Quick notes on
Dynamical Time

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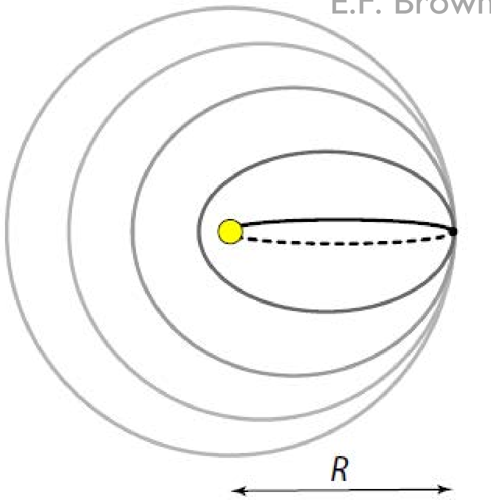
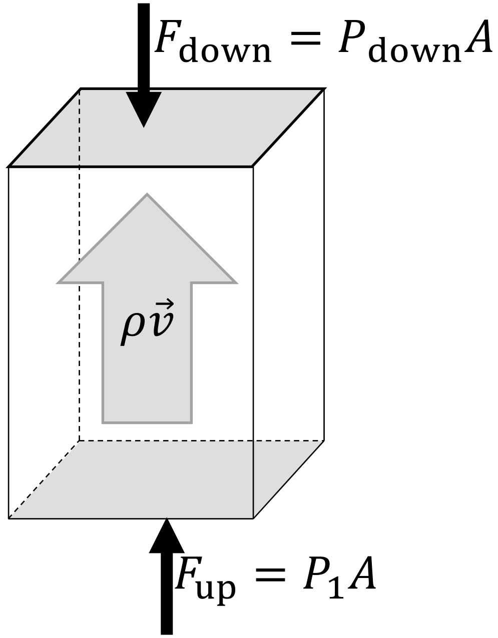
Free-Fall Time

- Hydrostatic equilibrium is pressure balancing gravity.
  In the absence of pressure, what's the timescale for bulk motion?
- Equation of motion for a chunk of matter orbiting the center of the star:
  - $F_{\text{centripetal}} = F_{\text{gravity}} \quad \longrightarrow \quad \frac{mv^2}{r} = \frac{GMm}{r^2} \quad \longrightarrow \quad v^2 = \frac{GM}{r}$
- The velocity is the orbit length divided by the orbit period τ
- Stretching an ellipse to absurdity will result in free-fall scenario
  - $v = {}^{2r}/_{\tau} = {}^{2r}/_{2t_{\text{ff}}}$ …so: $t_{\text{ff}}^2 = \frac{r^3}{GM}$
  - But the ellipse is stretched, so the "radius" is $r = R/2$, so: $t_{\text{ff}}^2 = \frac{R^3}{8GM}$
  - Recalling $\bar{\rho} = \frac{3M}{4\pi R^3} \;\rightarrow\; \frac{R^3}{M} = \frac{3}{4\pi\bar{\rho}}$
  - So: $t_{\text{ff}}^2 = \frac{3}{32\pi G\bar{\rho}} \quad \longrightarrow$ **Free-Fall time:** $\boldsymbol{t_{\text{ff}} = \sqrt{\frac{3}{32\pi}}\frac{1}{\sqrt{G\bar{\rho}}}}$

E.F. Brown



Sound-Crossing Time



- Collapse would cause outgoing pressure waves, moving at the sound speed
- Force for a fluid element from the pressure gradient
  - $\vec{F}_{\text{net}} = \vec{F}_{\text{up}} - \vec{F}_{\text{down}} = A(P_2 - P_1) = A\Delta P = -m\vec{a} = -m\frac{d\vec{v}}{dt}$
  - $\rightarrow \ \Delta P = \frac{-m}{A}\frac{d\vec{v}}{dt} \quad \rightarrow \frac{\Delta P}{\Delta z} = \frac{dP}{dz} = -\rho\frac{dv}{dt} \quad \rightarrow dP = -\rho dv\frac{dz}{dt}$
- Conservation of mass: $\dot{m} = \rho A v = constant$ …For fixed A, $\rho v = constant$
  - $d(\rho v) = 0 = v d\rho + \rho dv \quad \rightarrow \quad v d\rho = -\rho dv$
- Returning to pressure gradient
  - $dP = v d\rho\frac{dz}{dt} = v d\rho v = v^2 d\rho \quad \rightarrow v = \sqrt{\frac{dP}{d\rho}} = \sqrt{\frac{P}{\rho}}$ for ideal gas because $P \propto \rho T$
  - This process is isentropic, not isothermal, so need Laplace's correction: $v \equiv c_s = \sqrt{\frac{\gamma P}{\rho}}$
- Sound crossing time: $t_{\text{sc}} = \frac{R}{c_s} = R\sqrt{\frac{\rho}{\gamma P}} \sim R\sqrt{\frac{\rho_c}{P_{c,\rho\text{const}}}} \sim R\frac{1}{\sqrt{R^2 G\overline{\rho}}} \sim \frac{1}{\sqrt{G\overline{\rho}}}$

Dynamical Time

- Free fall time: $t_{\mathrm{ff}} \propto \frac{1}{\sqrt{G\overline{\rho}}}$
- Sound crossing time: $t_{\mathrm{sc}} \propto \frac{1}{\sqrt{G\overline{\rho}}}$
- The environment-related scaling is called the **dynamical time:** $t_{\mathrm{dyn}} \equiv \frac{1}{\sqrt{G\overline{\rho}}}$
- Essentially, how long does it take for structural changes to propagate.
- Since $t_{\mathrm{ff}} \sim t_{\mathrm{sc}}$, a star is able to maintain equilibrium.
  I.e., the time for action (t_{ff}) is roughly the same as
  the time for a response (t_{sc})
- This is true when $\gamma = 5/3$, but it's a delicate balance.
  A bit lower in γ will result in collapse, as we'll see later