Homework Assignment 5 **ASTR4201, Fall 2020**

Corresponds to Chapter 5 of "To Build a Star" (TBS) by E.F. Brown

1. TBS exercise 5.1 Team: 3 Lead: Josh

2. TBS exercise 5.2 Team: 2 Lead: Quinn

3. *See below* Team: 1 Lead: Gavin
   Considering the terms of the semi-empirical mass formula, which parameter is responsible for the following feature of the nuclear landscape:
   a. Location of the valley of stability for low A?
   b. Bend of valley of stability away from N=Z for large A?
   c. Large A/Z ratios?
   d. Lack of ultra-high A nuclides?
   e. Existence of nuclides in the first place?

4. *See below* Team: 3 Lead: Harshil
   Calculate the experimental binding energy difference between 15N and 15O [See http://amdc.impcas.ac.cn/masstables/Ame2016/mass16.txt]. Assuming this is due to the Coulomb term of the SEMF, what radius corresponds to A=15? Note that compared to a point-charge, a uniformly charged sphere has $U_{sphere} = (3/5)U_{point}$. Compare this to the usual approximation for the nuclear radius (using r_0=1.2fm).

5. TBS exercise 5.3 Team: 2 Lead: Michael

6. *See below* Team: 1 Lead: Anthony
   There's a maximum in the Binding energy per nucleon (found in TBS 5.3 and shown in the Quick Notes). For measured masses, this is at ^{58}Fe. Why isn't everything made of ^{58}Fe?

7. *See below* Team: 3 Lead: Ryan
   In a single-degenerate scenario, a Type-1a supernova converts a white-dwarf's mass of roughly ^{12}C (B = 92.162 MeV) to roughly ^{60}Fe (B = 525.351 MeV). How much energy is this? How does this compare to the gravitational binding energy of the white dwarf? What does this say about the power source of Type-1a's?

8. TBS exercise 5.4 Team: 4 Lead: Jacob

9. *See below* Team: 4 Lead: Gula
   If the ^{3}He+^{3}He cross section were purely geometric, what would the S-factor be for a 100keV interaction energy? What about for p+d? Compare to the measured values of 5MeV*b for ^{3}He+^{3}He and 1MeV*b for p+d.

10. TBS exercise 5.5 Team: 1 Lead: Brit

11. TBS exercise 5.6 Team: 5 Lead: Justin

12. *See below* Team: 2 Lead: Sam

    Which of the following reactions are possible without non-standard model physics? For invalid reactions, indicate what the issue is.

    a. $^{137}Cs \rightarrow {}^{137m}Ba + e^-$

    b. $^{12}C(\alpha,\gamma)^{16}O$

    c. $^{144}Sm(\gamma,\alpha)^{140}Nd$

    d. $^{126}Te(n,\gamma)^{127}I$

13. *See below* Team: 5 Lead: Robert

    For the CNO cycle at near-solar temperatures, the process piles-up at ^{14}N because this rate is the slowest. Where would the process pile-up at if the temperature were very high?

Problem 5.1

ASTR 4201 HW #5

5.1 Given two particles with masses m_1 and m_2 at positions R_1 and R_2, center of mass coordinates are

$$\vec{R_{cm}} = \frac{m_1 \vec{R}_1 + m_2 \vec{R}_2}{m_1 + m_2}$$

The energy of a particle in a 2-body system is

$$E = \tfrac{1}{2} M \dot{R}_{cm}^2 + \tfrac{1}{2} \mu \dot{r}^2 + V(r)$$

where $M = m_1 + m_2$, $\mu = \frac{m_1 m_2}{m_1 + m_2}$

In center-of-mass coordinates, R_{cm} is at the origin, so $R_{cm} = 0$ and

$$E = \tfrac{1}{2} \mu \dot{r}^2 + V(r)$$

which only relies on relative coordinates. For a single particle of reduced mass $\mu \approx m_n/2$, relative coordinates are

$$\boxed{R_{cm} = \tfrac{1}{2}(R_1 + R_2) \text{ and } r_1 = -r_2}$$

In relative coordinates, momentum is $p = \mu \dot{r}$, so the energy can be rewritten as

$$E = \frac{1}{2\mu} p^2 + V(r)$$

Plugging in the single particle reduced mass gives

$$E = \frac{1}{m_n} p^2 + V(r)$$

The uncertainty principle states

$$\Delta x \Delta p \geq \hbar/2$$

If $p \sim \Delta p$, then

$$p \gtrsim \frac{\hbar}{2\Delta x}$$

Since strong force disappears at $\gtrsim 2$ fm

and is strongly repulsive at $\ll 1$ fm,
We can assume $\Delta x = 2\text{ fm} - \ll 1\text{ fm} = 2\text{ fm}$.
Then, $\boxed{p \geq \frac{\hbar}{4}}$.

