Homework Assignment 4 **ASTR4201, Fall 2020**

Corresponds to Chapter 4 of "To Build a Star" (TBS) by E.F. Brown

1. *See below* Team: 1 Lead: Gavin
   Using semiclassical arguments and the proposal from Bohr that angular momentum is quantized ($L = n\hbar$), derive the -13.6eV Rydberg energy for the most-bound state of hydrogen that is listed in equation 4.1.
   Hint: The virial theorem is for more than just gravity.

2. *See below* Team: 2 Lead: Michael
   Figure 4.3 shows that there would be relatively wide gaps in the solar spectrum from absorption lines from various series. Calculate the wavelength at which the relatively wide gaps in the solar spectrum would appear for the Lyman, Balmer, and Paschen series. Do any of these gaps appear in Figure 4.1?

3. TBS exercise 4.1 Team: 3 Lead: Ryan

4. *See below* Team: 3 Lead: Josh
   Using semiclassical arguments and assuming angular momentum is quantized ($L = n\hbar$), derive the Bohr radius.

5. *See below* Team: 4 Lead: Gula
   Box 4.1 uses the estimate that the volume of all atoms has to be less than half of that available in the gas in order for atoms to not overlap. If we assume the atoms are spheres, is this reasonable? How much volume could the spheres take up?
   Hint: We are essentially talking about the atomic packing factor.

6. *See below* Team: 1 Lead: Anthony
   A nuclear form of the Saha equation exists, where we use nucleon density instead of electron density, nucleon mass instead of electron mass, and a nuclear reaction energy release as opposed to ionization energy release. Assuming the astrophysical *r*-process happens with a neutron density of 10^{20} neutrons/cm^3 and a temperature of 1 GK, calculate the neutron separation energy (Energy release from an (n,γ) reaction) that results in (n,γ)-(γ,n) equilibrium. Assume the neutron-capture parent and daughter have the same degeneracy.

7. *See below* Team: 2 Lead: Sam
   The hydrogen in the photosphere of the sun is mostly neutral. Would fluorine be similar or more ionized? Why?

8. *See below* Team: 2 Lead: Quinn
   If an electron is confined to a volume defined by its de Broglie wavelength and is non-relativistic, what is the number density in terms of the temperature? Do you recognize this quantity?

9. *See below* Team: 5 Lead: Justin
   The term in equation 4.6 that is raised to the 3/2 power is known as the quantum concentration. When an environment approaches this density, the matter is degenerate (i.e. Boltzmann statistics are no longer good). Compare this number density for electrons and nucleons at 1 GK. Then, convert to mass density, assuming that 1 g/mol is close enough, and compare to the average density of a white dwarf and of a neutron star.

10. TBS exercise 4.2 Team: 4 Lead: Jacob
    Hint: Consider the quantity $(1-x)^2/x$

11. *See below* Team: 3 Lead: Harshil
    Verify that the general solution for the driven harmonic oscillator (equation at the top of Box 4.2 on page 51) is correct.

12. *See below* Team: 1 Lead: Brit
    Show that $(\omega_0^2 - \omega^2) \approx 2\omega_0(\omega_0 - \omega)$ for $\omega \approx \omega_0$.

13. *See below* Team: 5 Lead: Robert
    How much narrower would you expect a spectral line in an A1 supergiant star to be relative to the same spectral line in an A1 main sequence star? Compare to the inset of Figure 4.7 and comment on the possible causes for any discrepancies. Assume the contribution to the line width from pressure depends linearly on the pressure and assume the A1 supergiant has the same mass as the A1 main sequence star.

Problem 4.1

Exercise 4.1

Angular momentum $L = n\hbar$ (quantized), $L = mvr$

$E = K + \Omega$, Virial theorem $2\langle K\rangle = -\langle \Omega\rangle \rightarrow \langle\Omega\rangle = -2\langle K\rangle$

$E = K + (-2K)$

$E = -K$

$K = \frac{1}{2}m_e v^2$

$L = n\hbar = mvr \rightarrow v = \frac{n\hbar}{m_e r}$

$K = \frac{1}{2}m_e\left(\frac{n\hbar}{m_e r}\right)^2$

$\hookrightarrow \frac{1}{2}\cancel{m_e}\left(\frac{n^2\hbar^2}{m_e^{\cancel{2}} r^2}\right)$

$K = \frac{1}{2}\frac{n^2\hbar^2}{m_e r^2} \longrightarrow r = \frac{n^2\hbar^2}{m_e k e^2}$ (Bohr radius)

