ANNALS OF PHYSICS: 4, 166–179 (1958)

Velocity Dependence of the Two-Nucleon Interaction*

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Invariance arguments are used to derive the most general velocity dependent charge independent two-nucleon interaction in the nonrelativistic approximation. If one stays on the energy shell, the only essentially new term in the two-nucleon interaction is the quadratic spin-orbit potential. Off the energy shell, the quadratic spin-momentum potential must be treated as independent. In meson theory, the quadratic terms arise as $(\mu/M)^2$ (μ is the pion mass, M the nucleon mass) corrections to the second-order static potential in contrast to the linear spin-orbit potential which originates as a (μ/M) correction to the fourth-order static potential.

1. INTRODUCTION

Attempts to fit the unpolarized and polarized two-nucleon scattering data up to 150 Mev with an arbitrary charge-independent combination of static potentials (both central and tensor) have all failed (*1*). Recently it has been shown that the addition of a spin-orbit potential to the static potential (*2*) yields a good fit of all the unpolarized and polarized scattering data up to 150 Mev. While the addition of the spin-orbit potential improves considerably the fit with the two-nucleon scattering data at even higher energies (up to 300 Mev), it seems likely that a precision fit of the 300-Mev scattering will require additional terms in the two-nucleon interaction (containing higher powers of the nucleon momentum than the first).

While the combination of central plus tensor plus spin orbit potentials proposed by Gammel and Thaler (*2*) is completely phenomenological in character, the potential of Signell and Marshak (*2*) builds to some extent on meson theory. The latter potential was derived by adding a phenomenological short-range attractive spin-orbit potential to the meson-theoretic Gartenhaus potential (*3*). Even the spin-orbit potential proposed by Signell and Marshak has some slight status in meson theory in that an extension of the Gartenhaus method taking into account the nucleon recoil, leads to an attractive spin-orbit potential in fourth order.

As part of our program to derive a phenomenological two-nucleon interaction which will give a precision fit of all scattering data up to 300 Mev, we

* Assisted in part by the U. S. Atomic Energy Commission.

have investigated the most general form of a charge-independent potential between two nucleons in the nonrelativistic approximation. In Section 2, we extend the considerations of Eisenbud and Wigner (*4*) beyond the linear dependence on the momentum of the nucleon which they treated. We arrive at a somewhat more general velocity dependent two-nucleon potential than the one postulated by Puzikov *et al.* (*5*) in connection with their discussion of the general scattering matrix. In view of the fact that the pion theory of nuclear forces, if the S wave coupling is properly suppressed, gives correct qualitative predictions for the central, tensor and spin-orbit forces, we attempt in Section 3 to identify the meson-theoretic origin of the new velocity dependent terms in the potential which are derived in Section 2. Interestingly enough, it turns out that while the linear spin-orbit potential is a (μ/M) (μ is the pion mass, M the nucleon mass) correction to the static potential in fourth order, the chief new term (the quadratic spin-orbit potential) is a $(\mu/M)^2$ correction to the static potential in second order. Meson theory therefore predicts in lowest order a quadratic spin-orbit potential which is more shallow but of longer range than the linear spin-orbit potential.

2. GENERAL EXPRESSION FOR THE TWO-NUCLEON POTENTIAL

We inquire into the most general form of a nonrelativistic potential between two nucleons. This problem was treated many years ago by Eisenbud and Wigner (*4*) in the approximation in which all terms higher than linear in the momentum of the nucleons were neglected. We generalize their arguments to the general case.

First, let us assume charge independence of the two-nucleon interaction. Then, as far as the isotopic spin is concerned, the potential can be expressed as a linear combination of 1 and $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$, where the $\boldsymbol{\tau}$ are the isotopic spin operators. We therefore need not consider the isotopic-spin dependence further. Thus we can regard the potential as a function of the coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$, momenta $\mathbf{p}_1$ and $\mathbf{p}_2$, and spin operators $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$, where the suffices 1 and 2 refer to the two nucleons. This potential must satisfy the following conditions (*4*):

Condition I) Translation Invariance

This implies that $\mathbf{r}_1$ and $\mathbf{r}_2$ must enter in the potential V only through the combination $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$.

Condition II) Galilean Invariance

This requires that $\mathbf{p}_1$ and $\mathbf{p}_2$ must appear together in the form $\mathbf{p} = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2)$, namely V is independent of the total momentum $\mathbf{p}_1 + \mathbf{p}_2$.

