Quick notes on
Virial Equilibrium

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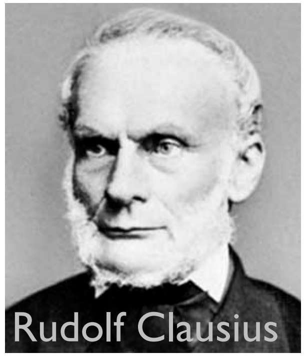
Ohio University - ASTR4201 - Fall 2020

Virial Theorem

- Describes the energy balance in a stable (not rapidly contracting/expanding), bound (matter isn't escaping) system of N particles
- Consider the total kinetic energy of the star:
  - $K = \sum_1^N K_i = \sum_1^N \frac{1}{2} m_i v_i^2 = \sum_1^N \frac{1}{2} p_i v_i = \sum_1^N \frac{1}{2} \vec{p}_i \cdot \frac{d\vec{r}_i}{dt}$
  - $\frac{d}{dt}\left(\sum_1^N \vec{p}_i \cdot \vec{r}_i\right) = \sum_1^N \vec{r}_i \cdot \frac{d\vec{p}_i}{dt} + \sum_1^N \vec{p}_i \cdot \frac{d\vec{r}_i}{dt}$
  - …so, $K = \frac{1}{2}\left(\frac{d}{dt}\left(\sum_1^N \vec{p}_i \cdot \vec{r}_i\right) - \sum_1^N \vec{r}_i \cdot \frac{d\vec{p}_i}{dt}\right) \equiv \frac{1}{2}\left(\frac{d}{dt} G - \sum_1^N \vec{r}_i \cdot \frac{d\vec{p}_i}{dt}\right) = \frac{1}{2}\left(\frac{dG}{dt} - \sum_1^N \vec{r}_i \cdot \vec{F}_i\right)$
- Recall Work = Force*Distance and Work = -(Change in potential energy)
  - so, $\vec{F}_i = -\nabla_i \Omega$
  - and therefore, $2K = \frac{dG}{dt} + \sum_1^N \vec{r}_i \cdot \nabla_i \Omega$
- Things are stable, so we can consider average properties,
  - $\langle 2K \rangle = \left\langle \frac{dG}{dt} + \sum_1^N \vec{r}_i \cdot \nabla_i \Omega \right\rangle = \left\langle \frac{dG}{dt} \right\rangle + \left\langle \sum_1^N \vec{r}_i \cdot \nabla_i \Omega \right\rangle$

Seems like a fun guy.


Rudolf Clausius

Virial Theorem part deux

- We now have our time average: $\langle 2K \rangle = \left\langle \frac{dG}{dt} + \sum_1^N \vec{r}_i \cdot \nabla_i \Omega \right\rangle = \left\langle \frac{dG}{dt} \right\rangle + \left\langle \sum_1^N \vec{r}_i \cdot \nabla_i \Omega \right\rangle$
- Recall how to take a time average:
  - $\langle f \rangle = \frac{\sum f(t)\Delta t}{\sum \Delta t} = \frac{1}{\tau} \int f(t)dt$ …over all time: $\langle f \rangle = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau f(t)dt$
- Back to our virial:
  - $\langle 2K \rangle = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau \left\langle \frac{dG}{dt} \right\rangle dt + \left\langle \sum_1^N \vec{r}_i \cdot \nabla_i \Omega \right\rangle = \left\langle \sum_1^N \vec{r}_i \cdot \nabla_i \Omega \right\rangle$ because $\left\langle \frac{dG}{dt} \right\rangle$ is finite and $\tau \to \infty$
- For potentials $\Omega = Cr^j$: $\vec{r} \cdot \nabla\Omega = r\frac{\partial}{\partial r}\Omega = r\frac{\partial}{\partial r}Cr^j = jrCr^{j-1} = jCr^j = j\Omega$
  - So, $\left\langle \sum_1^N \vec{r}_i \cdot \nabla_i \Omega \right\rangle = \langle j\Omega \rangle$ … **for gravity**, $\Omega \propto r^{-1}$ …so: $\left\langle \sum_1^N \vec{r}_i \cdot \nabla_i \Omega \right\rangle = -\langle \Omega \rangle$
  - Meaning, $\mathbf{2}\langle \boldsymbol{K} \rangle = -\langle \boldsymbol{\Omega} \rangle$
  - Total energy in this case is: $\langle E \rangle = \langle K \rangle + \langle \Omega \rangle = -\frac{1}{2}\langle \Omega \rangle + \langle \Omega \rangle = \frac{1}{2}\langle \Omega \rangle < 0$ (for a bound system)

Consequences of the Virial Theorem

- Consider a contracting star or cloud of gas.
  Since $\Omega \propto -\frac{1}{r}$, decreasing r results in a more negative Ω.
- Since energy is conserved, $\langle E \rangle = \langle K \rangle + \langle \Omega \rangle = Constant$,
  and therefore if r decreases, $\langle \Omega \rangle$ goes more negative, meaning $\langle K \rangle$ must increase
- For a monatomic ideal gas, $\langle K \rangle = \frac{3}{2} N k_B T$
- Since N is fixed, $\langle K \rangle$ increasing must mean that T is increasing
- Therefore, the temperature of a contracting cloud of gas is increasing
  - Meaning, a collapsing cloud of gas in space becomes hot enough to ignite hydrogen (a star)
  - Meaning, when a star runs through a core fuel (e.g. hydrogen) and contracts because of the loss in radiation pressure, the newly contracted core will have a higher temperature, ultimately enabling fusion of a harder-to-fuse fuel (e.g. helium).