$K = \frac{1}{2}\frac{n^2\hbar^2}{m_e\left(\frac{n^2\hbar^2}{m_e k e^2}\right)^2} \rightarrow K = \frac{1}{2}\frac{n^2\hbar^2}{m_e\left(\frac{n^4\hbar^4}{m_e^2 k^2 e^4}\right)} = \frac{1}{2}\frac{n^2\hbar^2}{\left(\frac{n^4\hbar^4}{m_e k^2 e^4}\right)}$

$K = \frac{1}{2}\left(\frac{\cancel{n^2}\cancel{\hbar^2}}{1}\cdot\frac{m_e k^2 e^4}{n^{\cancel{4}2}\hbar^{\cancel{4}2}}\right)$

$K = \frac{1}{2}\left(\frac{m_e k^2 e^4}{n^2\hbar^2}\right)$

$E = -K = -\frac{m_e k^2 e^4}{2\hbar^2}\times\left(\frac{1}{n^2}\right)$

multiply numerator and denominator by $\frac{c^2}{c^2}$

$E = \left(-\frac{1}{2}\right)\left(\frac{m_e c^2 k^2 e^4}{\hbar^2 c^2}\right)\left(\frac{1}{n^2}\right)$: $\frac{e^2}{\hbar c}$ = fine structure constant

$E = \left(-\frac{1}{2}\right)(m_e c^2)(k^2)\left(\frac{e^2}{\hbar c}\right)^2\left(\frac{1}{n^2}\right)$: most bound state of hydrogen at $n=1$

$E_1 = \left(-\frac{1}{2}\right)(m_e c^2)(k^2)\left(\frac{e^2}{\hbar c}\right)^2$ $\qquad \frac{e^2}{\hbar c} = \frac{1}{137} = 0.00729$

$E_1 = \left(-\frac{1}{2}\right)(m_e c^2)(k^2)(0.00729)^2$ $\qquad m_e c^2 = 0.511\ \mathrm{MeV}/c^2 = 511000\ \mathrm{eV}$

$E_1 = \left(-\frac{1}{2}\right)(511000\ \mathrm{eV})(k^2)(5.314\times10^{-5})$

$E_1 = (-13.6\ \mathrm{eV})(\cancel{k^2})\times\left(\frac{1}{n^2}\right)$

what happens to this k^2?

$\hookrightarrow$ k is in SI, using the fine structure constant means k won't be there.

$$\lambda_{m\to n} = \lambda_0\left(\frac{1}{m^2} - \frac{1}{n^2}\right)^{-1} \quad \lambda_0 = 91.2\,nm \;;\; n > m$$

in our case $n = \infty$ because of how close the absorption lines are to each other

Lyman $(1 \to n)$

$$\lambda_{1\to n} = (91.2 \times 10^{-9})\left(\frac{1}{1^2} - \frac{1}{\infty^2}\right)^{-1}$$
$$= (91.2 \times 10^{-9})\left(\frac{1}{1}\right)^{-1}$$
$$= 912 \times 10^{-10}\,m$$ ← outside of visible light

Balmer $(2 \to n)$

$$\lambda_{2\to n} = (91.2 \times 10^{-9})\left(\frac{1}{2^2} - \frac{1}{\infty^2}\right)^{-1}$$
$$= (91.2 \times 10^{-9})\left(\frac{1}{4} - 0\right)^{-1}$$
$$= 365\,nm$$ ← outside of visible light

absorption lines do not appear in Figure 4.1 because they are outside the visible light spectrum.

Paschen $(3 \to n)$

$$\lambda_{3\to n} = (91.2 \times 10^{-9})\left(\frac{1}{3^2}\right)^{-1}$$
$$= (91.2 \times 10^{-9})\left(\frac{1}{9}\right)^{-1}$$
$$= 821\,nm$$ ← outside of visible light

Problem 4.4

ASTR 4201 HW #4

Josh Olson

$F_c = ma_c$

$$\frac{e^2}{4\pi\epsilon_0 r^2} = \frac{\mu v^2}{r} \quad \text{where } \mu = \frac{m_p m_e}{m_p + m_e}$$

$$L = n\hbar = \mu r v \rightarrow v = \frac{L}{\mu r}$$

$$\frac{e^2}{4\pi\epsilon_0 \cancel{r^2}} = \frac{\cancel{\mu} L^2}{\mu^{\cancel{2}} r^{\cancel{3}}}$$

$$\frac{e^2}{4\pi\epsilon_0} = \frac{L^2}{\mu r}$$

$$r = \frac{4\pi\epsilon_0 L^2}{\mu e^2} \rightarrow L = n\hbar$$

Bohr radius is represented with a.