Thus, we may write

$$V \equiv V(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathbf{r}, \mathbf{p}). \tag{1}$$

Condition III) Symmetry Condition

This implies that V is invariant when particles 1 and 2 are interchanged, i.e.,

$$V(\sigma_1, \sigma_2, \mathbf{r}, \mathbf{p}) = V(\sigma_2, \sigma_1, -\mathbf{r}, -\mathbf{p}). \tag{2}$$

Condition IV) Rotation Invariance

Condition V) Space Reflection Invariance

This requires that:

$$V(\sigma_1, \sigma_2, \mathbf{r}, \mathbf{p}) = V(\sigma_1, \sigma_2, -\mathbf{r}, -\mathbf{p}) \tag{3}$$

or, combining (2) and (3), we have

$$V(\sigma_1, \sigma_2, \mathbf{r}, \mathbf{p}) = V(\sigma_2, \sigma_1, \mathbf{r}, \mathbf{p}) \tag{4}$$

Condition VI) Time Reversal Invariance

This is equivalent to the requirement:

$$V(\sigma_1, \sigma_2, \mathbf{r}, \mathbf{p}) = V^T(-\sigma_1, -\sigma_2, \mathbf{r}, -\mathbf{p}). \tag{5}$$

The meaning of V^T is here defined somewhat differently from the conventional definition. V^T implies that the order of operators is reversed, i.e., if $A_1, A_2, \cdots, A_n$ represent either, σ, $\mathbf{p}$, $\mathbf{r}$ or $f(r^2)$ (an arbitrary function of r^2), then

$$(A_1 A_2 \cdots A_n)^T = A_n \cdot A_{n-1} \cdots A_2 \cdot A_1 .$$

Condition VII) Hermiticity

$$V = V^+. \tag{6}$$

Actually, Eq. (5) pre-supposes this Hermiticity condition.

Our problem is now to find the most general form for the potential V which satisfies conditions I–VII above.

Due to the properties of the spin operator σ, V can be expressed as a sum of a σ-independent term, a term linear in σ_1 or σ_2 and a term bilinear in σ_1 and σ_2.

The σ-independent term is quite simple. From condition (IV), namely rotation invariance, V must be a function of r^2, p^2, $(\mathbf{r} \times \mathbf{p})^2$, and $\mathbf{p}\cdot\mathbf{r} + \mathbf{r}\cdot\mathbf{p}$. However, $\mathbf{p}\cdot\mathbf{r} + \mathbf{r}\cdot\mathbf{p}$ must appear bilinearly, if we arrange the potential symmetrically with respect to right and left, because of the time-reversal condition (VI) [see Eq. (5)]. This symmetrization with respect to right and left is always possible, if we suppose that the potential V can be expanded into a power series in the momentum $\mathbf{p}$ and if we use the commutation relation between $\mathbf{r}$ and $\mathbf{p}$. Thus, $(\mathbf{p}\cdot\mathbf{r} + \mathbf{r}\cdot\mathbf{p})$ must appear at least quadratically in this symmetrized form. Furthermore, as will be seen later [see Eq. (11)], $(\mathbf{p}\cdot\mathbf{r} + \mathbf{r}\cdot\mathbf{p})^2$ can be expressed as a linear combination of r^2, p^2, and $L^2 = (\mathbf{r} \times \mathbf{p})^2$. Hence, we conclude that the σ-independent part V_0 of the general potential V can be written in the form

$$V_0 = V_0(r^2, p^2, L^2) \tag{7}$$

if we arrange V_0 symmetrically with respect to right and left.

Next, we consider the part V_1, which depends linearly on σ_1 and σ_2. By Eq. (4), V_1 must contain σ_1 and σ_2 in the combination $\sigma_1 + \sigma_2 = 2\mathbf{S}$ and the invariants $(\mathbf{S}\cdot\mathbf{r})$ and $(\mathbf{S}\cdot\mathbf{p})$ are forbidden by Eq. (3). Hence, the only possible combination is

$$V_1 = (\mathbf{L}\cdot\mathbf{S})\, V_1(r^2, p^2, L^2) \tag{8}$$

where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and $(\mathbf{L}\cdot\mathbf{S})$ in Eq. (8) need not be symmetrized with respect to right and left, because it commutes with r^2, p^2, and L^2 in V_1; furthermore, the same arguments are used in the derivation of the r^2, p^2, L^2 dependence of V_1 as of V_0.

Finally, let us consider the bilinear term V_2. In this case, σ_1 and σ_2 can appear in the combinations:

$$(\sigma_1 \cdot \sigma_2), \qquad (\sigma_1\cdot\mathbf{r})\cdot(\sigma_2\cdot\mathbf{r}), \qquad (\sigma_1\cdot\mathbf{p})\cdot(\sigma_2\cdot\mathbf{p}),$$
$$(\sigma_1\cdot\mathbf{L})(\sigma_2\cdot\mathbf{L}) + (\sigma_2\cdot\mathbf{L})(\sigma_1\cdot\mathbf{L}),$$

or

$$(\sigma_1\cdot\mathbf{p})(\sigma_2\cdot\mathbf{r}) + (\sigma_2\cdot\mathbf{p})(\sigma_1\cdot\mathbf{r}) + \text{h.c.}$$

all of which satisfy conditions (III), (IV), and (V). However, the last term will change sign under the time-reversal. Thus, it must appear in the form:

