Homework Assignment 6 **ASTR4201, Fall 2020**

Corresponds to Chapter 6 of "To Build a Star" (TBS) by E.F. Brown

1. TBS exercise 6.1 Team: 3 Lead: Harshil
2. TBS exercise 6.2 Team: 4 Lead: Gula
3. TBS exercise 6.3 Team: 2 Lead: Michael
4. TBS exercise 6.4 Team: 3 Lead: Ryan
5. TBS exercise 6.5 Team: 5 Lead: Robert
6. TBS exercise 6.6 Team: 1 Lead: Anthony
7. TBS exercise 6.7 Team: 2 Lead: Sam
8. TBS exercise 6.8 Team: 4 Lead: Jacob
9. TBS exercise 6.9 Team: 5 Lead: Justin
10. TBS exercise 6.10 Team: 2 Lead: Quinn
11. TBS exercise 6.11 Team: 3 Lead: Josh
12. TBS exercise 6.12 Team: 1 Lead: Gavin
13. *See below* Team: 1 Lead: Brit
    Sirius, the brightest star in the night sky, is actually a double star system consisting of a main sequence star (Sirius A) and a white dwarf companion (Sirius B). The mass of Sirius A is quite well known, e.g. from Kepler's laws, to be ~2 solar masses. What would you estimate its radius to be and why?

Problem 6.1

Boat + weight :

$$V_{disp} = \frac{m_b + m_w}{\rho_{water}}.$$

Boat : $V_{disp} = \frac{m_b}{\rho_{water}}$

Weight : $V_{disp} = \frac{m_w}{\rho_{weight}}.$

Total $V_{disp} = \frac{m_b}{\rho_{water}} + \frac{m_w}{\rho_{weight}}$

$$\frac{m_w}{\rho_{weight}} < \frac{m_w}{\rho_{water}}$$

$\therefore$ Volume displaced would decrease.

6.2 Problem 6.2

$$T = T_0 \left(\frac{P}{P_0} \right)^{(\gamma-1)/\gamma} \quad \text{---- } 6.13$$

$$P = \rho \left(\frac{k_B}{\mu m_u} \right) T \text{ ---- } 2.5 \quad \text{and} \quad P_0 = \rho_0 \left(\frac{k_B}{\mu m_u} \right) T_0$$

Substitude 2.5 in 6.13

$$T = T_0 \left[\frac{\rho \left(\frac{k_B}{\mu m_u} \right) T}{P_0} \right]^{(\gamma-1)/\gamma}$$

$$T = \frac{T_0}{P_0} \left[\rho \left(\frac{k_B}{\mu m_u} \right) \right]^{(\gamma-1)/\gamma} \cdot T^{(\gamma-1)/\gamma}$$

$$T^{\left(1 - \frac{(\gamma-1)}{\gamma}\right)} = T^{\frac{\gamma-\gamma+1}{\gamma}} = T^{1/\gamma}$$

$$T = \left[\frac{T_0}{P_0} \left(\rho \frac{k_B}{\mu m_u} \right)^{(\gamma-1)/\gamma} \right]^{\gamma} \text{ --- } 3$$

$$= \left[\frac{\cancel{T_0}}{\rho_0 \left(\frac{k_B}{\mu m_u} \right) \cancel{T_0}} \right]^{\gamma} \left[\rho \frac{k_B}{\mu m_u} \right]^{\gamma-1}$$

$$= \frac{1}{\rho_0^{\gamma} \left(\frac{\cancel{k_B}}{\cancel{\mu m_u}} \right)^{\gamma}} \cdot \rho^{\gamma-1} \cdot \frac{\cancel{k_B^{\gamma}} \cdot k_B^{-1}}{\cancel{\mu^{\gamma}} \cdot \cancel{m_u^{\gamma}} \cdot \mu^{-1} \cdot m_u^{-1}}$$