$$a = \frac{4\pi\epsilon_0 n^2\hbar^2}{\mu e^2} = 5.29 \times 10^{-11} \text{ m}$$

Problem

Atomic packing factor = APF = $\frac{N_{atom} V_{atom}}{V_{unit cell}}$

$N_{atom} = 1$ we have one atom insida Unitcell

$V_{atom} = \frac{4}{3} \pi r^3$

$V_{unit cell} = a^3$ where $a = 2r$

$$\frac{1 \cdot \frac{\cancel{4}}{3} \pi \cancel{r^3}}{\underset{2}{\cancel{8}} \cancel{r^3}} = \frac{\pi}{6} \simeq 0.5236$$

A sphere that just fits into a cubic box takes up only about 50% of the Volume of the box

yes, it is resnoble to assume the atoms are sphere.

~~To find the Volume of the sphere~~

~~The~~ kipler conjecture says that close packing (cubic or hexagonal, which have equivalent packing densities) is the densest possible

The Volume of the sphere = 74% of the Volume of the box.

Anthony D'Alessandro

HW 4

Problem 4.6

6.

$$\frac{N_{i+1}}{N_i} \simeq 2\frac{1}{n}\left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2} \frac{g_{i+1,gs}}{g_{i,gs}} \exp\left(\frac{-E_{ion}}{k_B T}\right)$$

$$\frac{N_{i+1}}{N_i} \to 1 \qquad \frac{g_{i+1,gs}}{g_{i,gs}} \to 1$$

$$1 = 2\frac{1}{n}\left(\frac{m k_B T}{2\pi\hbar}\right)^{3/2} \exp\left(\frac{-E_{ion}}{k_B T}\right)$$

$$\frac{n}{2} = \left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2} \exp\left(\frac{-E_{ion}}{k_B T}\right)$$

$$\frac{\frac{n}{2}}{\left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2}} = \exp\left(\frac{-E_{ion}}{k_B T}\right) \to k_B T \cdot \ln\left(\frac{\frac{n}{2}}{\left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2}}\right) = -E_{ion}$$

$$(1.38\times10^{-23})(10^9)\cdot \ln\left(\frac{5\times10^{25}}{(3.31\times10^{26})^{3/2}}\right) = E_{ion}$$

$m = 1.67\times10^{-27}$ kg

$T = 1\times10^9$ K

$\hbar^2 = 1.11\times10^{-68}\ \frac{m^4 kg^2}{s^2}$

$n = 10^{26}$ neutrons/m^3

$k_B = 1.38\times10^{-23}$ J/K

4.47×10^{-13} J

$\to \boxed{2.79\,MeV}$

Problem 4.7

Samuel Fehringer 8 October 2020

The hydrogen in the photosphere of the sun is mostly neutral. Would fluorine be similar or more ionized? Why?

Fluorine would have a similar if not lower rate of ionization, as it has a higher ionization energy of 17.4 eV than Hydrogen's 13.6 eV. With photons in the Sun's photosphere typically having a wavelength in the green part of the visible spectrum at about 500 nm, Hydrogen's ionization energy corresponds to much shorter wavelength of about 91 nm. Fluorine would require an even shorter wavelength.

Short Answer: no because you would need more energy

Problem 4.8

8.) De Broglie:

$\lambda = \frac{h}{np}$

number density

$n = \frac{1}{\left(\frac{h}{p}\right)^3}$

$\Rightarrow n = \frac{1}{\left(\frac{h}{\sqrt{3mKT}}\right)^3}$

$n = \frac{(3mKT)^{3/2}}{h^3}$

$KE = \frac{3}{2}KT = \frac{1}{2}mv^2 = \frac{p^2}{2m}$

$\frac{3}{2}KT = \frac{p^2}{2m}$

$3mKT = p^2$

$\boxed{p = \sqrt{3mKT}}$

m is the mass of an electron.

This is similar to the quantum concentration in the Saha Equation.