$$[(\sigma_1\cdot\mathbf{p})(\sigma_2\cdot\mathbf{r}) + (\sigma_2\cdot\mathbf{p})(\sigma_1\cdot\mathbf{r}) + \text{h.c.}]\cdot(\mathbf{p}\cdot\mathbf{r} + \mathbf{r}\cdot\mathbf{p}) \tag{9}$$

if we symmetrize the potential so that V_2 is symmetrical with respect to right and left again. However, (9) is not independent of other quantities, as we can see from the following identity:

$$\begin{aligned}\tfrac{1}{2}(L_\mu L_\nu + L_\nu L_\mu) \equiv \delta_{\mu\nu}(L^2 - \tfrac{1}{2}) - \tfrac{1}{2}(r^2 p_\mu p_\nu + p_\mu p_\nu r^2) \\ -\tfrac{1}{2}(r_\mu r_\nu p^2 + p^2 r_\nu r_\mu) + \tfrac{1}{4}(r_\mu p_\nu + r_\nu p_\mu + p_\nu r_\mu + p_\mu r_\nu)(\mathbf{p}\cdot\mathbf{r} + \mathbf{r}\cdot\mathbf{p})\end{aligned} \tag{10}$$

with

$$L^2 = \tfrac{1}{2}(r^2 p^2 + p^2 r^2) - [\tfrac{1}{2}(\mathbf{r}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{r})]^2 + \tfrac{3}{4} \tag{11}$$

The last term on the right-hand side of (10) need not be symmetrized, because $[r_\mu p_\nu, (\mathbf{r}\cdot\mathbf{p})] \equiv 0$. Hence, we can write:

$$\begin{aligned}V_2 = (\sigma_1 \cdot \sigma_2)F_1 + (\sigma_1\cdot\mathbf{r})(\sigma_2\cdot\mathbf{r})F_2 + (\sigma_1\cdot\mathbf{p})(\sigma_2\cdot\mathbf{p})F_3 \\ + \tfrac{1}{2}[(\sigma_1\cdot\mathbf{L})(\sigma_2\cdot\mathbf{L}) + (\sigma_2\cdot\mathbf{L})(\sigma_1\cdot\mathbf{L})]F_4 + \text{h.c.}\end{aligned} \tag{12}$$

where the F_i are functions of r^2, p^2, and L^2 only, i.e.,

$$F_i = F_i(r^2, p^2, L^2). \tag{13}$$

Then, the most general velocity-dependent nonrelativistic potential is

$$V = V_0 + V_1 + V_2, \tag{14}$$

where V_0, V_1, and V_2 are given by Eqs. (7), (8), and (12), respectively, and in addition, all the spin-independent functions there (V_0, V_1, and F_i) are real, according to the Hermiticity condition Eq. (6). Equations (7) and (8) are not new except for the possible p^2 and L^2 dependence. The new terms in Eq. (12) are the last two, $(\sigma_1\cdot\mathbf{p})(\sigma_2\cdot\mathbf{p})$[1] and $[(\sigma_1\cdot\mathbf{L})(\sigma_2\cdot\mathbf{L}) + (\sigma_2\cdot\mathbf{L})(\sigma_1\cdot\mathbf{L})]$. (In the next section, we shall show that these new terms are really implicit in present field theory.) It appears that we have essentially six independent potentials, one from (7), a second from (8) and the remaining four from (12). On the other hand, it has been shown (*5*) that the scattering matrix on the energy shell for two nucleons has essentially five independent components if we make use of the unitarity condition [equivalent to the Hermiticity condition Eq. (6)]. On the basis of an analysis of the scattering matrix, Puzikov *et al.* (*5*) have argued that the most general nonrelativistic potential between nucleons which reproduces the scattering matrix is:

$$V = V_1(r) + V_2(r)\mathbf{L}\cdot\mathbf{S} + (\sigma_1\cdot\sigma_2)V_3(r) + S_{12}V_4(r) + (\sigma_1\cdot L)(\sigma_2\cdot L)V_5(r). \tag{14a}$$

Equation (14a) differs from our Eq. (14) in two respects:

(1) Our general expression seems to contain a sixth type of term depending on $(\sigma_1\cdot\mathbf{p})(\sigma_2\cdot\mathbf{p})$.

(2) All of our coefficients are presumed to be functions of r^2, p^2, L^2 whereas Puzikov *et al.* regard their coefficients as functions of r^2 only.

It can readily be shown that the $(\sigma_1\cdot\mathbf{p})(\sigma_2\cdot\mathbf{p})$ potential is redundant provided one is interested only in the scattering matrix (see Appendix). This means that on the energy shell, the $(\sigma_1\cdot\mathbf{p})(\sigma_2\cdot\mathbf{p})$ term yields the same result as a linear combination of the other terms; however, off the energy shell, it must be considered as an independent term. One need not therefore invoke a $(\sigma_1\cdot\mathbf{p})(\sigma_2\cdot\mathbf{p})$ potential to completely define the S-matrix. However, for calculations which require a knowledge of the matrix element off the energy shell or of the wave function, the presence of the $(\sigma_1\cdot\mathbf{p})(\sigma_2\cdot\mathbf{p})$ term will give a different answer from an equivalent potential which does not contain this term but which yields the same scattering matrix element on the energy shell. This reflects the essential arbitrariness in deducing potentials from the S matrix.