Problem 6.2

$$T = \frac{\rho^{\gamma-1}}{\rho_0^{\gamma}} \frac{\mu m_u}{K_B} \quad \text{....} 3$$

putting eq ③ in eq (2.5)

$$P = \rho \left(\frac{\cancel{K_B}}{\cancel{\mu} m_u}\right) \cdot \frac{\rho^{\gamma-1}}{\rho_0^{\gamma}} \cdot \frac{\cancel{\mu} m_u}{\cancel{K_B}}$$

$$= \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

Problem 6.3

6.3

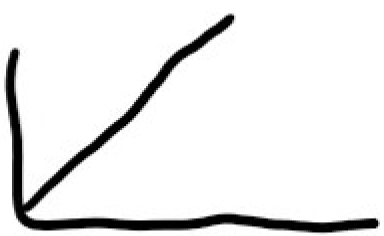
Max slope is $2/s$, anything greater than that is unstable

slope is $1/s$

stable

Slope is 0

stable



slope is $3/s$

unstable

Slope is $-1/s$

stable

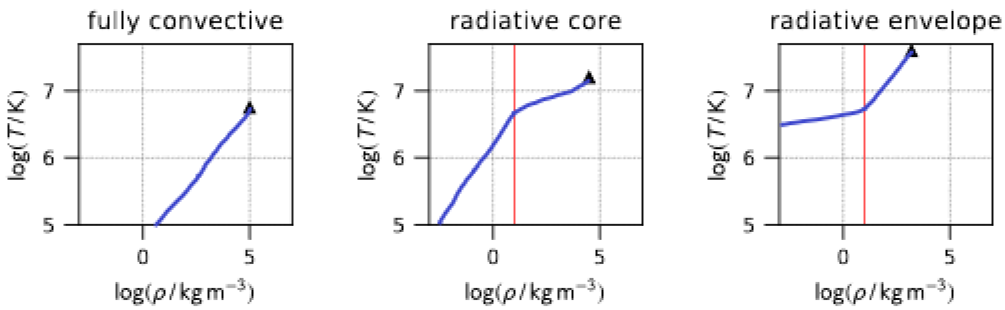
6.4

$$\frac{P}{T}\left(\frac{\partial T}{\partial P}\right)_s = \left(\frac{\partial \ln T}{\partial \ln P}\right)_s = \frac{\gamma - 1}{\gamma}.$$

$\gamma = 5/3 \rightarrow (\gamma - 1)/\gamma = 2/5 = \underline{0.4}$

Slope = 0.4 convection

Slope < 0.4 radiative



Const 0.4 slope

Shallow slope in core

0.4 envelope slope

0.4 slope in core

Shallow envelope slope

$$X = \ln\left(\frac{M}{R^3}\right)$$

$$X = \ln(M) - \ln(R^3)$$

$$\frac{-X + \ln(M)}{3} = \ln(R)$$

$$e^{\frac{\ln(M) - X}{3}} = R$$

$$T = \frac{G \mu m_u}{5 K_B} \frac{M}{R}$$

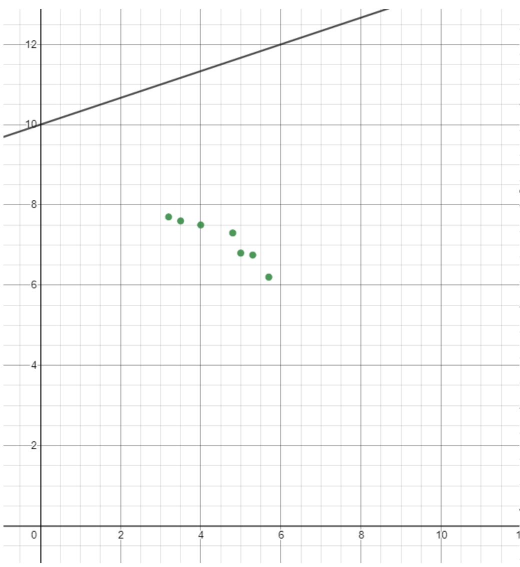
$$\ln(T) = \ln(CM) - \ln(R)$$

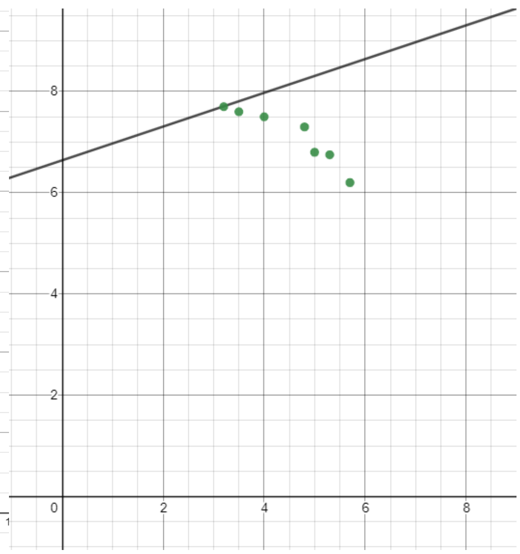
$$\ln(T) = \ln(CM) - \ln(M)\frac{1}{3} - \frac{X}{3}$$

