Homework Assignment 3 **ASTR4201, Fall 2020**

Corresponds to Chapter 3 of "To Build a Star" (TBS) by E.F. Brown

1. TBS exercise 3.1 Team: 1 Lead: Britt

2. TBS exercise 3.2 Team: 2 Lead: Sam

3. TBS exercise 3.3 Team: 3 Lead: Harshil
   Hint: Consult solutions for linear first order differential equations.

4. TBS exercise 3.4 Team: 4 Lead: Gula
   Hint: Consult the solution for common first order differential equations.

5. TBS exercise 3.5 Team: 1 Lead: Anthony

6. TBS exercise 3.6 Team: 2 Lead: Michael

7. TBS exercise 3.7 Team: 5 Lead: Justin
   Hint: Chapter 1 briefly discussed the radiative energy density, which is the total radiative energy divided by the volume.

8. TBS exercise 3.8 Team: 5 Lead: Robert

9. TBS exercise 3.9 Team: 3 Lead: Josh

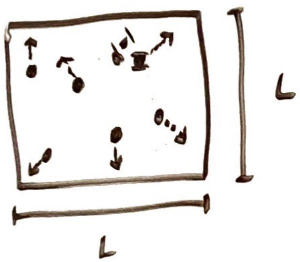
10. TBS exercise 3.10 Team: Lead: Gavin

11. TBS exercise 3.11 Team: 3 Lead: Ryan

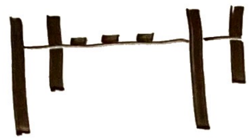
12. TBS exercise 3.12 Team: 2 Lead: Quinn
    Hints: For the first part, keep in mind that $P_0(\mu)=1$ and that any function is equal to itself multiplied by 1. For the second part, trigonometric identities and u-substitution are your friend.

13. TBS exercises 3.13-14 Team: 4 Lead: Jacob
    Hints: Consult Boxes 3.1 and 3.2.

[top view]



[side view]



puck diameter = 1

number of pucks = N

box length & width = L

estimate mean free path, ℓ

Problem 3.1

need to show how probability equation works for 2D:

$\Rightarrow$ have area coverage equivalent to ratio of $\frac{\text{puck diameter}}{\text{length traveled}}$

$\Rightarrow$ $P = \frac{\text{[obstacle diameter]}}{\text{[puck diameter]}}$

$\Rightarrow$ [obstacle diameter] = [areal density of pucks] * [area covered by puck] * [puck diameter]

$\Rightarrow$ [area covered by puck] = [puck diameter] * [length traveled]

$\Rightarrow$ plugging in : $P = \frac{(N/L^2) * (d\,\aleph) * d}{d}$

(I'm calling [length traveled], $\aleph$, because I find using ℓ for two different things confusing)

$\rightarrow P = \left(\frac{N}{L^2}\right) * (d\,\aleph)$

$\Rightarrow$ for $P = 1$, $\aleph$ = mean free path = ℓ

$\Rightarrow \frac{L^2}{Nd} = \ell$

$\Rightarrow d = 1$, so, $\ell = \frac{L^2}{N}$

Problem 3.2

Samuel Fehringer 9/21/2020
ASTR 4201

3.2 In the Sun, free electrons scatter photons.
The cross-section for this is...

$$\sigma_{Th} = \left(\frac{8\pi}{3}\right)\left(\frac{e^2}{m_e c^2}\right)^2 = 6.65 \times 10^{-29}\ m^2$$

What is the mean free path against this process for a photon at the average density of the solar interior?

$\mu_\odot = 0.617$ $M_\odot = 1.198 \times 10^{57}$ amu $V_\odot = 1.4 \times 10^{27}\ m^3$

$n_\odot = 1.387 \times 10^{30}$ particles/m3

mean free path $l = \frac{1}{n\sigma} = 0.0108\ m$

Problem 3.3

3.3 $\Delta I_\nu = -n\sigma I_\nu \Delta s$ — n = density of obstacles

$dI_\nu = -n\sigma I_\nu \, ds$ — σ = cross-sectional area of particles

$$\int_{I_0}^{I_\nu} \frac{dI_\nu}{I_\nu} = -n\sigma \int_0^s ds$$

$$\ln\left(\frac{I_\nu}{I_0}\right) = -n\sigma(s) \qquad (n\sigma = \kappa_\nu^{abs}\rho)$$