As regards our more general dependence on r^2, p^2, L^2, the situation is as follows. We can idealize and suppose that the two nucleons are spinless so that the most general potential is given by Eq. (7). In this fictitious scattering problem, we decompose the scattering matrix into partial waves and suppose that the scattering matrix (or equivalently the phase-shift) corresponding to each partial wave is known. Then, from the work of Jost and Kohn (*7*), we know that the

[1] Breit (*6*) has investigated some of the properties of the $(\sigma_1\cdot\mathbf{p})(\sigma_2\cdot\mathbf{p})$ potential.

potential is uniquely determined for each partial wave, if there is no bound-state. If there are bound-states, the potential cannot be determined uniquely, but there are many equivalent potentials. For definiteness, suppose furthermore that there are no bound states and so the potential can be determined uniquely for every partial wave. Thus, if the scattering matrix is known, we can determine the potential uniquely and it must be a function of r^2 and L^2 only. Hence, the p^2 dependence of V_0 in (7) is actually redundant for the purpose of explaining the idealized scattering experiment. However, again, for processes involving off the energy shell matrix elements, the p^2 dependence of the potential must be considered explicitly.

The above argument can be extended to the coefficients in Eqs. (8) and (12). Consequently, if we are only interested in the analysis of scattering experiments, the most general nonrelativistic potential can be written in the form:

$$V' = G_1 + (\sigma_1\cdot\sigma_2)G_2 + (\mathbf{L}\cdot\mathbf{S})G_3 + (\sigma_1\cdot\mathbf{r})(\sigma_2\cdot\mathbf{r})G_4 + [(\sigma_1\cdot\mathbf{L})(\sigma_2\cdot\mathbf{L}) + (\sigma_2\cdot\mathbf{L})(\sigma_1\cdot\mathbf{L})]G_5$$

where the G_i are only functions of r^2 and L^2. We therefore conclude that Puzikov *et al.* are correct in stating that only five independent potentials are required to determine the nonrelativistic S matrix between two nucleons but that the coefficients should be regarded as functions of L^2 as well as of r^2.

3. MESON THEORY OF NONSTATIC POTENTIALS

In section 2, we showed that the general nonstatic potential may contain $(\sigma_1\cdot\mathbf{p})(\sigma_2\cdot\mathbf{p})$ or $(\sigma_1\cdot\mathbf{L})(\sigma_2\cdot\mathbf{L})$ terms as well as the $\mathbf{L}\cdot\mathbf{S}$ term. In this section, we investigate whether any of these potentials are predicted by meson field theory and if so, what are the qualitative features of these predictions.

As the pion-nucleon interaction Hamiltonian, we take

$$H = H_p + H_{S1} + H_{S2}, \tag{15}$$

where

$$H_P = i\left(\frac{f}{\mu}\right)\bar{\psi}\gamma_5\gamma_\mu\frac{\partial}{\partial x_\mu}(\boldsymbol{\tau}\cdot\boldsymbol{\phi})\psi, \tag{16a}$$

$$H_{S1} = \rho_1\frac{f^2}{\mu}\bar{\psi}\phi^2\psi, \tag{16b}$$

$$H_{S2} = \rho_2\left(\frac{f}{\mu}\right)^2\bar{\psi}\gamma_4\boldsymbol{\tau}\cdot\left(\boldsymbol{\phi}\times\frac{\partial}{\partial t}\boldsymbol{\phi}\right)\psi, \tag{16c}$$

where Eq. (16a) is the P-wave interaction, (16b) the isotopic spin independent S-wave interaction and (16c) the isotopic spin dependent S-wave interaction.

The notation is standard for the wave functions and operators, f is the renormalized coupling constant and ρ_1 and ρ_2 are factors which are explained below.

The factors ρ_1 and ρ_2 are determined as follows. It is well known that the PS-PS interaction can be brought into the form (15) and (16) by means of a Foldy-Dyson transformation (*8*). The first three terms of the transformed Hamiltonian coincide with Eqs. (16a)–(16c) provided we take (G is the PS-PS coupling constant):

$$f = \left(\frac{\mu}{2M}\right) \cdot G, \qquad \rho_1 = \frac{2M}{\mu}, \qquad \rho_2 = 1. \tag{17}$$

If account is taken of the higher order corrections (*9*), the values of ρ_1 and ρ_2 may be changed. Indeed, in the present state of meson theory, it is more reasonable to treat ρ_1 and ρ_2 as arbitrary parameters which are to be determined by experiment.