$$\ln(T) = \ln\left(C\frac{M}{M^{1/3}}\right) - \frac{X}{3}$$

$$\ln(T) = \ln(C M^{2/3}) - \frac{X}{3}$$

The equation is rewritten in terms of x, which is the x axis from the figure in the book. After that, plotting becomes very easy. If you set C to the variables defined by the virial relation, the plot misses the data points, but the slope is one third and the constants can be tweaked to fit the points manually.





Problem 6.6

Anthony D'Alessandro HW 6

6.6

(6.37)
$$\mathcal{N}_e = \left(\sum_i X_i \frac{Z_i}{A_i}\right)^{-1}$$

(6.36)
$$P = \frac{2}{5}\left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{2m} n^{5/3}$$

$$n_e = \frac{\rho}{m\mathcal{N}_e} = \frac{\rho}{m_u}\sum X_i \frac{Z_i}{A_i}$$

So,
$$P(\rho) = \frac{2}{5}\left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{2m}\left(\frac{\rho}{m_u}\sum\left(X_i \frac{Z_i}{A_i}\right)\right)^{5/3}$$

From Virial Scalings,
$$P = \frac{3}{20\pi} \frac{GM^2}{R^4}$$

$$a = \frac{2}{5}\left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{2m} \qquad b = \frac{3G}{20\pi}$$

removing constants & setting P equal

$$\left(\frac{\rho}{m_u}\right)^{5/3} = \frac{M^2}{R^4}$$

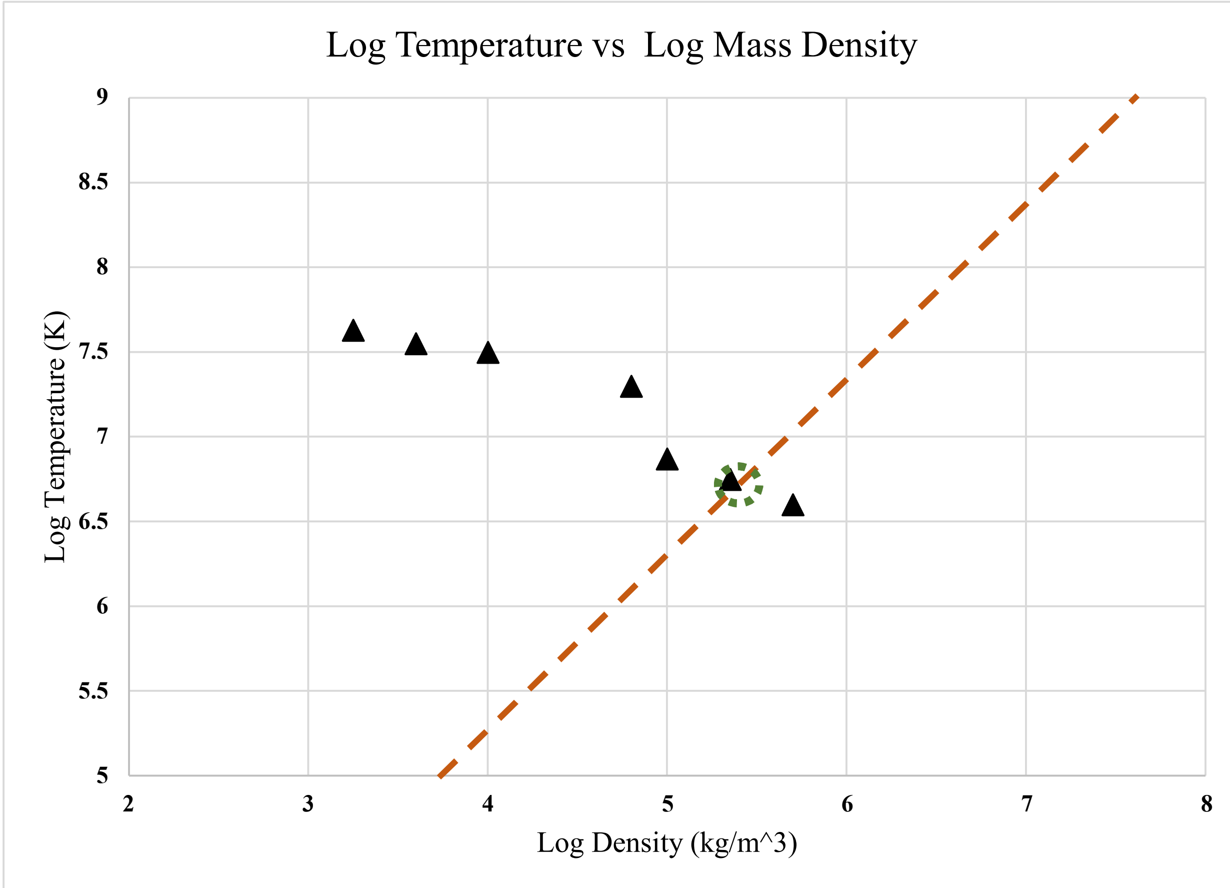
$$\frac{\rho^{5/3}}{m^{5/3}} = \frac{M^2}{R^4}$$

