Quick notes on
Orbital Angular Momentum

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Energy Quantization in Hydrogen

- Back in Quantum Mechanics, you likely suffered through finding the eigenvalues of the Hydrogen atom.
  There are more fun semi-classical ways to guess the result. Below is one.
- The kinetic energy of the electron is $E = \frac{1}{2}m\left(\frac{2\pi r}{\tau}\right)^2$
  - From Kepler's 3rd law, $\tau^2 \propto r^3$ … so, $E \propto r^{-1}$
  - For an equilibrium orbit, $\frac{mv^2}{r} = \frac{q^2}{r^2} \rightarrow mv^2 r = q^2 \rightarrow L^2 = mrq^2 \rightarrow L^2 \propto r$
  - So, $E \propto L^{-2}$
- Our electron particle is also a wave with $\lambda = \frac{h}{mv} = \frac{hr}{L}$
  - For a standing wave, one orbit is an integer number of wavelengths: $n\lambda = 2\pi r$
  - So $n\frac{hr}{L} = 2\pi r$ …meaning $L = n\hbar$
- Therefore, $E \propto n^{-2}$, as found empirically and through QM

Key Role of Orbital Angular Momentum

- Why do semi-classical methods based on orbital angular momentum work?
- The electrostatic potential is spherically symmetric, $U \propto r^{-1}$, so it doesn't change when you rotate the hydrogen atom
- Noether's theorem tells us that L is therefore conserved
- (an analogue of) The Ehrenfest theorem tells us that the time-averaged behavior of the QM operator for L will follow the behavior of the classical L
- …which jives with the Correspondence ~~gut feeling~~ Principle of Bohr that QM calculations reproduce classical calculations for large quantum numbers

"An expert is a person who has found out by their own painful experience all the mistakes that one can make in a very narrow field"

Degeneracy of Hydrogen Atom Energy Levels

- Why are there $4n^2$ states for each energy E_n?
- For any state, the proton and electron can independently either be spin-up or spin-down, creating 4 combinations
- A long time ago, you (painfully) solved the Hydrogen atom eigenfunctions with QM and found $\psi_{nlm} = R_{nl}(r)Y_l^m(\theta, \varphi)$, where n is the principal quantum number and $n > l \geq |m| \geq 0$ … so $l_{\max} = n - 1$
  - Roughly, n is the size of the orbital, l is it's shape, and m is its orientation
- Since m runs from $-l$ through 0 to l, there are $2l + 1$ for each m
- So, for each n, we have $\sum_{l=0}^{n-1} 2l + 1 = 1 + 3 + 5 + 7 + \cdots + 2n - 1 = n^2$ via the odd number theorem
- So, there are $4n^2$ states with energy $E = -13.6n^{-2}$ eV