$$\ln\left(\frac{I_\nu}{I_0}\right) = -\kappa_\nu^{abs}\rho(s)$$

$$\boxed{I_\nu = I_0 e^{(-\kappa_\nu^{abs}\rho(s))}}$$

$$I_\nu = \frac{1}{e} \cdot I_0$$

$$\frac{\cancel{I_0}}{e} = \cancel{I_0} e^{(-\kappa_\nu^{abs}\rho(s))}$$

$$e^{-1} = e^{(-\kappa_\nu^{abs}\rho(s))}$$

(taking ln)

$$-1 = -\kappa_\nu^{abs}\rho(s)$$

$$\boxed{s = \frac{1}{\kappa_\nu^{abs}\rho}}$$

Scanned with CamScanner

Problem 3.4

3.4

$$\frac{dI_\nu}{ds} = \rho J_\nu - \rho k_\nu^{abs} I_\nu$$

$$dI_\nu' + \rho k_\nu^{abs} I_\nu = \rho J_\nu \quad \text{----} \ (1)$$

This is 1^{st} order differential equation

$$y' + p(x) y = q(x)$$

We can solve eq (1) as follow

$$I = e^{\int p(x) . dx}$$

In eq (1) $p(x) = \rho k_\nu^{abs}$

$$I = e^{\int \rho k_\nu^{abs} . ds}$$

$$I = e^{\rho k_\nu^{abs} . s} \quad \text{----} \ (2)$$

multiply both sides of equation (1) by eq (2)

$$e^{\rho k_\nu^{abs} . s} (I_\nu' + \rho k_\nu^{abs} I_\nu) = \rho J_\nu e^{\rho k_\nu^{abs} . s}$$

$$I_\nu' e^{\rho k_\nu^{abs} . s} + \rho k_\nu^{abs} I_\nu e^{\rho k_\nu^{abs} . s} = \rho$$

$$= \rho J_\nu e^{\rho k_\nu^{abs} . s} \quad \text{----} \ (3)$$

We can rewrite eq(3) as

$$\frac{d}{ds}\left(I_\nu e^{\rho k_\nu^{abs} \cdot s}\right) = \rho J_\nu e^{\rho k_\nu^{abs} \cdot s}$$

$$d\left(I_\nu e^{\rho k_\nu^{abs} \cdot s}\right) = \left(\rho J_\nu e^{\rho k_\nu^{abs} \cdot s}\right) ds \quad \ldots (4)$$

Integrating both sides of eq(4)

$$I_\nu e^{\rho k_\nu^{abs} \cdot s} = \int \rho J_\nu e^{\rho k_\nu^{abs} \cdot s} . ds$$

$$= \rho J_\nu \int e^{\rho k_\nu^{abs} \cdot s} . ds$$

now Let

$$u = \rho k_\nu^{abs} . s$$

$$du = \rho k_\nu^{abs} . ds \Rightarrow ds = \frac{1}{\rho k_\nu^{abs}} . du$$

$$\therefore I_\nu e^{\rho k_\nu^{abs} \cdot s} = \cancel{\rho} J_\nu \int \frac{1}{\cancel{\rho} k_\nu^{abs}} e^u . du$$

$$= \frac{J_\nu}{k_\nu^{abs}} e^u + C$$

$$= \frac{J_\nu}{k_\nu^{abs}} e^{\rho k_\nu^{abs} \cdot s} + C$$

where c = constant = I_0

$$I_\nu = \frac{J_\nu}{k_\nu^{abs}} + I_0 e^{-s k_\nu^{abs} \cdot s}$$

as $s \longrightarrow \infty$

$$I_\nu = \frac{J_\nu}{k_\nu^{abs}} + I_0 e^{-\infty}$$

$$= \frac{J_\nu}{k_\nu^{abs}}$$

Problem 3.5

HW3 Final

3.5

Using given mean free path: $l = .01m$

τ_ν = optical depth = $\int_0^s \frac{ds}{l}$

s = radius of sun = $6.957\times10^8 m$

$$= \frac{1}{l}\int_0^s ds$$

$$= \frac{1}{l}\left[s\right]_0^{6.957\times10^8}$$

$$= \frac{1}{.01m}\left[6.957\times10^8 m - 0\right] = \boxed{6.957\times10^{10} m = \tau_\nu}$$

should be unitless, so no m as a unit

Problem 3.6

inversely proportional

$$F_\nu = -\frac{4\pi}{3}\left[\frac{1}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial T}\right]\frac{dT}{dr}$$

max at second because larger at corresponding point on B_ν graph

Problem 3.7

3 Homework 3

Question 7: EXERCISE 3.7 — Let's dissect eq. (3.13) to see how it sets the luminosity.