$$\frac{\left(\frac{M}{R^3}\right)^{5/3}}{m^{5/3}} = \frac{M^2}{R^4}$$

$$\boxed{R = M^{-1/3}}$$

Problem 6.7

M	log p	log T
0.09	5.7	6.6
0.15	5.35	6.75
0.3	5	6.87
2	4.8	7.3
10	4	7.5
25	3.6	7.55
100	3.25	7.63



Problem 6.8

$$R^4 = M^2 \left[\frac{5\, G m_e}{h^2} \left(\frac{3}{8\pi}\right)^{1/3} (m_u \mu_e)^{5/3} \right] \rho^{-5/3}$$

$$R^4 = M^2 [C] \left(\tfrac{4}{3}\pi\right)^{5/3} M^{-5/3} R^5$$

$$M^{1/3} [C]^{-1} \left(\tfrac{3}{4\pi}\right)^{5/3} = R$$

Degenerate

$$\rho = \frac{M}{\frac{4}{3}\pi R^3}$$

$$R = \sqrt{M} [C]^{1/4} \rho^{-5/12}$$

$$\rho = \frac{1.176\, m_u}{a_B^3} = \boxed{1.3187 \times 10^4\ kg/m^3}$$

$$\mu_e = [X_H (\tfrac{1}{1}) + X_{He} (\tfrac{2}{4})]^{-1} = 1.176$$

$\rho_{Pb} = 1.1343 \times 10^4\ kg/m^3$

$\rho_{Au} = 1.93 \times 10^4\ kg/m^3$

This density is very plausible given common densities of metals

$\rho_{Fe} = 7.87 \times 10^3\ kg/m^3$

$$\rho = \frac{\mu_e m_u}{a_B^3} \Rightarrow R = \sqrt{M} \left[\frac{5\, G m_e}{h^2} \left(\frac{3}{8\pi}\right)^{1/3} (a_B)^5 \right]$$

or $\boxed{R = \sqrt{M} \,(6.089 \times 10^{-7}\ m/\sqrt{kg})}$

$$R = M^{1/3} \left(\tfrac{3}{4\pi}\right)^{5/3} \frac{1}{[C]} = M^{1/3} (8.833 \times 10^{16}\ m\,kg^{1/3}) = \sqrt{M}\,(6.13 \times 10^{-7}\ m/kg^{1/2})$$

$$M^{-5/6} = 6.89 \times 10^{-24}\ kg^{-5/6}$$

$$\boxed{M = 6.2211 \times 10^{27}\ kg \approx 3.25\ M_{Jupiter}}$$

$M_{Jupiter} = 1.898 \times 10^{27}\ kg$

Problem 6.9

6 Homework 6

Question 9: EXERCISE 6.9 — Use the virial relations for density and temperature to estimate how the ratio P_{rad}/P_{gas} depends on the mass of the star.