So, the kinetic energy of the particle is
$$K = \frac{1}{m_n}\left(\frac{\hbar}{4}\right)^2 = \boxed{\frac{1}{m_n} \cdot \frac{\hbar^2}{16}}$$

The energy then becomes
$$E = \frac{1}{m_n}\frac{\hbar^2}{16} + V$$
Since $E_d \ll V$ for deuteron, E_d can be assumed to be 0. Thus,
$$0 = \frac{1}{m_n}\frac{\hbar^2}{16} + V$$
$$\boxed{V = -\frac{1}{m_n}\frac{\hbar^2}{16}}$$

Problem 5.2

5.2.) $\Omega_{const\ den} = -\frac{3}{5}G\frac{M^2}{R}$ [2.22]

Ignoring constants $\frac{-3G}{5}$ and plugging in the nucleus equivalent for M, [2.22] turns into,

$$\Omega = \frac{A^2}{R}$$

using the equation for the radius of the nucleus,

$$r_A = R_0 A^{\frac{1}{3}}$$

we ignore the constant R_0 and plug in $A^{\frac{1}{3}}$ for R and get,

$$\Omega = \frac{A^2}{A^{\frac{1}{3}}}$$

$$\Omega_{const\ den} = A^{5/3}$$

This is how the binding energy would scale with A in this case.

Problem 5.3

a) Asymmetry; the location of the valley of stability for low A is caused by asymmetry because at very low A, the system will have N=Z, which will cause the system to be stable.

b) Coulomb; the bend of the valley of stability away from N=Z for large A will be caused by the coulomb because there are more proton charges being added into the system, since neutrons don't carry any charge, the addition of more protons will change the energy of the system based on the charges being added.

c) Coulomb; large A/Z ratios will be caused by the coulomb because there is going to be a large neutron excess in the system.

d) Surface; the lack of ultra-high A nuclides is caused by the surface parameter because the A goes by a factor of $A^{\frac{2}{3}}$, which will penalize the A and the binding energy will not be able to support at high A values.

e) Volume; for nuclides to even exist, there needs to be a factor of A. The parameter that is going to guarantee the A is the volume.

Problem 5.4

4.

	15N	15O
B.E.	115491.9 (Kev)	111955.38 (kev)
Z	7	8

$\Delta B.E. = 3536.52 \text{ keV} \approx 5.658 \times 10^{-13} \text{ V}$

$|\Delta U_{sphere}| = U_N - U_O$

$= \frac{3e^2 Z_N(Z_N-1)}{20\pi\varepsilon_0 R} - \frac{3e^2 Z_O(Z_O-1)}{20\pi\varepsilon_0 R}$

$= \frac{3e^2}{20\pi\varepsilon_0 R}\left[Z_n(Z_n-1) - Z_o(Z_o-1)\right]$

$\approx \frac{1.3805\times10^{-28}}{R}\left[7(6) - 8(7)\right]$

$\approx \frac{1.933\times10^{-27}}{R} = \Delta B.E.$

$\therefore R = \frac{1.933\times10^{-27}}{5.658\times10^{-13}} \approx 3.416\times10^{-15} \text{ m}$

$R_{theoretical} = r_0 A^{1/3} = (1.2 \text{fm})(15)^{1/3}$

$= (1.2\times10^{-15}\text{m})(15)^{1/3}$

$\approx 2.96\times10^{-15}\text{ m}$

$\approx 2.96 \text{ fm}$

$|\Delta R| = |3.416 - 2.96|\text{ fm} = 0.456 \text{ fm}$

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Michael Ickes Exercise 5.3

Problem 5.5

Taking derivative)