1. To keep the algebra simple, assume that$F$ is constant throughout the star and that aT^4 is linear in r — that is, $aT^4 = aT_c^4(1 - r/R)$. Since F is constant, you can express it in terms of the luminosity at the surface L. Use this to transform eq. (3.13) into an expression for L in terms of R and T_c (along with ρ, κ_R, and c).

2. Write the luminosity as $L = E_\gamma/\tau$, where E_γ is the total radiative energy of the star, and τ some as-yet-undetermined diffusion timescale. Give an estimate of E_γ in terms of the mean temperature T and the radius R of the star.

3. Finally, assume that the photon mean free path $\ell = (\rho\kappa_R)^{-1}$ is constant. Substitute the results from parts 1 and 2 into equation (3.13). After simplifying, you should end up with a simple expression for τ in terms of c, R, and ℓ. For Thomson scattering, what is τ (express in years)?

1. Since F is a constant throughout the star and is the luminosity L over the area A,

$$F = \frac{L}{A} = \frac{L}{4\pi R^2} \tag{23}$$

$$F = -\frac{1}{3}\frac{c}{\rho\kappa_R}\frac{d}{dr}aT^4 \tag{24}$$

$$\tag{25}$$

and $aT^4 = aT_c^4\,(1 - r/R)$ giving

$$L = \frac{4\pi}{3}\frac{c}{\rho\kappa_R}aRT_c^4 \tag{26}$$

2. Solving for E_γ, and realizing that the radiative energy density U is

$$U = aT_c^4 = \frac{E_\gamma}{\frac{4\pi}{3}R^3} \tag{27}$$

and hence E_γ is

$$E_\gamma = \frac{4\pi}{3}aR^3T_c^4 \tag{28}$$

making the luminosity L

$$L = \frac{4\pi}{3\tau}aR^3T_c^4 \tag{29}$$

3. Setting the equations for the luminosity equal to each other,

$$\frac{4\pi}{3\tau} a R^3 T_c^4 = \frac{4\pi}{3} \frac{c}{\rho \kappa_R} a R T_c^4 \tag{30}$$

keeping in mind that $\ell = (\rho\kappa_R)^{-1}$ and canceling like terms,

$$\cancel{\frac{4\pi}{3}} \cancel{a T_c^4} \frac{R^{\cancel{3}2}}{\tau} = \cancel{\frac{4\pi}{3}} \cancel{a T_c^4} \cancel{R} \ell c \tag{31}$$

rearranging, then τ is

$$\tau = \frac{R^2}{\ell c} \tag{32}$$

substituting $R = R_\odot = 696.5 \times 10^6$ m for the Sun, with $\ell \approx 0.01$ m obtains $\tau \approx 5130$ y.

Problem 3.8

Comparing the two equations:

$$F = -\frac{1}{3}c\ell\frac{dU}{dr},$$ and $$F = -\frac{1}{3}\frac{c}{\rho\kappa_R}\frac{d}{dr}aT^4.$$

The substitution $\ell = \frac{1}{\rho\kappa_\nu}$ relates the mean free path to the opacity function. By changing from a function of temperature to the energy density, the mean Rossland mean opacity is exchanged for the frequency dependent version. The frequency dependence is absorbed into the mean using the black body spectrum as a weight function. This means the mean opacity depends on how the black body spectrum's shape changes across all frequencies.

$$\frac{1}{\kappa_R} = \left(\int_0^\infty \frac{\partial B_\nu}{\partial T}\, d\nu\right)^{-1} \int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B_\nu}{\partial T}\, d\nu.$$

" DARK
" 480.0s
1. Create filtered

ASTR 4201 HW #3 Problem 3.9

Josh Olso

$\Sigma \langle x_i \rangle = \langle \Sigma x_i \rangle$

1. $d = \langle \text{travel}^{m}_{\text{right}} - \text{travel}^{N-m}_{\text{left}} \rangle$