From the analysis of low-energy pion-nucleon scattering by a type of Kroll-Ruderman theorem, Klein (*10*) has derived the following value for the renormalized ρ_1 and ρ_2 :

$$\rho_1 \cong 0.14, \qquad \rho_2 \cong 0.56.$$

By using a modification of the Chew-Low formalism, Drell *et al.* (*11*) have obtained:

$$\rho_1 \cong 0.4 \qquad \rho_2 \cong 0.4$$

We have calculated the value of the renormalized ρ_2 directly from the Panofsky ratio by using the Kroll-Ruderman theorem (*12*) for the photo-pion process and the corresponding theorem for the low-energy S-wave pion-nucleon scattering (*10*). If we take 1.6 as the experimental value of the Panofsky ratio (*13*), we find

$$\rho_2 \cong 1.3.$$

These estimates of ρ_1 and ρ_2 are in reasonable agreement and imply that the value of ρ_1 given by (*17*) is much too large whereas ρ_2 is approximately correct. Somehow the higher order corrections to the PS-PS theory severely damp the nonisotopic spin dependent S-wave pion-nucleon interaction without appreciably affecting the isotopic spin dependent S-wave interaction, an effect which is not completely understood at the present time (*14*).

If one uses the emperically determined values of ρ_1 and ρ_2, the three contributions (16a), (16b), and (16c) to the pion-nucleon interaction Hamiltonian are all of the same order and we must take account of all of them in order to calculate the two-nucleon interaction. One of the chief reasons for the failure of the Lévy

potential (*1*) was the dominant role played by H_{s1} with the perturbation value for ρ_1, i.e., $\rho_1 = 2M/\mu \cong 13.5$ (instead of $\lesssim 1$).

We shall not undertake a detailed calculation of the two-nucleon interaction on the basis of the interaction Hamiltonian (15) and (16) since all available methods are quite unreliable. Our aim is rather modest. We purpose merely to use (15) and (16) to spell out the various types of nonstatic contributions to the two-nucleon interaction in the lowest orders in which they arise. We shall see that all the velocity dependent potentials considered in Section 2 are contained in meson field theory.

For our purposes, it suffices to employ the Tamm-Dancoff method (*15*). We expand the potential in powers of μ/M.

$$V = V^{(0)} + V^{(1)} + V^{(2)} + \cdots . \tag{18}$$

The superscript here indicates the power of μ/M and we regard (15) and (16) as being of zero order in μ/M.

Up to fourth order with respect to the coupling constant f, the Feynman diagrams are shown in Fig. 1. The zeroth order potential $V^{(0)}$ has been calculated by several authors (*16*, *17*) and it yields the static terms in the two-nucleon interaction which we shall not discuss here. We next consider $V^{(1)}$, namely the correction of order μ/M to $V^{(0)}$. It will give rise to a spin-orbit force.

As we see later, the second order diagram (I), does not give any contribution to $V^{(1)}$. The ($\mathbf{L}\cdot\mathbf{S}$) potential arising from diagram (II) has already been calcu-

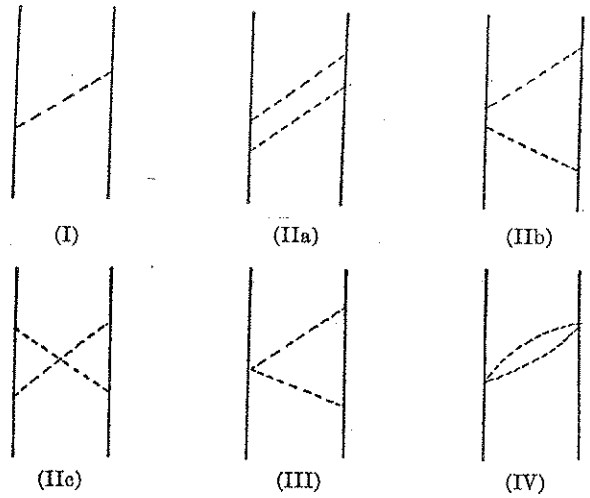

FIG. 1. Feynman diagrams.

lated (*18*) and it gives

$$V^{(1)}(\mathrm{II}) = -(2\mu)\left(\frac{\mu}{M}\right)\left(\frac{f^2}{4\pi}\right)^2 \frac{1}{x^6}(1+x)(1+x+x^2)e^{-2x} \times (3+2\tau_1\cdot\tau_2)(\mathbf{L}\cdot\mathbf{S}), \tag{19}$$

where

$$x \equiv \mu r.$$

It is worthwhile to note that the $\mathbf{L}\cdot\mathbf{S}$ potential (19) is of order μ/M and not of order $(\mu/M)^2$ which is the order of the usual Thomas $\mathbf{L}\cdot\mathbf{S}$ potential. The Brueckner-Watson diagram (IIb) does not contribute to the $\mathbf{L}\cdot\mathbf{S}$ potential in this order.

