftrightarrow {}^{12}\text{C}^*$$

to make a small abundance of ^{12}C in an excited state (denoted by the *). While most of the $^{12}\text{C}^*$ decays back into $^8\text{Be} + {}^4\text{He}$, a small fraction transitions to the ground state, $^{12}\text{C}^* \to {}^{12}\text{C} + \gamma$. As a result, there is a net conversion $3\,{}^4\text{He} \to {}^{12}\text{C}$—the TRIPLE-ALPHA REACTION.

The triple-alpha reaction is incredibly temperature-sensitive: $\partial \ln \varepsilon_{3\alpha}/\partial \ln T \approx 40$ at $T = 10^8$ K. This sensitivity, combined with the mildly degenerate conditions of the core, makes the ignition of ^{4}He somewhat unstable for solar-mass stars.

Once core ^{4}He has ignited, the star settles onto a "helium main sequence;" observationally this is the HORIZONTAL BRANCH, so called because these stars lie in a clump on a Hertzsprung-Russell diagram. The luminosity on the horizontal branch is about $(30\text{–}100)\,L_\odot$. The higher luminosity and the much lower energy release from the triple-alpha reaction make the horizontal branch lifetime much shorter than that of the main-sequence (e.g., the horizontal branch lifetime is $\sim 10^8$ yr for a solar-mass star).

EXERCISE 7.1 — Following exercises 5.4 and 6.11, find the heat released per kilogram from fusing 3 ^{4}He nuclei ($B = 28.296$ MeV) into ^{12}C ($B = 92.162$ MeV). Take the core mass to be $0.45\,M_\odot$ (the minimum core mass needed for the ignition of helium). For a luminosity of $30\,L_\odot$, find the lifetime for core helium burning.

As the mass of ^{12}C builds up in the core, the reaction $^{12}\mathrm{C} + {}^4\mathrm{He} \to {}^{16}\mathrm{O}$ begins to compete with the triple alpha reaction. As a result, the core becomes composed of a ^{12}C/^{16}O mixture.

The asymptotic giant branch and emergence of a white dwarf

With the depletion of ^{4}He, the core—now composed of ^{12}C and ^{16}O—again contracts, while the growing luminosity from the H- and He-burning shells again inflate the envelope to large radii. Observationally, this phase is the ASYMPTOTIC GIANT BRANCH: on an HR diagram, the stars move on a track that approaches the giant branch. The hydrogen-rich envelope is consumed at its base by the H- and He-burning shells and is expelled at the surface by an increasingly strong wind. After the envelope is gone, the hot core—observed as a white dwarf—slowly cools. For a solar-mass star, the expected final mass of the core, and hence of the white dwarf, is $\approx 0.6\,M_\odot$.

7.2 Massive stars

For stars with main-sequence masses $\gtrsim (8\text{–}10)\,M_\odot$, the fusion of ^{12}C commences while the core is non-degenerate and at a temperature $\approx 8 \times 10^8$ K. At this temperature, electron-positron pairs form and annihilate ($e^- + e^+ \longleftrightarrow \gamma\gamma$); occasionally instead of producing photons, the reaction

$$e^- + e^+ \longleftrightarrow \nu_e + \bar{\nu}_e$$

occurs instead and generates a neutrino-antineutrino pair. The mean free path for the neutrinos is larger than the radius of the star; as a result, the neutrinos stream out and take energy from the core. As the core temperature increases, these neutrinos carry away most of the heat from the core.

Within the core, ^{12}C is consumed by the reactions

$$^{12}\mathrm{C} + {}^{12}\mathrm{C} \to \begin{cases} {}^{23}\mathrm{Na} + \mathrm{p} \\ {}^{20}\mathrm{Ne} + {}^4\mathrm{He} \end{cases} .$$

The p and ^{4}He capture onto other nuclei that are present. At slightly higher temperatures, $^{20}\mathrm{Ne} + \gamma \to {}^{16}\mathrm{O} + {}^4\mathrm{He}$ releases ^{4}He nuclei that subsequently capture onto other ^{16}O, ^{20}Ne, and ^{24}Mg. The next significant burning stage is

$$^{16}\mathrm{O} + {}^{16}\mathrm{O} \to \begin{cases} {}^{31}\mathrm{P} + \mathrm{p} \\ {}^{28}\mathrm{Si} + {}^4\mathrm{He} \end{cases};$$

as with $^{12}\mathrm{C} + {}^{12}\mathrm{C}$, the p and ^{4}He combine with ambient nuclei with the end result being a distribution of isotopes about ^{28}Si.