$= \langle \ell m - \ell(n-m) \rangle$

$= \ell \langle m \rangle - \ell \langle n-m \rangle$

$= \ell \langle m \rangle - \ell \langle n \rangle + \ell \langle m \rangle$

$= \ell(2\langle m \rangle - \langle n \rangle)$

$\boxed{d = \ell(2np - n)}$

2. $\langle d \rangle = \ell(2np - n)$

$d = R$, all steps are forward so p

$R = \ell(-n)$

$\boxed{n = R/\ell}$

3. $[\langle (m - \langle m \rangle)^2 \rangle]^{1/2} = [n_{edge} p (1-p)]^{1/2}$

$R/\ell = [n_{edge} p (1-p)]^{1/2}$

$R^2/\ell^2 = n_{edge} p (1-p)$

$\boxed{n_{edge} = \frac{R^2}{p \ell^2 (1-p)}}$

4. $d = n_{edge} \ell$

$= \frac{R^2}{p \ell^2 (1-p)} \ell$

$\boxed{d = \frac{R^2}{p \ell (1-p)}}$

$t = \frac{d}{c} = \boxed{\frac{R^2}{c p \ell (1-p)}}$

Problem 3.10

Question: Does matter with a gray opacity in thermal equilibrium also have a gray emissivity j_ν?

- Matter with a gray opacity would not also have a gray emissivity. To prove this, we look at equation (3.8) which shows $\frac{j_\nu}{k_\nu^{abs}} = B_\nu(T)$. Once we take the opacity to be gray, it will lose its frequency dependence, while the emissivity and blackbody will keep their dependence on frequency. Now that opacity has no dependence on frequency, we can show the relationship between the emissivity and absorption depth (opacity) by equation (3.21), showing $j_\nu = k^{abs} B_\nu$. This equation will show that just because the opacity is gray, does not mean that the emissivity will also be gray.

Problem 3.11

Exercise 3.11

$P_{ph} \approx g/k$

110% m_s and 110% r_s all else equal

$g = MG/r^2$

$g = 1.1\, M_s G/(1.1 R_s)^2$

$g = \frac{1}{1.1}\, M_s G/R_s^2$

$g = .91\, g_s$

$P_{ph} \approx .91\, g_s/k$

$\underline{P_{ph} \approx .91\, P_{ph}}$

Pressure at photosphere of larger star is about $\underline{.91}$ of the pressure at the Sun's

Problem 3.12

3.12.) $$\boxed{\int_{-1}^{1} P_n(\mu)\, P_m(\mu)\, d\mu = \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1} & m = n \end{cases}}$$ equation $[3.23]$

show that

$$\boxed{\frac{1}{4\pi}\int I\, d\Omega = I_0}$$ and $$\boxed{\int \mu I\, d\Omega = \left(\frac{4\pi}{3}\right) I_1 \quad \text{for } I = I_0 + I_1\mu}$$

theta goes from 0 to π.

$\frac{1}{4\pi}\int I\, d\Omega = I_0$ but $I = I_0 + I_1\mu$

$$\frac{1}{4\pi}\int I\, d\Omega = \frac{1}{4\pi}\iint (I_0 + I_1\mu)\sin\theta\, d\theta\, d\phi$$

rearrange integrals:

$$= \frac{1}{4\pi}\int_0^{\pi}(I_0 + I_1\mu)\sin\theta\, d\theta \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$= \frac{2\pi}{4\pi}\Big[\int_0^{\pi} I_0 \underbrace{P_0(\mu)\; P_0(\mu)}_{1} \sin\theta\, d\theta + \int_0^{\pi} I_1 \underbrace{P_0(\mu)\; P_1(\mu)}_{0 \text{ b/c of orthogonality}} \sin\theta\, d\theta\Big]$$

$$= \frac{1}{2}\left[\int_0^{\pi} I_0 \sin\theta\, d\theta\right]$$

$$= \frac{1}{2}\left[I_0[-\cos(x)]_0^{\pi}\right]$$

$$= \frac{1}{2}\left[I_0[2]\right]$$

$$\boxed{\frac{1}{4\pi}\int I\, d\Omega = I_0}$$

$$\int \mu I\, d\Omega = \int_0^{\pi} \mu\,(I_0 + I_1\mu)\sin\theta\, d\theta \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$= 2\pi\left[\int_0^{\pi} I_0\mu\sin\theta\, d\theta + \int_0^{\pi} I_1\mu^2\sin\theta\, d\theta\right]$$

$$= 2\pi\Big[\int_0^{\pi} I_0 \underbrace{P_1(\mu)\, P_0(\mu)}_{0} \sin\theta\, d\theta + \int_0^{\pi} I_1 \underbrace{P_1(\mu)\, P_1(\mu)}_{\frac{2}{2(1)+1} = \frac{2}{3}} \sin\theta\, d\theta$$