The next contribution to the $\mathbf{L}\cdot\mathbf{S}$ potential comes from diagram III:

$$V^{(1)}(\mathrm{III}) = -(4\mu)\left(\frac{\mu}{M}\right)\rho_2\cdot\left(\frac{f^2}{4\pi}\right)^2(\tau_1\cdot\tau_2)\frac{1}{x^6}(1+x)^2e^{-2x}(\mathbf{L}\cdot\mathbf{S}). \tag{20}$$

It is interesting to note that Eqs. (19) and (20) have entirely different origins. Equation (19) results from a correction of order μ/M in the evaluation of the matrix element of (16a), whereas Eq. (20) is due to the nucleon recoil effect in the energy-denominator when (16c) is considered. The diagram (III) arising from the isotopic spin independent Hamiltonian, H_{s1} (16b), does not yield an $\mathbf{L}\cdot\mathbf{S}$ potential in order μ/M due to a rather accidental cancellation between a contribution from the correction of the matrix element of (16a) and one from the recoil effect in the energy-denominator. Furthermore, the diagram (IV) does not give any $\mathbf{L}\cdot\mathbf{S}$ potential in this order.[2]

Thus, the total $\mathbf{L}\cdot\mathbf{S}$ potential in order μ/M is:

$$V^{(1)}_{L-S} = V^{(1)}(\mathrm{II}) + V^{(1)}(\mathrm{III}), \tag{21}$$

where $V^{(1)}(\mathrm{II})$ and $V^{(1)}(\mathrm{III})$ are given by (19) and (20). The effect of multiple meson scattering on the $\mathbf{L}\cdot\mathbf{S}$ potential has also been calculated by one of the authors (S. O.), using dispersion relations. The results will be published separately (*19*).

If we compare Eqs. (19) and (20) with the spin-orbit potential used by Signell and Marshak (*2*) [and by Gammel and Thaler (*2*)] to fit the two-nucleon scattering data, we observe that the attractive short-range character of this potential is correctly predicted by meson theory in the lowest order in which it is calculated. This is encouraging although it may be an accident. Quantititave agreement is not obtained and should not be expected.

[2] This diagram gives rise to a repulsive spin-orbit potential in order $\rho_1^2(\mu/M)^2$. When one sets $\rho_1 = (2M/\mu)$, this becomes the Lévy-Klein spin-orbit potential in fourth order which must be discarded because effectively $\rho_1 \lesssim 1$ (see above).

Finally, let us consider the $(\mu/M)^2$ correction $V^{(2)}$. In this order, for the first time, both $(\sigma_1\cdot\mathbf{p})(\sigma_2\cdot\mathbf{p})$ and $(\sigma_1\cdot\mathbf{L})(\sigma_2\cdot\mathbf{L})$ terms appear. The calculation of these terms using the fourth order diagrams (II) is quite complicated. However, $(\mu/M)^2$ corrections also appear in the second order diagram (I). We give here only the contribution from (I), although the contribution from (II) has also been calculated. We find:

$$V(\mathrm{I}) = V^{(0)}(\mathrm{I}) + V^{(1)}(\mathrm{I}) + V^{(2)}(\mathrm{I}) + O\left[\left(\frac{\mu}{M}\right)^3\right], \tag{22}$$

where

$$V^{(0)}(\mathrm{I}) = -(\sigma_1\cdot\nabla)(\sigma_2\cdot\nabla)A_2(r), \tag{23}$$

$$V^{(1)}(\mathrm{I}) = -\frac{1}{M}(\sigma_1\cdot\nabla)(\sigma_2\cdot\nabla)\left[-\frac{1}{2}(\Delta A_3) + i(\nabla\cdot\mathbf{p})A_3\right], \tag{24}$$

$$\begin{aligned} V^{(2)}(\mathrm{II}) = {} & -\frac{1}{M^2}(\sigma_1\cdot\nabla)(\sigma_2\cdot\nabla)\left[\frac{1}{4}(\Delta A_2) - (\tilde{\mathbf{p}})^2A_2 - (\nabla\cdot\tilde{\mathbf{p}})^2A_4\right], \\ & -\frac{1}{M^2}(\sigma_1\cdot\tilde{\mathbf{p}})(\sigma_2\cdot\tilde{\mathbf{p}})A_0 - \frac{1}{2M^2}[(\sigma_1\cdot\nabla)(\sigma_2\cdot\tilde{\mathbf{p}}) \\ & + (\sigma_2\cdot\nabla)(\sigma_1\cdot\tilde{\mathbf{p}})](\tilde{\mathbf{p}}\cdot\nabla)A_2, \end{aligned} \tag{25}$$

with

$$A_n(r) = -\frac{1}{(2\pi)^3}\left(\frac{f}{\mu}\right)^2(\tau_1\cdot\tau_2)\int d^3\mathbf{k}\,\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\omega_k{}^n} \tag{26}$$

and in (25), $\tilde{\mathbf{p}}$ means

$$\tilde{p}_\mu A \equiv \tfrac{1}{2}(p_\mu A + Ap_\mu)$$

so that:

$$\tilde{p}_\mu\tilde{p}_\nu A \equiv \tfrac{1}{4}(p_\mu p_\nu A + p_\mu A p_\nu + p_\nu A p_\mu + A p_\mu p_\nu).$$

$V^{(0)}(\mathrm{I})$ is the well-known Yukawa-potential. The reason why we have written down the explicit form for V is that $V^{(1)}(\mathrm{I})$ is not hermitian and appears to violate the hermiticity condition given in the previous section. However, this is only apparent. It has already been pointed out (*20*) by one of the authors that such a non-Hermitian potential has the same origin as the non-Hermitian potential which arises in the reduction of Dirac equation to the two-component Pauli equation.