EXERCISE 7.2— At the onset of ^{16}O burning in a $25\,M_\odot$ star, the central density (Table 7.1) is $3.6 \times 10^6\,\mathrm{g\,cm^{-3}}$ ($3.6 \times 10^9\,\mathrm{kg\,m^{-3}}$). What is the dynamical time of the core?

The strong Coulomb barrier inhibits the fusion of nuclei beyond ^{16}O; instead, photodissociation reactions such as $^{28}\mathrm{Si} + \gamma \to {}^{24}\mathrm{Mg} + {}^4\mathrm{He}$ liberate n, p, and ^{4}He. These light nuclei then capture onto heavier nuclei, and the composition gradually becomes composed of isotopes about ^{56}Fe. This is NUCLEAR STATISTICAL EQUILIBRIUM: the composition is in the lowest energy state (most bound) for the ambient density and temperature. As a result, there is no further release of nuclear energy possible. The (mostly ^{56}Fe) core contracts and becomes degenerate; its mass gradually increases from the burning of surrounding material.

The amount of energy available from the fusion of heavy nuclei is low; as a consequence, the time required for the core to deplete the available fuel grows shorter and shorter, with the final stages occurring in a day (column labeled τ in Table 7.1). After the ignition of carbon, the core evolves too quickly for the envelope to keep up. Thus the external appearance of the star provides no window into the final days of burning.

Core collapse

When the core of a massive star reaches nuclear statistical equilibrium (NSE), there are no further sources of energy available. Fusion reactions in the shells surrounding the core add mass to it, causing it to contract. The increasing density raises the electron Fermi energy. When the Fermi energy approaches the rest mass of the electrons—$m_e c^2 = 0.511\,\mathrm{MeV}$—the electrons move relativistically. This alters the equation of state.

The reason is that the energy no longer goes as $p^2/2m$ for relativistic particles. The correct relation is

$$E = \sqrt{p^2c^2 + m^2c^4} = mc^2\sqrt{1 + \left(\frac{p}{mc}\right)^2};$$

when $p/mc \ll 1$, we can expand this as $E \approx mc^2 + p^2/2m$—that is, as the sum of the rest mass and the Newtonian form of the kinetic energy. We'll now explore the opposite limit, with $p \gg mc$, so that $E \approx pc$.

Table 7.1: Nuclear burning timescales for massive stars. Values taken from Woosley et al. [2002].

hydrogen				
M_{ZAMS} ($M_\odot$)	T_c (10^7 K)	ρ_c (g cm^{-3})	L ($10^3\,L_\odot$)	τ (Myr)
15	3.53	5.81	28	11.1
25	3.81	3.81	110	6.7
helium				
M_{ZAMS} ($M_\odot$)	T_c (10^8 K)	ρ_c (10^3 g cm^{-3})	L ($10^3\,L_\odot$)	τ (Myr)
15	1.78	1.39	41	1.97
25	1.96	0.76	182	0.84
carbon				
M_{ZAMS} ($M_\odot$)	T_c (10^8 K)	ρ_c (10^6 g cm^{-3})	L ($10^3\,L_\odot$)	τ (kyr)
15	8.34	2.39	83	2.03
25	8.41	1.29	245	0.52
oxygen				
M_{ZAMS} ($M_\odot$)	T_c (10^9 K)	ρ_c (10^6 g cm^{-3})	L ($10^3\,L_\odot$)	τ (yr)
15	1.94	6.66	87	2.58
25	2.09	3.60	246	0.40
silicon				
M_{ZAMS} ($M_\odot$)	T_c (10^9 K)	ρ_c (10^7 g cm^{-3})	L ($10^3\,L_\odot$)	τ (d)
15	3.34	4.26	87	18.3
25	3.65	3.01	246	0.7