Part 1)

$$B = (a_v - a_A n^2)A - a_s A^{2/3} - a_c\left(\frac{z^2}{A^{1/3}}\right) \quad n = 1 - \frac{2z}{A}$$

$$n^2 = \frac{4z^2}{A^2} - \frac{4z}{A} + 1$$

$$= a_v A - A a_A\left(\frac{4z^2}{A^2} - \frac{4z}{A} + 1\right) - a_s A^{2/3} - a_c\left(\frac{z^2}{A^{1/3}}\right)$$

$$= a_v A - \frac{4 A a_A z^2}{A^2} + \frac{4 A a_A z}{A} - A a_A - a_s A^{2/3} - a_c\left(z^2/A^{1/3}\right)$$

$$= (a_v A)\frac{d}{dz} - \left(\frac{4 a_A z^2}{A}\right)\frac{d}{dz} + (4 a_A z)\frac{d}{dz} - (A a_A)\frac{d}{dz} - (a_s A^{2/3})\frac{d}{dz} - \left(a_c \frac{z^2}{A^{1/3}}\right)\frac{d}{dz}$$

$$0 = -\frac{8 a_A z}{A} + 4 a_A - \frac{2 a_c z}{A^{1/3}}$$

$$a = \frac{8 a_A}{A} \quad b = 4 a_A \quad c = \frac{2 a_c}{A^{1/3}}$$

$$0 = -az + b - cz$$

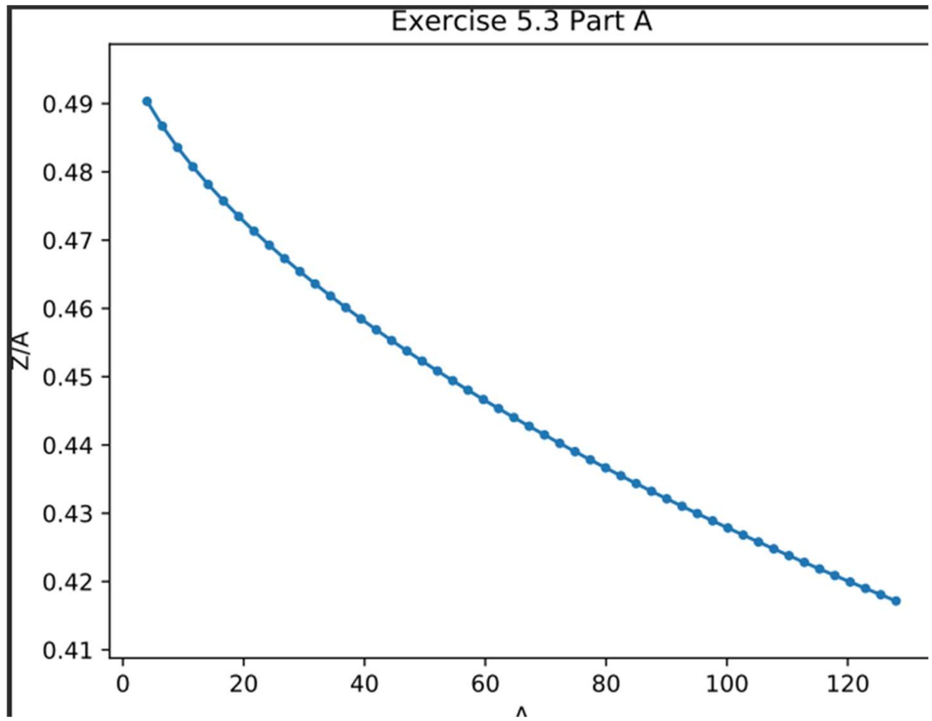
$$z(a+c) = b$$

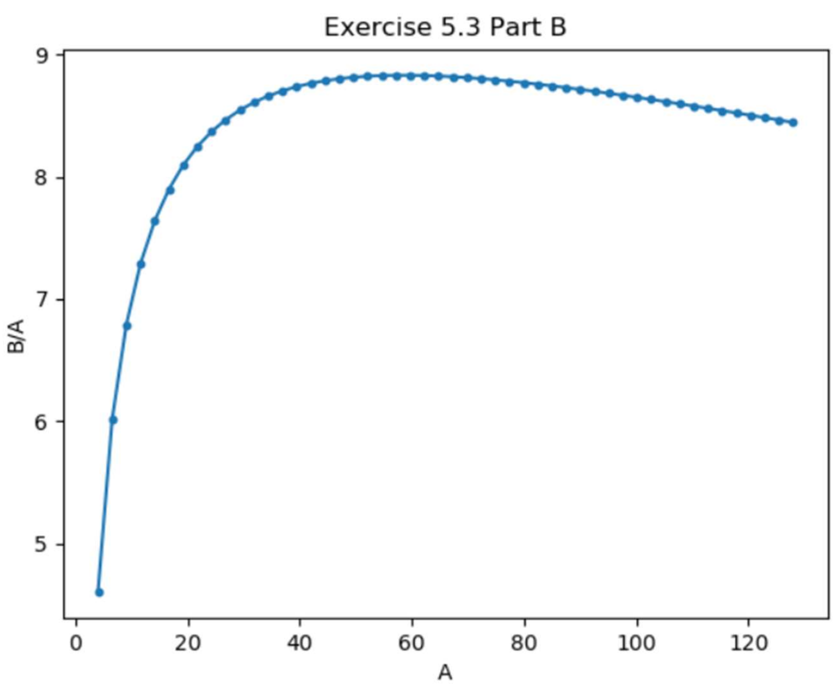
$$z = b/a+c \qquad z = 4 a_A\left(\frac{8 a_A}{A} + \frac{2 a_c}{A^{1/3}}\right)^{-1}$$

Code)

```
Set as interpreter
#!/usr/bin/python

import math
import numpy as np
import matplotlib.pyplot as plt

print("Exercise 5.3 from TBS")

av = 15.5
aA = 22.7
aS = 16.6
ac = 0.71


def Exercise5_3_PartA():
    A = np.linspace(4,128)
    Z = 4*aA*((((8*aA)/A)+((2*ac)/A**(1/3)))**(-1))
    print(Z)
    plt.scatter(A,Z/A, s=10)
    plt.plot(A,Z/A)
    plt.xlabel('A')
    plt.ylabel('Z/A')
    plt.title('Exercise 5.3 Part A')
    plt.show()
    return

def Exercise5_3_PartB():
    A = np.linspace(4,128)
    Z = Z = 4*aA*((((8*aA)/A)+((2*ac)/A**(1/3)))**(-1))
    B = A*(av-aA*(1-(2*Z/A))**2) - aS*(A**(2/3))-ac*(Z**(2)/A**(1/3))
    print(B)
    plt.scatter(A, B/A, s=10)
    plt.plot(A, B/A)
    plt.xlabel('A')
    plt.ylabel('B/A')
    plt.title('Exercise 5.3 Part B')
    plt.show()