The non-Hermiticity can be removed by amplitude renormalization (*20*). The Tamm-Dancoff equation gives

$$\left(E + \frac{1}{M}\Delta\right)\psi(\mathbf{r}) = V(\mathrm{I})\psi(\mathbf{r}) \tag{27}$$

where $\psi(\mathbf{r})$ is the Tamm-Dancoff amplitude in coordinate space. If we change the wave function from $\psi(\mathbf{r})$ to $\varphi(\mathbf{r})$ where

$$\varphi(\mathbf{r}) \equiv [1 + (\sigma_1\cdot\nabla)(\sigma_2\cdot\nabla)A_3]^{1/2}\psi(\mathbf{r}), \tag{28}$$

(27) become

$$\left(E + \frac{1}{M}\Delta\right)\varphi(\mathbf{r}) = [V(\mathrm{I}) - V^{(1)}(\mathrm{I})]\varphi(\mathbf{r}) \tag{29}$$

if we neglect terms of higher order than $(\mu/M)^3$ and f^4. Thus, the new potential becomes

$$\bar{V}(\mathrm{I}) \equiv V(\mathrm{I}) - V^{(1)}(\mathrm{I}) = V^{(0)}(\mathrm{I}) + V^{(2)}(\mathrm{I}) \tag{30}$$

and this is Hermitian. In this fashion, the potential of order μ/M disappears completely. The meaning of the transformation (28) can be found in Ref. *20*.

$V^{(2)}(\mathrm{I})$ in Eq. (25) contains a potential of the form

$$[(\sigma_1\cdot\mathbf{p})(\sigma_2\cdot\mathbf{r}) + (\sigma_2\cdot\mathbf{p})(\sigma_1\cdot\mathbf{r})] \times (\mathbf{p}\cdot\mathbf{r} + \mathbf{r}\cdot\mathbf{p}).$$

Then, as was explained in Section 2, this can be rewritten as a linear combination of $(\sigma_1\cdot\mathbf{p})(\sigma_2\cdot\mathbf{p})$ and $(\sigma_1\cdot\mathbf{L})(\sigma_2\cdot\mathbf{L})$ by Eq. (10). Since we are particularly interested in these potentials, we pick them out of Eq. (25) with the result:

$$\begin{aligned} V^{(2)}(\mathrm{I}) = (6\mu)\left(\frac{\mu}{2M}\right)^2\left(\frac{f^2}{4\pi}\right)(\tau_1\cdot\tau_2)\frac{1}{x^5}(3 + 3x + x^2)e^{-x} \\ \times \{\tfrac{1}{2}[(\sigma_1\cdot\mathbf{L})(\sigma_2\cdot\mathbf{L}) + (\sigma_2\cdot\mathbf{L})(\sigma_1\cdot\mathbf{L})] - (\sigma_1\cdot\sigma_2)\mathbf{L}^2\} \\ + \mu\left(\frac{\mu}{2M}\right)^2\left(\frac{f^2}{4\pi}\right)\cdot\left\{(\tau_1\cdot\tau_2)\frac{1}{x^3}(5 + 5x + 3x^2)e^{-x}\right. \\ \left. \times \frac{(\sigma_1\cdot\mathbf{p})(\sigma_2\cdot\mathbf{p})}{\mu^2} + \text{h.c.}\right\}, \end{aligned} \tag{31}$$

where $x = \mu r$ again.

Equation (31) shows that the quadratic spin-orbit potential arises in meson theory as a correction of order $(\mu/M)^2$ to the second-order static potential in contrast to the linear spin-orbit potential which originates as a (μ/M) correction to the fourth-order static potential. Meson theory therefore predicts that the quadratic spin-orbit potential is of larger range but more shallow than the linear spin-orbit potential. It is also interesting to note that meson theory predicts a difference in sign between the quadratic and linear spin-orbit potentials in the lowest orders in which they have been calculated. While it is expected that the higher order meson-theoretic corrections will be important, it is hoped that (Eq. 31) will be as useful a guide for phenomenological calculations with quadratic spin-orbit potentials[3] as Eq. (21) has been for the linear spin-orbit potential (*2*).