Recall that for a degenerate gas, we began filling energy states, starting with the lowest open levels until we have added all N electrons (eq. [6.33]):

$$N = \frac{2}{h^3}\int_V \mathrm{d}^3x \int_0^{E_\mathrm{F}} \mathrm{d}^3p.$$

We then change variables, $\mathrm{d}^3p = 4\pi p^2\,\mathrm{d}p = 4\pi c^{-3}\epsilon^2\,\mathrm{d}\epsilon$, where $\epsilon = pc$ is the energy of a single electron:

$$N = \frac{8\pi}{h^3c^3}V\int_0^{E_\mathrm{F}} \epsilon^2\,\mathrm{d}\epsilon = \frac{8\pi}{3h^3c^3}VE_\mathrm{F}^3.$$

Solving for the Fermi energy,

$$E_\mathrm{F} = hc\left(\frac{3}{8\pi}\frac{N}{V}\right)^{1/3}.$$

To get the total energy, we multiply each electron by its energy ϵ and integrate over phase space:

$$E = \frac{8\pi}{h^3c^3}V\int_0^{E_\mathrm{F}} \epsilon^3\,\mathrm{d}\epsilon = \frac{1}{4}\frac{8\pi}{h^3c^3}VE_\mathrm{F}^4 = \frac{3}{4}NE_\mathrm{F}.$$

For a relativistic gas, the pressure is $P = (1/3)(E/V)$ (cf. Box 1.2), so that

$$P = \frac{1}{4}nE_\mathrm{F} = \frac{1}{4}\left(\frac{3}{8\pi}\right)^{1/3} hcn^{4/3}, \tag{7.1}$$

with $n = \rho/\mu_e m_u$.

The Chandrasekhar mass

In exercise 6.6, we constructed a mass-radius relation for white dwarfs by combining the virial relations,

$$\begin{aligned} P &\propto \frac{GM^2}{R^4} \\ \rho &\propto \frac{M}{R^3} \end{aligned}$$

and the equation of state for a non-relativistic, degenerate, ideal gas. We found that $R \propto M^{-1/3}$. If we try that with our relativistic equation of state, eq. (7.1), we get

$$\frac{GM^2}{R^4} \propto P = \frac{1}{4}\left(\frac{3}{8\pi}\right)^{1/3} hc \left(\frac{\rho}{m_\mathrm{u}\mu_e}\right)^{4/3} \propto \frac{M^{4/3}}{R^4}.$$

The radius R cancels, and what we have is a relation $M \propto (hc/G)^{3/2}/m_\mathrm{u}^2$. This is rather odd: a gas with a relativistic equation of state in hydrostatic balance has a characteristic mass defined in terms of fundamental constants.

Let's investigate this further. Suppose we have a box with adjustable sides, which we pack with N degenerate electrons. We add some nuclei for mass, so that the total mass in the box is $\mu_e m_\mathrm{u} N$. The volume of the box $V \sim R^3$, and since the electrons are degenerate, the volume per electron is roughly λ^3, where $\lambda \sim h/p$ is the wavelength of the electrons. As a result, $N = (R/\lambda)^3$; further, the momentum of an electron is

$$p \sim \frac{h}{\lambda} \sim h\frac{N^{1/3}}{R}.$$

If our electrons were non-relativistic, the total, kinetic plus gravitational, energy of our box would be

$$E_\mathrm{total} = N\frac{p^2}{2m_e} - \frac{GM^2}{R} \sim N^{5/3}\frac{h^2}{R^2 m_e} - GN^2\mu_e^2 m_\mathrm{u}^2\frac{1}{R}.$$

For a given N, we can adjust R to make $E_\mathrm{total} < 0$, and indeed, if we satisfy the virial theorem, we will recover the $R \propto M^{-1/3}$ scaling.

If, however, the electrons are relativistic then the total energy is

$$\begin{aligned} E_\mathrm{total} = Npc - \frac{GM^2}{R} &= \frac{1}{R}\left[hcN^{4/3} - GN^2(\mu_e m_\mathrm{u})^2\right] \\ &= G(\mu_e m_\mathrm{u})^2\frac{N^{4/3}}{R}\left[\frac{hc}{G(\mu_e m_\mathrm{u})^2} - N^{2/3}\right]. \end{aligned}$$

Look at the term in $[\cdot]$. If $N < [hc/G/(\mu_e m_\mathrm{u})^2]^{3/2}$, then $E_\mathrm{total} > 0$; by making R larger, however, we can lower the energy until the electrons are no longer relativistic. If $N > [hc/G(\mu_e m_\mathrm{u})^2]^{3/2}$, then $E_\mathrm{total} < 0$; by making R smaller, however, we can keep reducing E_total indefinitely.