    return

Exercise5_3_PartA()
Exercise5_3_PartB()
```

Graphs)





Part c) Solving empirically, we can see that the maximum occurs around when A=70

Problem 5.6

Anthony D'Alessandro

ASTR 4201

Homework 5

6. There's a maximum in the Binding energy per nucleon (found in TBS 5.3 and shown in the Quick Notes). For measured masses, this is at 58Fe. Why isn't everything made of 58Fe?

This is because, although Fe58 would be the most stable element, it has not been subjected to the required energy it needs to fuse to a mass number this high. All matter would need to be given an immense amount of energy for this to happen and the universe is not nearly that uniform in energy that high. Also, it is to be noted that even if there was sufficient energy in the universe, there would still me a distribution of elements around Fe58.

Problem 5.7

Ch. 5 Q 7

$^{12}C \rightarrow {}^{60}Fe$ $\quad 5\,{}^{12}C \rightarrow {}^{60}Fe$

sum $(Z m_p + N m_n + e\, m_e - B/c^2) = E$

$5\,{}^{12}C$ $\quad 5\,(6 \cdot 938.3\ \mathrm{MeV}/c^2 + 6 \cdot 939.6 + 6 \cdot .511 - 92.162)$

$1\,{}^{60}Fe$ $\quad - (26 \cdot 938.3 + 34 \cdot 939.6 + 26 \cdot .511 - 525.351)$

$= 61.385\ \mathrm{MeV} / 5\,{}^{12}C$

$= 12.277\ \mathrm{MeV}/{}^{12}C$

$M_{WD} = 1.2_{\odot} = 2.39 \times 10^{30}$ kg

$M_{^{12}C} = 6 \cdot 938.3 + 6 \cdot 939.6 + 6 \cdot .511 - 92.162$

$= 11{,}178.304\ \mathrm{MeV}/c^2$

$= 2 \times 10^{-26}$ kg

$M_{WD}/M_{^{12}C} = 1.19 \times 10^{56}\ {}^{12}C$

Total $E = 1.466 \times 10^{57}$ MeV

$M_{WD} = 2.39 \times 10^{30}$ kg

$U_{g\ bind} = 3GM^2/5R$ $\quad R = 7000$ km

$U_b = 3.26 \times 10^{46}$ J

$= 2.033 \times 10^{56}$ MeV

E released over U_{bind},
Fusing ^{12}C to ^{60}Fe provides enough energy to supernova