APPENDIX

Let us consider the scattering matrix from the initial momentum $\mathbf{k}_i$ to the final momentum $\mathbf{k}_f$. Here, we treat only the part containing σ_1 and σ_2 bilinearly. Instead of $\mathbf{k}_i$ and $\mathbf{k}_f$, it is more convenient to use:

$$\mathbf{k} = \mathbf{k}_i - \mathbf{k}_f, \qquad \mathbf{k}' = \mathbf{k}_i + \mathbf{k}_f. \tag{A1}$$

The scattering matrix then contains terms of the type:

$$(\sigma_1\cdot\mathbf{k})(\sigma_2\cdot\mathbf{k}), \quad (\sigma_1\cdot\mathbf{k}')(\sigma_2\cdot\mathbf{k}'), \quad (\sigma_1\cdot\mathbf{k}\times\mathbf{k}')(\sigma_2\cdot\mathbf{k}\times\mathbf{k}'), \quad (\sigma_1\cdot\sigma_2). \tag{A2}$$

It is immediately evident that only three of the terms in (A2) are independent because of the relation

$$\begin{aligned} (\sigma_1\cdot\sigma_2) \equiv \frac{1}{k^2}(\sigma_1\cdot\mathbf{k})(\sigma_2\cdot\mathbf{k}) + \frac{1}{k'^2}(\sigma_1\cdot\mathbf{k}')(\sigma_2\cdot\mathbf{k}') \\ + \frac{1}{k^2k'^2}(\sigma_1\cdot\mathbf{k}\times\mathbf{k}')(\sigma_2\cdot\mathbf{k}\times\mathbf{k}'), \end{aligned} \tag{A3}$$

where we have used

$$\mathbf{k}\cdot\mathbf{k}' = \mathbf{k}_i^2 - \mathbf{k}_f^2 \equiv 0. \tag{A4}$$

It can be shown that no other bilinear term in σ appears in the scattering matrix other than those of (A2). For example, let us consider a term of the type

$$(\sigma_1\cdot\mathbf{k})(\sigma_2\cdot\mathbf{k}'). \tag{A5}$$

Under time reversal, $\mathbf{k}_i \rightleftarrows -\mathbf{k}_f$ and so

$$\mathbf{k} \to +\mathbf{k}, \qquad \mathbf{k}' \to -\mathbf{k}', \qquad \sigma_i \to -\sigma_i.$$

Thus, (A5) is not invariant under time reversal. The only possible way to achieve invariance is through the combination:

$$(\sigma_1\cdot\mathbf{k})(\sigma_2\cdot\mathbf{k}')(\mathbf{k}\cdot\mathbf{k}'). \tag{A6}$$

However, on the energy shell, $\mathbf{k}\cdot\mathbf{k}' = 0$ [see (A4)], and so (A6) vanishes.

Hence, the S-matrix can be expressed as a linear combination of three of the terms in (A2) insofar as the bi-linear terms with respect to σ_1 and σ_2 are concerned. Consequently, on the energy shell, a $(\sigma_1\cdot\mathbf{p})(\sigma_2\cdot\mathbf{p})$ term in the potential will be equivalent to a linear combination of $(\sigma_1\cdot\sigma_2)$, $(\sigma_1\cdot\mathbf{r})(\sigma_2\cdot\mathbf{r})$ and $(\sigma_1\cdot\mathbf{L})(\sigma_2\cdot\mathbf{L})$

[3] Mr. R. Bryan is carrying out such calculations at the present time in an attempt to improve the fit with the two-nucleon scattering data at 300 Mev.

terms. However, off the energy shell, (A4) is no longer true and (A6) is no longer zero. Thus, the scattering matrix off the energy shell has four independent bilinear terms. This implies that a $(\sigma_1 \cdot \mathbf{p})(\sigma_2 \cdot \mathbf{p})$ term is independent of $(\sigma_1 \cdot \mathbf{r})(\sigma_2 \cdot \mathbf{r})$, $(\sigma_1 \cdot \mathbf{L})(\sigma_2 \cdot \mathbf{L})$ and $(\sigma_1 \cdot \sigma_2)$ terms in the potential.

A similar argument can be given with respect to the discussion at the end of Section 2. In the Born approximation, an ordinary potential $V_1(r)$ yields a scattering matrix which can be written in the form:

$$f_1(k^2) \tag{A7}$$

and $p^2V_2(r) + V_2(r)p^2$ yields:

$$(k^2 + k'^2)f_2(k^2) \tag{A8}$$

and finally, $L^2V_3(r)$ leads to

$$(\mathbf{k}_i \times \mathbf{k}_f)^2 f_3(k^2) = \tfrac{1}{4}[k^2k'^2 - (\mathbf{k}\cdot\mathbf{k}')^2]f_3(k^2). \tag{A9}$$

On the energy shell (A4) holds and hence of (A7), (A8), and (A9), only two are independent (k^2 and k'^2 are two independent variables). Thus $p^2V_2(r) + V_2(r)p^2$ gives essentially the same result as a linear combination of $V_1(r) + L^2V_3(r)$, as was stated at the end of Section 2. However, off the energy shell, k^2, k'^2, and $(\mathbf{k}\cdot\mathbf{k}')^2$ are independent and so are the three quantities (A7), (A8), and (A9). Hence, off the energy shell $p^2V_2(r) + V_2(r)p^2$ is independent of $V_1(r)$ and $L^2V_3(r)$.

RECEIVED: February 11, 1958

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