4 Homework 4

Question 9: The term in equation 4.6 that is raised to the 3/2 power is known as the quantum concentration. When an environment approaches this density, the matter is degenerate (i.e. Boltzmann statistics are no longer good). Compare this number density for electrons and nucleons at 1 GK. Then, convert to mass density, assuming that 1 g/mol is close enough, and compare to the average density of a white dwarf and of a neutron star.

Given

$$\frac{N_{t+1}}{N_t} = \frac{2}{n_e}\left[\frac{m_e k_B T}{2\pi\hbar^2}\right]^{\frac{3}{2}} \frac{Q_{t+1}}{Q_t} \tag{33}$$

the quantum concentration is is:

$$\left[\frac{m_e k_B T}{2\pi\hbar}\right]^{\frac{3}{2}} \tag{34}$$

to simplify the equation multiplying it by 1 allows for a substitution:

$$1\left[\frac{mk_B T}{2\pi\hbar^2}\right]^{\frac{3}{2}} = \left[\frac{mk_B T}{2\pi\hbar^2}\frac{c^2}{c^2}\right]^{\frac{3}{2}} = \left[\frac{mc^2 k_B T}{2\pi\left(\hbar c\right)^2}\right]^{\frac{3}{2}} \tag{35}$$

with $\hbar c = 197\,\text{MeV fm}$ and mc^2 being the mass of the particle in question. Now, utilizing $m_e = 0.511 MeV/c^2$ for the electron and $m_N = 931.494 MeV/c^2$ for the nucleon and substituting into the above equation:

$$e^- : 7.67 \times 10^{-11}\ \text{fm}^3 \tag{36}$$
$$N : 5.97 \times 10^{-6}\ \text{fm}^3 \tag{37}$$

Converting to mass density, ρ, utilizing the molar density $\mu = 1$ g/mol:

$$\rho = \frac{n_p \mu}{N_A} \tag{38}$$
$$\rho_{e^-} = 1.27 \times 10^{-34}\ \text{g/fm}^3 = 1.27 \times 10^5\ \text{g/cm}^3 \tag{39}$$
$$\rho_N = 9.92 \times 10^{-30}\ \text{g/fm}^3 = 9.92 \times 10^9\ \text{g/cm}^3 \tag{40}$$

Comparing to the approximate mass density of white dwarfs and neutron stars:

$$\rho = \frac{m}{V} = \frac{m}{\frac{4\pi}{3}R^3} \tag{41}$$

$$\rho_{WD} = \frac{M_\odot}{\frac{4\pi}{3}R_\oplus^3} = 1.85 \times 10^6 \text{ g/cm}^3 \tag{42}$$

$$\rho_{NS} = \frac{M_\odot}{\frac{4\pi}{3}\left(10\text{ km}\right)^3} = 477.46 \times 10^{12} \text{ g/cm}^3 \tag{43}$$

Quantum concentration relates the point where Boltzmann statistics fail and Fermi-Dirac statistics are necessary to study a system. If the density of an object exceeds the quantum concentration then it is a degenerate object. In this case, the estimated density for a white dwarf star exceeds the quantum concentration for electron degeneracy, so it is electron degenerate. The neutron star exceeds the quantum concentration for neutron degeneracy, so it is neutron degenerate.

$n_I \rightarrow$ H_I neutral H (p & bound e^-)
$n_{II} \rightarrow$ H_{II} ionized H (p & free e^-)
density

Problem 4.10

4.2) $X = \frac{n_I}{n_I + n_{II}}$ $\quad 1 - X = \frac{n_{II}}{n_I + n_{II}}$

$n_I + n_{II} = 10^{15}\,cm^{-3} = n_T$

$n_e = n_{II} = n_T - n_I$

$n_e = n_T(1-x)$

$x = n_I / n_T \Rightarrow n_T x = n_I$

$$\frac{n_{II}}{n_I} = \frac{N_{II}}{N_I}\frac{V}{V} = \frac{N_{II}}{N_I} = \left[\frac{2}{n_e}\left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^{3/2}\right]\frac{Q_{II}}{Q_I}$$

$\beta = (k_B T)^{-1}$

$$\frac{Q_{II}}{Q_I} = \frac{g_{II,1}}{g_{I,1}}\frac{e^{-\beta E_{II,1}}}{e^{-\beta E_{I,1}}} = \frac{g_{II,1}}{g_{I,1}} e^{-\beta(E_{II,1} - E_{I,1})}$$

where $E_{II,1} - E_{I,1}$ is the energy required to ionize the electron in H_I from its ground state, $n=1$.