P_{rad} is the radiation pressure and P_{gas} is the gas pressure; they are given by

$$P_{rad} = \frac{a}{3}\bar{T}^4 \tag{62}$$

$$P_{gas} = \bar{\rho}\frac{k_B}{\mu m_u}\bar{T} \tag{63}$$

and their ratio is

$$\frac{P_{rad}}{P_{gas}} = \frac{\frac{a}{3}\bar{T}^4}{\bar{\rho}\frac{k_B}{\mu m_u}\bar{T}} \tag{64}$$

using the virial relations for density and temperature

$$\bar{\rho} = \frac{M}{\frac{4\pi}{3}R^3} \tag{65}$$

$$\bar{T} = \frac{1}{5}\frac{GM}{R}\frac{\mu m_u}{k_B} \tag{66}$$

Substituting these values into P_{rad}/P_{gas}

$$\frac{P_{rad}}{P_{gas}} = \frac{\frac{a}{3}\bar{T}^3}{\bar{\rho}\frac{k_B}{\mu m_u}} \tag{67}$$

$$= \frac{\frac{a}{3}\left(\frac{1}{5}\frac{GM}{R}\frac{\mu m_u}{k_B}\right)^3}{\frac{M}{\frac{4\pi}{3}R^3}\frac{k_B}{\mu m_u}} \tag{68}$$

$$= \frac{aG^3M^2}{500\pi}\left(\frac{\mu m_u}{k_B}\right)^4 \tag{69}$$

The ratio P_{rad}/P_{gas} depends on M^2

Problem 6.10

6.10.) $P_{rad} = \dfrac{aT^4}{3}$

$P_{gas} = \dfrac{k_B T}{\mu m_H}\rho$

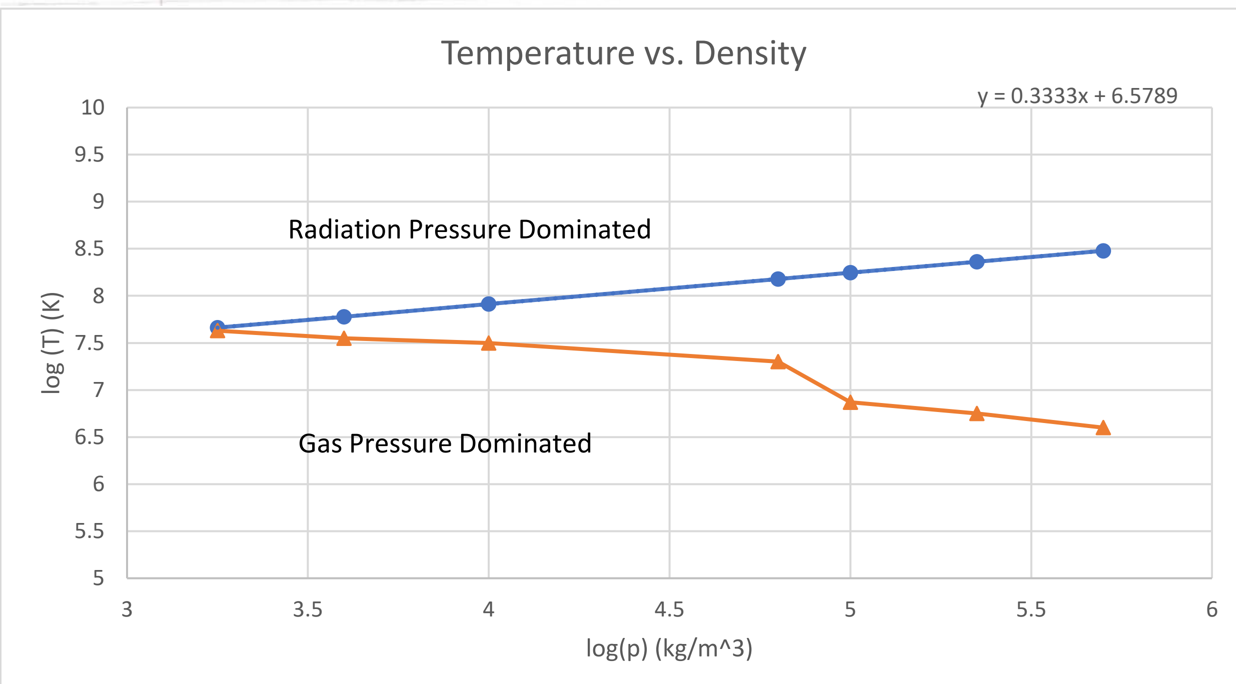
Given that $P_{gas} \approx P_{rad}$

$$\frac{aT^4}{3} = \frac{k_B T}{\mu m_H}\rho$$

$$T = \left(\frac{k_B \cdot 3}{a \mu m_H}\rho\right)^{1/3}$$

$k_B = 1.381 \times 10^{-23}$ J/K
$a = 7.5657 \times 10^{-16}$ J/m$^3 \cdot$K^4
$\mu = 0.6$
$m_H = 1.6735 \times 10^{-27}$ kg