There is no bound state with finite R for $M > (hc/G)^{3/2}(\mu_e m_\mathrm{u})^{-2}$.

Box 7.1 Instability for a relativistic equation of state

There is another way of looking at the onset of instability which is instructive (this treatment follows that in Cox [1980]). In exercise 2.10 you found that during a contraction or expansion, the equation of motion for a thin layer at the star's surface was

$$\delta\ddot{R} = \frac{GM}{R^2}\left[4 - 3\gamma\right]\frac{\delta R}{R}.$$

Here M and R are the total stellar mass and radius, and the adiabatic pressure-density relation is $P \propto \rho^{\gamma}$.

For a non-relativistic gas with $\gamma = 5/3$, we have $\delta\ddot{R} \propto -\delta R$: the star oscillates with a period that is comparable to the dynamical timescale of the star. If, however, $\gamma < 4/3$ the equation of motion is $\delta\ddot{R} \propto \delta R$, which has an exponential solution: squeeze the star slightly, and it will collapse!

Let's work out a more physical explanation for what is happening. Suppose we have a star in virial equilibrium. Then the central pressure and density are

$$\begin{aligned} P &\propto \frac{GM^2}{R^4} \\ \rho &\propto \frac{M}{R^3}. \end{aligned}$$

Now if we contract the star by a small amount, say $\delta R/R = -1\%$, then the density increases by an amount $\delta\rho/\rho = -3\delta R/R = 3\%$. How does the pressure respond? If the star contracts slowly, on a Kelvin-Helmholtz timescale, then there is time for heat to radiate away, so that the internal pressure can increase by the amount needed to maintain equilibrium: in this case $\delta P/P = -4\delta R/R = 4\%$. Under an *adiabatic* contraction, however, there is not enough time for the star to radiate away excess heat; as a consequence, the pressure and density are linked, so that $\delta P/P = \gamma\delta\rho/\rho = -3\gamma\delta R/R$.

If the adiabatic index is $\gamma = 4/3$, then during an adiabatic compression of $\delta R/R = -1\%$, the density increases by $3|\delta R/R| = 3\%$ and the pressure increases by $3\gamma|\delta R/R| = 4\%$, which is precisely the increase needed to maintain mechanical equilibrium. As a result, the star remains in hydrostatic balance at its new, smaller radius. This is why there was no mass-radius relation for $\gamma = 4/3$; it takes no energy to contract (or expand) the star.

For $\gamma > 4/3$, when the star contracts the central pressure increases by $3\gamma|\delta R/R| > 4|\delta R/R|$. As a result, the pressure becomes greater than the amount needed for hydrostatic balance. This excess pressure pushes the star outward and acts as a restoring

Box 7.1 continued

source. During an expansion, the pressure falls below the amount needed for hydrostatic equilibrium, so gravity halts the expansion and forces the star to contract. Hence, for $\gamma > 4/3$, the star responds to a radial perturbation by oscillating with a period comparable to the dynamical timescale (cf. exercise 2.10).

In contrast, if the star has $\gamma < 4/3$, then the increase in pressure during contraction is $3\gamma|\delta R/R| < 4|\delta R/R|$. The gas pressure does not increase enough to maintain hydrostatic equilibrium, and so the star's contraction accelerates. A small perturbation inwards leads to implosion.

Thus, there is a limit to the total mass that can be supported in hydrostatic equilibrium by degenerate electrons. An exact calculation for the maximum mass of a cold star yields

$$M_{\mathrm{Ch}} = 1.456 \left(\frac{2}{\mu_e}\right)^2 M_\odot. \tag{7.2}$$

When the mass reaches this limiting value, known as the CHANDRASEKHAR MASS[6], the electrons become relativistic and $\partial P/\partial\rho \to 4/3$; the star becomes unstable and collapses.