$E_{ion} = E_{II,1} - E_{I,1} = 13.6\,eV$

$\rightarrow (n=1)$

$g_{II,1} = 2 \times 2 \times n^2 = 4$

$g_{I,1} = 2 \times 2 \times n^2 = 4$

$$\frac{n_{II}}{n_I} = e^{-E_{ion}/k_B T}\frac{2}{n_e}\left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^{3/2}$$

$$= e^{-E_{ion}/k_B T}\frac{2}{n_T(1-x)}\left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^{3/2}$$

$$\frac{n_{II}}{n_I}(1-x) = f(T) = e^{-E_{ion}/k_B T}\frac{2}{n_T}\left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^{3/2}$$

$$\frac{(1-x)^2}{x} = \frac{n_{II}^2 \cancel{n_T}}{n_T^{\cancel{2}} n_I} = \frac{n_{II}}{n_I}\frac{n_{II}}{n_T} = \frac{n_{II}}{n_I}(1-x) = f(T)$$

See Saha Equation Quick Notes for plot

$$\frac{(1-x)^2}{x} = f(T)$$

$$x^2 - 2x + 1 = x f(T)$$

$$x^2 - x(2 + f(T)) + 1 = 0 \Rightarrow \boxed{x = \frac{(2 + f(T)) \pm \sqrt{(2+f(T))^2 - 4}}{2}}$$

$x = \frac{n_I}{n_I + n_{II}}$ is never greater than 1, $\Rightarrow$ $\boxed{x = \left(2 + f(T) - \sqrt{(2+f(T))^2 - 4}\right)/2}$

$$\frac{n_2}{n_1} = \frac{n_{I,2}}{n_{I,1}} = \frac{g_2}{g_1} e^{-(E_2 - E_1)/k_B T} \Rightarrow \boxed{x\frac{n_2}{n_1} = 2\left(2 + f(T) - \sqrt{(2+f(T))^2 - 4}\right)e^{-\frac{3E_{ion}}{4k_B T}}}$$

$\frac{g_2}{g_1} = \frac{2\times2\times2^2}{2\times2\times1^2} = 4$ $\quad E_n = -13.6\left(\frac{1}{n^2}\right)$ $\Rightarrow$

$E_2 - E_1 = -13.6\left(\frac{1}{2^2} - \frac{1}{1^2}\right)$

$= +\frac{3}{4}(13.6)$

Problem 4.11

11. $x(t) = \frac{F/m}{(\omega_0^2 - \omega^2)} \cos(\omega t) + A\cos(\omega_0 t) + B\sin(\omega_0 t)$

$x'(t) = -\frac{F/m}{(\omega_0^2 - \omega^2)} \sin(\omega t)\omega - A\omega_0 \sin(\omega_0 t) + B\omega_0 \cos(\omega_0 t)$

$x''(t) = -\frac{F/m}{(\omega_0^2 - \omega^2)} \cos(\omega t)\omega^2 - A\omega_0^2 \cos(\omega_0 t) - B\omega_0^2 \sin(\omega_0 t)$

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F}{m}\cos(\omega t)$$

$-\frac{F/m\ \omega^2}{(\omega_0^2-\omega^2)} \cos(\omega t) - \cancel{A\omega_0^2 \cos(\omega_0 t)} - \cancel{B\omega_0^2 \sin(\omega_0 t)}$

$+\frac{F/m}{(\omega_0^2-\omega^2)} \cos(\omega t)\omega_0^2 + \cancel{A\omega_0^2 \cos(\omega_0 t)} + \cancel{B\omega_0^2 (\omega_0 t)}$

$= \frac{F}{m}\cos(\omega t)\left[\frac{\omega_0^2}{(\omega_0^2-\omega^2)} - \frac{\omega^2}{(\omega_0^2-\omega^2)}\right]$

$= \frac{F}{m}\cos(\omega t)\left[\frac{\omega_0^2 - \omega^2}{\omega_0^2-\omega^2}\right]$

$= \frac{F}{m}\cos(\omega t)$

LHS = RHS

Scanned with CamScanner

want to show $(\omega_0^2 - \omega^2) \approx 2\omega_0(\omega_0 - \omega)$ for $\omega \approx \omega_0$

Problem 4.12

Taylor formula: $f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(c)}{j!}(x-c)^j$