Problem 5.8

1 Problem 5.4

Fusion of hydrogen into helium entails converting 4 hydrogen atoms (including the 4 electrons) into 1 helium atom (2 protons, 2 neutrons, 2 electrons) with B = 28:296 MeV. What is the heat evolved per hydrogen atom? Assume that the sun has been shining with its current luminosity over its life. What mass of hydrogen atoms would need to undergo fusion to supply this energy? How large is this mass relative to the total mass of the sun?

$4\ {}^1\text{H} \longrightarrow {}^4\text{He}$

Meaning 4 protons and electrons become a nucleus of 2 protons and 2 neutrons with 2 electrons. The change in mass is then:

$$\Delta M = 4M_P + 4M_e - (2M_P + 2M_N + 2M_e - BE(^4He) \tag{1}$$

or

$$\Delta M = 2M_P + 2M_e - 2M_N + BE(^4He) \tag{2}$$

From TBS (and a quick google search for the last three), we know the following:

Name	Symbol	Value	Units
Binding Energy of ^{4}He	BE(^{4}He)	28.296	MeV
Mass of the Proton	M_P	938.28	MeV
Mass of the Neutron	M_N	939.57	MeV
Mass of the Electron	M_e	0.5110	MeV
Luminosity of the Sun	$L_\odot$	3.86×10^{26}	J/s
Mass of the Sun	$M_\odot$	1.99×10^{30}	Kg
Lifetime of the Sun	$T_\odot$	1.42×10^{17}	sec
Mass of Hydrogen	M_H	1.67×10^{-27}	Kg
Mega Electron Volt	MeV	1.6×10^{-13}	J

Tab. 1

The heat evolved is the change in mass, or

$$\Delta M = 2(938.28MeV + 0.511MeV - 939.57MeV) + (28.296MeV) = 26.738MeV \tag{3}$$

The heat evolved per hydrogen would then be $\Delta M/4$, or 6.68 MeV.
To find the mass of hydrogen needed, M_n,

$$M_n = L_\odot \times T_\times \frac{M_H}{\Delta M(H) \times MeV} = 8.58 \times 10^{28} kg \tag{4}$$

Or about 4.3% of the mass of the sun, $1.99{\times}10^{30}$.

9.

① $^3He(^3He, 2p)^4He$

if $^3He + ^3He \rightarrow {}^4He + 2{}^1H$ cross-section is purely geometric

what is $S(E) = ?$ at $E = 100\,keV$

$$S(E) = \sigma(E) * E.$$

$$\sigma = \pi\left(\frac{\lambda}{2\pi}\right)^2 \sum_{l=0}^{l_{max}} (2l+1)T_l$$

$$R = (1.2)(A)^{1/3}$$

$T_l = 1$ for $l > l_{max}$

$$\mu = \frac{m_T * m_p}{m_T + m_p}$$

$$l_{max} = \frac{2\pi R}{\lambda} \quad \text{where } \lambda = \frac{h}{\sqrt{2\mu E}}$$

$$= \frac{2\pi\, 1.2 * (A_p^{1/3} + A_T^{1/3})}{h = 2\pi\hbar} \sqrt{2 * \frac{m_T m_p}{m_T + m_p} * \frac{931.5\,MeV}{c^2} * E_p}$$

$$= \frac{1.2 * (3^{1/2} + 3^{1/2})\,fm}{\hbar c = 197\,MeV\,fm} \sqrt{2\,\frac{9}{6} * 931.5 * 0.1}\;MeV$$

$$= 0.24 \simeq 0 \qquad l_{max} = 0$$

$$\sigma = \pi\left(\frac{\hbar c}{\sqrt{2 * \frac{9}{6}\, 931.5 * 0.1}}\right)^2 (2*0+1)*1$$