$$= 2\pi\left[I_1 \frac{2}{2(1)+1}\right]$$

$$= 2\pi I_1 \frac{2}{3}$$

$$\boxed{\int \mu I\, d\Omega = \frac{4\pi}{3} \cdot I_1}$$

Problem 3.13

3.13 $I_\nu(\mu,\tau) = I_{\nu,0}(\tau) + \mu I_{\nu,1}(\tau)$

$$J_\nu = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1} I_\nu \, d\mu \, d\varphi = \frac{1}{2}\int_{-1}^{1}\left(I_{\nu,0}(\tau) + \mu I_{\nu,1}(\tau)\right) d\mu$$

Note: $\int_{-1}^{1} \mu \, d\mu = \int_{-1}^{1} \mu^3 d\mu = 0$

$$\frac{1-(-1)^2}{2} = \frac{1-(-1)^4}{4} = 0$$

and: $\int_{-1}^{1} d\mu = 1-(-1) = 2$, $\int_{-1}^{1}\mu^2 d\mu = \frac{1}{3}\left(1-(-1)^3\right) = \frac{2}{3} = \frac{1}{3}\int_{-1}^{1} d\mu$

$$J_\nu = \frac{2}{2} I_{\nu,0}(\tau) + \frac{0}{2} \cdot I_{\nu,1}(\tau) = I_{\nu,0}(\tau)$$

$$K_\nu = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1}\mu^2\left(I_{\nu,0}(\tau) + \mu I_{\nu,1}(\tau)\right) d\mu \, d\varphi$$

$$= \frac{1}{2}\left[\int_{-1}^{1}\mu^2 I_{\nu,0}(\tau) d\mu + \underbrace{\int_{-1}^{1}\mu^3 I_{\nu,1}(\tau) \, d\mu}_{0}\right] = \frac{1}{3}\cdot\frac{2}{2} I_{\nu,0}(\tau) = \boxed{\frac{1}{3} J_\nu}$$

3.14) $\mu = \cos\theta$ $J_\nu = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1} I_\nu \, d\mu \, d\varphi$

$$H_\nu = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1} \mu I_\nu \, d\mu \, d\varphi$$

1) Isotropic $\Rightarrow$ $I = \text{const} = I_0 \Rightarrow H_\nu = \frac{1}{4\pi}\int_0^{2\pi}\cancel{\int_{-1}^{1} \mu I_0 \, d\mu} \, d\varphi$ $\rightarrow 0$

$$\int_{-1}^{1} \mu \, d\mu = \frac{1-(-1)^2}{2} = 0$$

$$\Rightarrow H_\nu = 0 \Rightarrow \frac{H_\nu}{J_\nu} = 0$$

2) $\mu = 1$, $I_\nu(\mu) = a_\nu \delta(\mu - 1)$

$$J_\nu = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1} a_\nu \delta($$

$$\int_{-1}^{1} f(\mu)\,\delta(\mu-1)\,d\mu = f(1)$$

$$f(\mu) = 1 \Rightarrow J_\mu = \frac{a_\nu}{2}$$

$$H_\nu = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1} \mu a_\nu \delta(\mu-1)\,d\mu\,d\varphi \qquad \Rightarrow \boxed{\frac{H_\nu}{J_\nu} = 1}$$

$$f(\mu) = \mu \Rightarrow f(1) = 1 \Rightarrow H_\nu = \frac{a_\nu}{2}$$

3) $U_\nu = \frac{4\pi}{c} J_\nu$; $F_\nu = 4\pi H_\nu$

Grey atmosphere $\Rightarrow$ $F = \int F_\nu \, d\nu = \text{constant}$; $\tau_0 = 2/3$

$$U(\tau) = \frac{3}{2} F (\tau + \tau_0)$$

$$\boxed{\frac{H_\nu}{J_\nu} = \frac{F_\nu}{c U_\nu} = \frac{1}{3(\tau + 2/3)}}$$

$$\boxed{\begin{aligned} \frac{H_\nu}{J_\nu}(\tau=0) &= \frac{1}{2} \\ \frac{H_\nu}{J_\nu}(\tau=2/3) &= \frac{1}{4} \\ \frac{H_\nu}{J_\nu}(\tau=10) &= \frac{1}{32} \end{aligned}}$$