$f = \omega_0^2 - \omega^2$

at $\omega = \omega_0$ $\Rightarrow$

j	jth deriv.	at ω_0	$*(x-c)^j$
0	$\omega_0^2 - \omega^2$	0	0
1	-2ω	$-2\omega_0$	$-2\omega_0(\omega - \omega_0)$

so, $f(x) = 0 - 2\omega_0(\omega - \omega_0)$

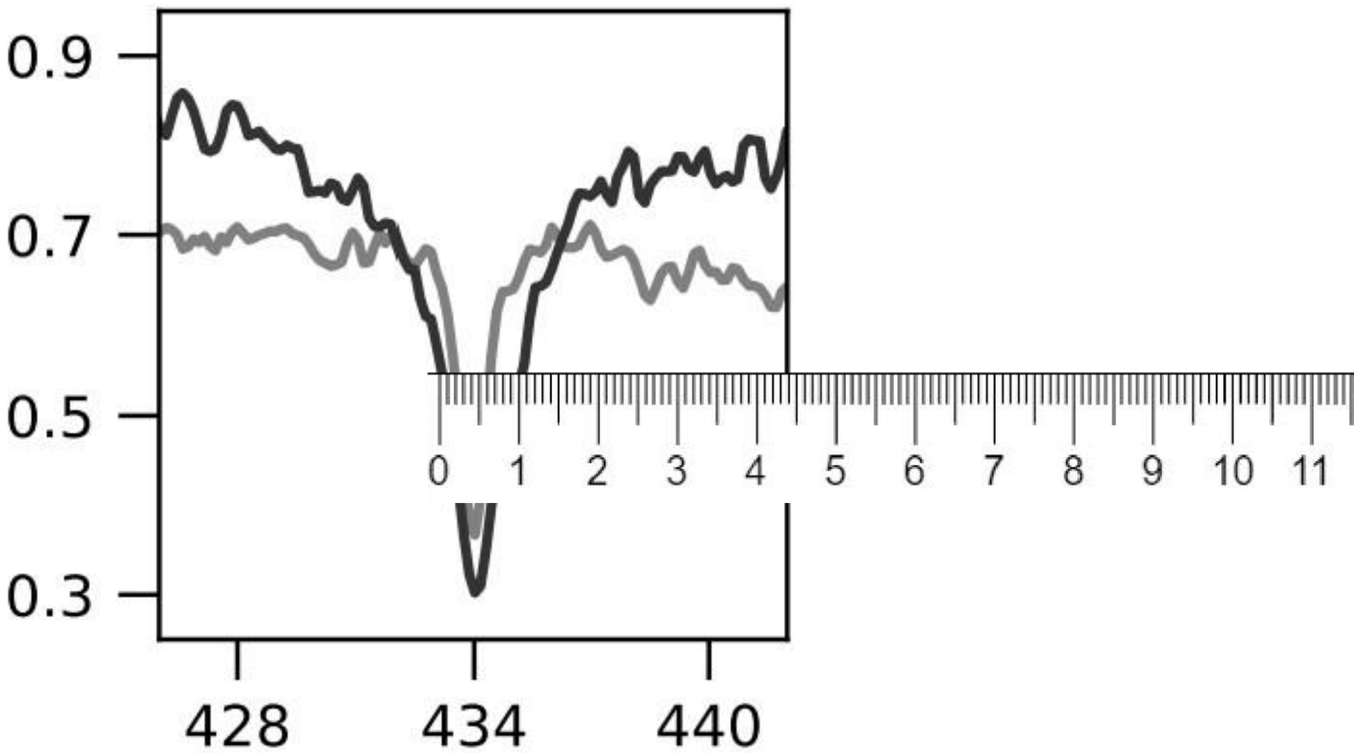
$= 2\omega_0(\omega_0 - \omega)$

so, $(\omega_0^2 - \omega^2) \approx 2\omega_0(\omega_0 - \omega)$ for $\omega \approx \omega_0$

as desired

Problem 4.13

- Supergiant stars have radii between 30 and 1000 times the sun, and masses between 8 to 12 times the sun.
- $P = (g/\kappa)\tau$
- "The surface gravity sets the pressure at the photosphere, the location where the optical depth is of order unity and where photons can escape from the star."
- $\tau \sim 1$, $g = GM/r^2$
- min P1/P2 = g1/g2 = 8/10^6, max P1/P2 = g1/g2 = 12/900,
- line width should be between 8*10^-4 and 1.3% larger for supergiant stars



- In reality, the spreading appears much larger due to factors unrelated to the surface gravity.