$$= \pi * \left(\frac{197\,MeV\,fm}{16.716}\right)^2 = 436.29\ fm^2$$

$$= 4.36\,b$$

$S(E) = 4.36 \quad * \, 0.1 \, MeV = 0.436 \, MeV*b$

② $^{2}H + {}^{1}H \longrightarrow {}^{3}He + \gamma$

$$l_{max} = \frac{\cancel{2\pi} * 1.2*((2)^{1/3} * (1)^{1/3})}{\cancel{2\pi}\hbar} \sqrt{2 * \frac{2}{3} * \frac{931.5}{c^2} \cdot 0.1 \, MeV}$$

$$= 0.135 \simeq 0$$

$$\sigma = \pi \left(\frac{197 \, \cancel{MeV} fm}{\sqrt{2 * \frac{2}{3} * 931.5 * 0.1 \, \cancel{MeV}}} \right)^2 * (2*0+1) * 1$$

$$= 1308.87 \, fm^2 = 13.09 \, b$$

$S(E) = 13.09 \, b * 0.1 \, MeV = 1.309 \, MeV \, b$

* for $^{3}He + {}^{3}He \longrightarrow {}^{3}He + 2\,{}^{1}H$

$S(E) = 0.436 \, MeV \qquad S(E)_{exp} = 5 \, MeV \, b$

for $^{2}H + {}^{1}H \longrightarrow {}^{3}He + \gamma$

$S(E) = 1.309 \, MeV \, b \qquad S(E)_{exp} = 1 \, MeV \, b$

Problem 5.10

TBS 5.5

first note:

have Coulomb repulsion

set KE = PE

then $K = \frac{Q^2}{4\pi\epsilon_0 R}$

rearrange:

$$r_{min} = \frac{k q Q}{K_i}$$

TBS 5.5

$$r_{min} = \frac{k q Q}{K_i}$$

$k = 9 * 10^9 \; Nm^2/C^2$

$q = 1e \Rightarrow e = 1.6 * 10^{-19} \; C$

$Q = q$

$K_i = 1 \; keV \Rightarrow 1 \; keV = 1.6 * 10^{-16} \; J$

$$r_{min} = \frac{(9 * 10^9)(1.6 * 10^{-19})^2}{(1.6 * 10^{-16})}$$

$$r_{min} = 1.44 * 10^{-12} \; m$$

$$\frac{(Nm^2/C^2)(C^2)}{J} = \frac{Nm^2C^2}{JC^2} = \frac{Nm^2C^2}{NmC^2}$$

(checking units)

Problem 5.11

5 Homework 5

Question 11: EXERCISE 5.6 - Suppose we wish to approximate a function $f(x)$ at a point x_0 with a power-law, $p(x; A, n) = Ax^n$. Impose the condition $p(x_0; A, n) = f(x_0)$and $dp/dx|_{x=x_0} = df/dx|_{x=x_0}$ to find the parameters A and n, and show that

$$n = \frac{d \ln f}{d \ln x}. \tag{44}$$

Apply this to the reaction rate, eq. (5.8), and thus derive eq. (5.9).

Given the fact that $p(x) = f(x)$, and $p(x_0; A, n) = f(x_0)$ and $dp/dx|_{x=x_0} = df/dx|_{x=x_0}$, taking the derivative of $p(x)$ yields:

$$\frac{df}{dx} = Anx^{n-1} = \frac{n}{x}Ax^n = \frac{n}{x}f(x) \tag{45}$$

rearranging by moving the dx and $f(x)$,

$$\frac{df}{f} = n\frac{dx}{x} \tag{46}$$

and now knowing that $d \ln x/dx = 1/x$ and using the physics cheat of multiplying by dx yields:

$$\frac{d \ln x}{dx} = \frac{1}{x} \tag{47}$$

$$d \ln x = \frac{dx}{x} \tag{48}$$

and therefore the solution for n:

$$d \ln f = nd \ln x \tag{49}$$

$$n = \frac{d \ln f}{d \ln x} \tag{50}$$

solving for A:

$$p'(x) - p(x) = f'_{x=x_0} - f_{x=x_0} \tag{51}$$

$$Anx^{n-1} - Ax^n = f'_{x=x_0} - f_{x=x_0} \tag{52}$$

substituting $x = x_0$ yields the value for A:

$$A = \frac{f'_{x=x_0} - f_{x=x_0}}{nx_0^{n-1} - x_0^n} = \frac{f'_{x=x_0} - f_{x=x_0}}{x_0^n\left(\frac{n}{x_0} - 1\right)} \tag{53}$$