[6] Derived by S. Chandrasekhar at age 20(!) while traveling from India to England in 1930

When the core of a massive star begins its collapse, the electron Fermi is $\sim$ MeV, which is sufficient to induce electron captures on iron-group nuclei. These captures increase μ_e and reduce M_{Ch}. As the core begins the final plunge, the rapidly rising temperature induces the photodissociation of iron-group nuclei into neutrons, protons, and helium nuclei. This process is endothermic, which further robs the core of pressure support and accelerates the collapse. The effective $\gamma = \partial P/\partial\rho < 4/3$ on account of the photodissociation and electron captures, and the core implodes.

When the core density approaches $0.16\,\mathrm{fm}^{-3}$, the nucleons begin to repel one another on account of the strong nuclear force. At this point the collapse halts, sending a shockwave outwards. The core now consists mostly of neutrons and is termed a NEUTRON STAR.

EXERCISE 7.3— What is the mass density if the number density of nucleons is $0.16\,\mathrm{fm}^{-3}$? What is the gravitational binding energy for an object with a mass $1.4\,M_\odot$ at this density?

The outward going shockwave soon stalls as the outer layers of the star fall inward. The energy needed to blow the envelope off is about 1% of the gravitational binding energy of the core, so there is plenty of energy available to disperse the envelope if this energy can be tapped. Most of the gravitational binding energy released by the imploding core is carried outwards by neutrinos. During the collapse, the neutrino mean free path becomes smaller than the core radius for two reasons: the weak interaction cross-section increases as the nucleons reach temperatures $\gtrsim 10^{10}$ K, and the mean free path $\ell = (n\sigma)^{-1}$ decreases with density. As a result the neutrinos become trapped and must diffuse of the collapsing core. As the neutrinos diffuse out, they transfer a small fraction of their energy to the material heating it. This tends to push the shock outward. A competition arises between the ram pressure of infalling matter and the heating from the neutrinos. If the neutrinos can transfer enough energy to the envelope, then the envelope will be blown off in a supernova. If not, then matter will continue to accumulate onto the neutron star. The maximum mass of a neutron star is uncertain[7], but on physical grounds is likely $< 3\,M_{\odot}$. If the shock is not re-energized, then conceivably the entire star could implode into a black hole.

[7] By timing pulsars in a binary system, the orbital parameters and hence the mass of the neutron star can be deduced; the largest measured mass is $2\,M_{\odot}$.

7.3 *Stellar resurrection*

In the previous section, we learned that stars with $M \lesssim (8–10)\,M_{\odot}$ eventually become white dwarfs composed of carbon and oxygen and supported by electron degeneracy pressure; and that more massive stars have cores that collapse, either to form neutron stars supported by the strong nuclear interaction or to collapse fully into black holes.

Both the white dwarfs and neutron stars that emerge from the ashes of isolated stars slowly cool and dim. The cooling of white dwarfs can be modeled accurately enough that observations of white dwarfs in clusters can be used to infer the ages of and distances to their host clusters. No such capability is possible with isolated neutron stars: most are too dim to be observed, and there are vast uncertainties about the composition of the deep interior, where the density is several times higher than that of an atomic nucleus. Rather, efforts have been on using observations of the handful of isolated neutron stars with measured surface temperatures to constrain models of nuclear matter.

Many observed neutron stars are endowed with strong magnetic field $\gtrsim 10^8$ T. If the neutron star spins rapidly enough, then a tremendous voltage is generated is the surface that accelerates charges above the polar caps. In turn, these accelerated charges emit photons that fan outward from the poles. As the neutron star spins, the beams of radiation are swept around; a distant observer therefore observes light pulsing at the rotation frequency of the star. These systems, known as PULSARS, were discovered by Jocelyn Bell and Anthony Hewish in 1967.

Interestingly, the radio emission from several pulsars, including the Crab, was independently detected by C. Schisler at the Ballistic Missile Early Warning Site, Clear Air Force Station, Alaska.

EXERCISE 7.4 — The Crab pulsar pulsates at a frequency of 33 Hz. For a star of $1\,M_{\odot}$, find the maximum radius such that material at the equator remains bound to the star. Based on these results, argue that the Crab pulsar cannot be a white dwarf.