From the graph I can see that Gas Pressure is most important in low mass Stars; Radiation Pressure is most important in high mass stars.



Problem 6.11

6.11

$$\left(6.6795 \text{ MeV/H atom}\right)\left(1.6022\times10^{-13}\text{ J/MeV}\right)\left(\frac{1}{1.67\times10^{-27}}\text{ H atom/kg}\right)$$

$$= 6.41\times10^{14}\text{ J/kg}$$

$$70\% \cdot 10\% = 7\%$$

$$L_\odot = 3.828\times10^{26}\text{ W} = 3.828\times10^{26}\text{ J/s}$$

$$M_\odot = 1.989\times10^{30}\text{ kg} \times .07 = 1.392\times10^{29}$$

$$\frac{L_\odot}{M_\odot} = \text{energy-mass loss}$$

$$= \frac{3.828\times10^{26}\text{ J/s}}{1.392\times10^{29}\text{ kg}}$$

$$= .00275 \ \frac{\text{J}}{\text{kg}\cdot\text{s}}$$

$$\text{MS lifetime} = \frac{6.41\times10^{14}\text{ J/kg}}{.00275\text{ J/kg}\cdot\text{s}} = 2.33\times10^{17}\text{ s}$$

$$= 7.39\times10^{9}\text{ years}$$

Gavin Shaw

Problem 6.12

Exercise 6.12

Equation 6.18 : $\frac{dT}{dr} = -\frac{L}{4\pi r^2}\frac{3\rho k}{4acT^3}$

$\frac{dT}{dr} \approx \frac{T_c}{R}$

$\rho \approx \bar{\rho}$

$L/4\pi r^2 \approx L/4\pi R^2$

$T \approx T_c$

$T_c = \frac{1}{2}\left(\frac{GM}{R}\frac{\mu m_u}{k_b}\right)$ Eq 2.14

k = opacity (constant)

$\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3}$

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a) $\frac{T_c}{R} = -\frac{L}{4\pi R^2}\frac{3\bar{\rho}k}{4acT_c^3}$

$$\frac{\left[\frac{1}{2}\left(\frac{GM}{R}\frac{\mu m_u}{k_b}\right)\right]}{R} = -\frac{L}{4\pi R^2}\frac{3\left(\frac{M}{\frac{4}{3}\pi R^3}\right)k}{4ac\left[\frac{1}{2}\left(\frac{GM}{R}\frac{\mu m_u}{k_b}\right)\right]^3}$$

$$\frac{\frac{GM\mu m_u}{2Rk_b}}{R} = -\frac{L}{4\pi R^2}\frac{\frac{3Mk}{\frac{4\pi R^3}{3}}}{4ac\left(\frac{GM\mu m_u}{2Rk_b}\right)^3}$$

$$\frac{GM\mu m_u}{2Rk_b}\cdot\frac{1}{R} = -\frac{L}{4\pi R^2}\frac{\left(\frac{3Mk}{1}\cdot\frac{3}{4\pi R^3}\right)}{4ac\left(\frac{GM\mu m_u}{2Rk_b}\right)^3}$$

$$\frac{GM\mu m_u}{2R^2k_b} = -\frac{L}{4\pi R^2}\left[\frac{\frac{9Mk}{4\pi R^3}}{4ac\left(\frac{GM\mu m_u}{2Rk_b}\right)^3}\right]$$

$$\frac{GM\mu m_u}{2R^2k_b} = -\frac{L}{4\pi R^2}\left(\frac{9k\,k_b^3}{\pi ac\,G^3M^2\mu^3m_u^3}\right)$$