Applying the equation for n to eq. 5.8 (given below):

$$(5.8): \frac{2^{13/6}}{\sqrt{3m}} \frac{E_G^{1/6} S(E_{pk})}{(k_B T)^{2/3}} \exp\left[-3\left(\frac{E_G}{k_B T}\right)^{1/3}\right] \tag{54}$$

and recognizing that a constant C is present:

$$C = \frac{2^{13/6}}{\sqrt{3m}} \frac{E_G^{1/6} S(E_{pk})}{(k_B)^{2/3}} \tag{55}$$

$$f = CT^{-2/3} \exp\left[-3\left(\frac{E_G}{k_B T}\right)^{1/3}\right] \tag{56}$$

and the rate $r(T) \propto f/C$, then n can be calculated as:

$$\frac{\partial \ln r(T)}{\partial \ln T} = \frac{\partial \ln\left[T^{-2/3} \exp\left[-3\left(\frac{E_G}{k_B T}\right)^{1/3}\right]\right]}{\partial \ln T} \tag{57}$$

$$= T\left[-\frac{2}{3T} + -3\frac{\partial}{\partial T}\left(\frac{E_G}{K_B T}\right)^{1/3}\right] \tag{58}$$

$$= -\frac{2}{3} + T\left(\frac{E_G}{K_B T^4}\right)^{1/3} \tag{59}$$

$$= -\frac{2}{3} + \left(\frac{E_G}{K_B T}\right)^{1/3} \tag{60}$$

where the last equation is given as (5.9), or

$$(5.9): n = \frac{\partial \ln r}{\partial \ln T} = -\frac{2}{3} + \left(\frac{E_G}{K_B T}\right)^{1/3} \tag{61}$$

Problem 5.12

Samuel Fehringer

ASTR 4201 Homework 5.12

$^{137}Cs \rightarrow {}^{137m}Ba + e^-$

This does not work, the lepton number is not conserved.

^{137}Cs: 137 nucleons	lepton number is 55	charge is 0
$^{137}\ \ Ba + e^-$: 137 nucleons	lepton number is 57	charge is -1e

$^{12}C(\alpha, \gamma)^{16}O$

This does work, because the two needed electrons are provided by the environment

^{12}C and α: 16 nucleons	lepton number is 6	charge is +2e
^{16}O: 16 nucleons	lepton number is 8	charge is 0

$^{144}Sm(\gamma, \alpha)^{140}Nd$

This works

^{144}Sm: 144 nucleons	lepton number is 62	charge is +2e
^{140}Nd and α: 144 nucleons	lepton number is 62	charge is +2e

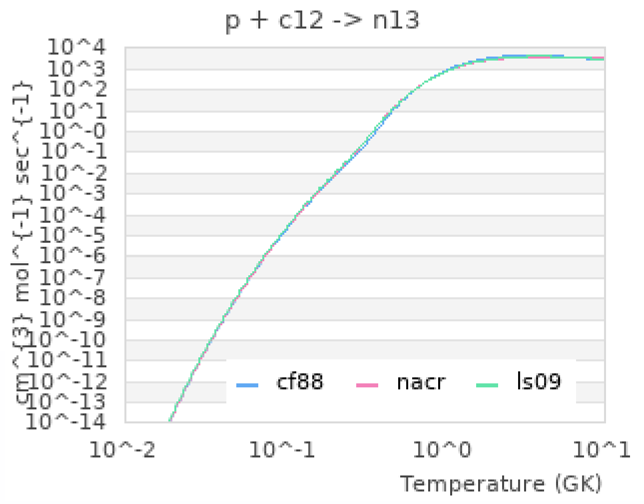
$^{126}Te(n, \gamma)^{127}I$

This does not work, because charge is not conserved.

^{126}Te and n: 127 nucleons	lepton number is 52	charge is -1
^{127}I: 127 nucleons	lepton number is 53	charge is 0

Problem 5.13

As energy increases the rates of proton capture increase at a steady rate as high energy protons are able to pass through the coulomb barrier easily.



Conversely the rate of weak beta remains nearly constant. This implies weak decays would limit the rates of the following steps, and data from https://reaclib.jinaweb.org/ implies that n13->c13 is more limiting than o15->n15, though the scales are slightly unclear.