MANY STARS ARE IN BINARY SYSTEMS. If the binary happens to survive the evolution off the main sequence, it can often happen that the orbit is close enough for matter to be tidally stripped from the companion and ACCRETED onto the compact star (i.e., white dwarf, neutron star, or black hole). As matter falls into the gravitational potential, it liberates a considerable amount of energy. This makes the system bright.

EXERCISE 7.5 — Let's estimate the luminosity and surface temperature of an accreting neutron star. Assume a mass of $1.4\,M_\odot$ and a radius of 10 km. How much gravitational energy (in MeV) is released when a proton falls onto the surface (use a Newtonian approximation for the gravitational potential). How does this compare to the energy released (per proton) from the fusion of hydrogen into helium? Now suppose the neutron star is accreting at $10^{17}\,\mathrm{g\,s^{-1}}$, which is a typical rate for many observed systems. What would be the luminosity generated by this accretion? Suppose the luminosity were emitted thermally from the surface of the neutron star. What would be the surface effective temperature? In what band (e.g., visible, IR, UV, X-ray) would you want to observe this system?

When sufficient material[8] has accumulated on the surface, thermonuclear reactions can ignite in the accreted layer. This ignition is typically thermally unstable and leads to an explosion. On a white dwarf, this explosion is manifest as a NOVA[9] as the white dwarf abruptly brightens and then dims over several weeks to months. The mass of the burning layer is typically $\left(10^{-5}\text{ to }10^{-4}\right) M_\odot$; at typical accretion rates $\lesssim 10^{-9}\,M_\odot\,\mathrm{yr}^{-1}$ the time between the explosions is thousands of years or longer. The amount of mass necessary for ignition decreases strongly with the mass of the white dwarf, however, so that the time between explosions can be years to decades. In these systems the novae are observed to reoccur and they are called—appropriately enough—RECURRENT NOVAE. On a neutron star, the explosion is observed as an X-RAY BURST that lasts (10–100) s. The strong gravity means the amount of material needed for ignition is much less: roughly $10^{-12}\,M_\odot$. As a consequence, the time between bursts can be as short as hours to days.

[8] The accreted matter is usually mostly hydrogen, but if the companion star is evolved it could be enriched in helium or even, if the companion star is itself a white dwarf, carbon and oxygen.

[9] from the Latin *novus* meaning "new"

Bibliography

Bradley W. Carroll and Dale A. Ostlie. *An Introduction to Modern Astrophysics*. Addison-Wesley, 2d edition, 2006.

J. P. Cox. *Theory of Stellar Pulsation*. Princeton University Press, 1980.

Michael C. Cushing, J. Davy Kirkpatrick, Christopher R. Gelino, et al. The Discovery of Y Dwarfs using Data from the Wide-field Infrared Survey Explorer (WISE). *ApJ*, 743:50, December 2011. DOI: 10.1088/0004-637X/743/1/50.

Richard P. Feynman, Robert B. Leighton, and Matthew Sands. *The Feynman Lectures on Physics*, volume 3. Addison-Wesley, 1989.

G. H. Jacoby, D. A. Hunter, and C. A. Christian. A library of stellar spectra. *ApJS*, 56:257–281, October 1984. DOI: 10.1086/190983.

J. D. Kirkpatrick, I. N. Reid, J. Liebert, et al. Dwarfs Cooler than "M": The Definition of Spectral Type "L" Using Discoveries from the 2 Micron All-Sky Survey (2MASS). *ApJ*, 519:802–833, July 1999.

Francis LeBlanc. *An Introduction to Stellar Astrophysics*. John Wiley & Sons, Ltd, West Sussex, UK, 2010.

D. Mihalas. *Stellar Atmospheres*. W. H. Freeman, 2d edition, 1978.

Bill Paxton, Lars Bildsten, Aaron Dotter, Falk Herwig, Pierre Lesaffre, and Frank Timmes. Modules for experiments in stellar astrophysics (MESA). *ApJS*, 192:3, January 2011.

C. H. Payne. *Stellar Atmospheres; a Contribution to the Observational Study of High Temperature in the Reversing Layers of Stars*. PhD thesis, RADCLIFFE COLLEGE., 1925.

S. E. Woosley, A. Heger, and T. A. Weaver. The evolution and explosion of massive stars. *Reviews of Modern Physics*, 74:1015–1071, November 2002. DOI: 10.1103/RevModPhys.74.1015.