• Divide both sides by this term

$$\frac{GM\mu m_u}{2R^2k_b}\cdot\frac{\pi ac\,G^3M^2\mu^3m_u^3}{9k\,k_b^3} = -\frac{L}{4\pi R^2}$$

$$L = -\frac{4G^4 M^3 \mu^4 m_u^4 \pi a c}{18 \kappa k_b^4}$$

- We see luminosity depends on mass to some power
- Make all variables / coefficients into one constant "C"

$$C = -\frac{4G^4 \mu^4 m_u^4 \pi a c}{18 \kappa k_b^4}$$

$$\boxed{L = (C) M^3}$$

b) Comparing expression for Luminosity found in (a) with table 2.2

c) Find expression for stellar lifetime as function of mass then calibrate to Sun's main sequence lifetime, $\tau_\odot \sim 10$ Gyr

t_{ms} (time for main sequence lifetime)

$$t_{ms} = \frac{M}{L}$$

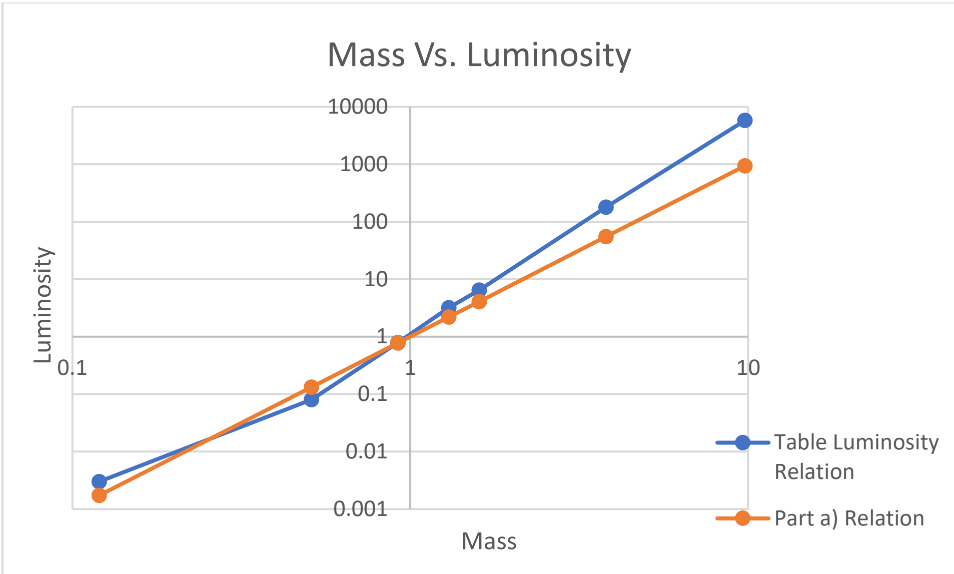
$L \sim M^{3\text{[illegible]}}$ (mass-Luminosity relationship for Main-sequence stars)

$$t_{ms} \sim \frac{M}{M^{3\text{[illegible]}}} = \frac{1}{M^{\text{[illegible]}}} = M^{-2\text{[illegible]}}$$

$$\frac{t_{ms}}{t_\odot} \sim \left(\frac{M}{M_\odot}\right)^{-2\text{[illegible]}}$$ • calibrating step

$$t_{ms} = \left(\frac{M}{M_\odot}\right)^{-2\text{[illegible]}} (t_\odot)$$

$$\boxed{t_{ms} = \left(\frac{M}{M_\odot}\right)^{-2\text{[illegible]}} (10 \text{ Gyr})}$$



Mass Vs. Luminosity
10000
1000
100
10
1
0.1
0.01
0.001
0.1
1
10
Luminosity
Mass
Table Luminosity Relation
Part a) Relation

Given: Sirius A mass ~2 solar masses

Want: radius

Problem 6.13

Know:

- ★ main sequence stars fusing hydrogen to helium two different ways:
  - high mass → CNO cycle
  - low mass → p-p chain
- ★ for high mass, $R \propto M$ since the sun (our reference) is on low/high boundary
- ★ type of burning is dependent on core temp (CNO high, p-p low)
- ★ Sirius A has high core temp since on main sequence and is 2 solar masses

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so, for Sirius A, $R \propto M$

then, $R \sim 2$ solar radii

⇒ actual value is 1.7 solar radii, so close enough